A Groupoid Approach to Fourier Integral Operators

Groupoids and Operator algebras
Orléans

Joint works with J.M. Lescure and D. Manchon (UCA)

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Some examples of Lie groupoids

- Spaces $X ightrightarrows X$ and Lie groups $G ightrightarrows \{e\}$. 
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- Spaces $X \rightrightarrows X$ and Lie groups $G \rightrightarrows \{e\}$.
- Pair groupoids: $X \times X \rightrightarrows X$ and fibred pair groupoids: $X \times X \rightrightarrows_X X$. 
- Vector bundles $E \rightrightarrows X$.
- Transformation groupoids: $G \times X \rightrightarrows X$: $s(\mathbf{g},x) = x$, $r(\mathbf{g},x) = \mathbf{g} \cdot x$, $(\mathbf{g},hx) = (\mathbf{gh},x)$, $\iota(\mathbf{g},x) = (\mathbf{g}^{-1},\mathbf{gx})$.
- The tangent groupoid of a manifold: $T\!M = \left(TM \times \{0\}\right) \sqcup \left(0 \times _B [0,1] \times M \times M\right) \rightrightarrows \left([0,1] \times M\right)$.
- The $b$-groupoid of a manifold with boundary: $(M \setminus \partial M) \times (M \setminus \partial M) \cup \partial M \times \partial M \times R \rightrightarrows M$.
- Analogous groupoids for stratified spaces and (singular) foliations...
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- Spaces $X \xrightarrow{\sim} X$ and Lie groups $G \xrightarrow{\sim} \{e\}$.
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### $G$-PDOs: Examples

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Can we find a notion of $G$-FIOs yielding a similar table?
FIOs are useful in the theory of linear PDE on compact manifolds: strictly hyperbolic problems, asymptotics of spectra, singularities of \( \text{Tr}(e^{-itP}) \in \mathcal{D}'(\mathbb{R}) \), Egorov theorem . . .

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Lagrangian distributions (Hörmander)

Let $X$ be a $C^\infty$ manifold of dimension $n$ and $\Lambda \subset T^*X \setminus 0$ a conic lagrangian sub-mfd. The set $I^m(X, \Lambda)$, $m \in \mathbb{R}$, consists of distributions $u \in \mathcal{D}'(X)$ of the form:

$$u = \sum_{j \in J} \int e^{i\phi_j(x, \theta_j)} a_j(x, \theta_j) d\theta_j \mod C^\infty(X)$$

where

- $(x, \theta_j) \in \mathcal{V}_j \subset U_j \times \mathbb{R}^{N_j}$ with $U_j$ a coord. patch, $\mathcal{V}_j$ open and homogeneous;
- $\phi_j : \mathcal{V}_j \to \mathbb{R}$ is a non-degenerate phase parametrizing $\Lambda$;
- $a_j(x, \theta_j) \in S^{m+(nx-2N_j)/4}(U_j \times \mathbb{R}^{N_j})$ and $\text{supp}(a_j) \subset \mathcal{V}_j \setminus 0$.

$I^m(X, \Lambda) = \text{Lagrangian distributions on } X \text{ subordinated to } \Lambda$. 


Convolution and $G$-ops

**Convolution in $C_c^\infty(G)$**

$$f \ast g(\gamma) = \int_{\gamma_1 \gamma_2 = \gamma} f(\gamma_1)g(\gamma_2) = m_*(f \otimes g|_{G^{(2)}}), \quad f, g \in C_c^\infty(G)$$

**$G$-operators**

Continuous linear maps $P : C_c^\infty(G) \to C^\infty(G)$ such that

$$P(f \ast g) = P(f) \ast g \quad \text{for any } f, g \in C_c^\infty(G),$$

Equivalently, $P = (P_x)_{x \in M}$ with $P_x : C_c^\infty(G_x) \to C^\infty(G_x)$, $P_{r(\gamma)} R_\gamma = R_\gamma P_{s(\gamma)}$ and

$$(Pf)|_{G_x} = P_x(f|_{G_x}), \quad f \in C_c^\infty(G).$$

(Notations: $G_x = s^{-1}(x)$, $G^x = r^{-1}(x)$).

$G$-PDOs = $G$-Ops $P$ with $P_x$ a pseudodifferential operator on $G_x$ for any $x$. Actually:

$$G - \text{PDOs} = I(G, M)$$
Convolution of distributions

FIOs are lagrangian distributions. To get a satisfactory approach on groupoids (calculus, continuity, Egorov, evolution equations...), we need:

- to understand the convolution of distributions;
- to select a suitable set of Lagrangian submanifolds.

There are two ways of extending convolution product of functions to distributions:

- Under transversality conditions
- Under wave front sets conditions
Let $G \rightrightarrows M$ be a Lie groupoid.

- For any $u \in \mathcal{D}'(G)$ and $f \in C^\infty_c(G)$, define $s_*(uf) \in \mathcal{D}'(M)$ by:
  \[
g \in C^\infty(M), \quad \langle s_*(uf), g \rangle = \langle u, f.s^*g \rangle.
  \]

This provides an isomorphism:

\[
s_* : \mathcal{D}'(G) \overset{\sim}{\longrightarrow} \mathcal{L}_{C^\infty(M)}(C^\infty_c(G), \mathcal{D}'(M)). \tag{1}
\]

- Consider the subspace of $s$-transversal distributions on $G$:
  \[
  \mathcal{D}'_s(G) = \{ u \in \mathcal{D}'(G) ; \text{Im}(s_*(u)) \subset C^\infty(M) \}. \tag{2}
  \]

Then

\[
s_* : \mathcal{D}'_s(G) \overset{\sim}{\longrightarrow} \mathcal{L}_{C^\infty(M)}(C^\infty_c(G), C^\infty(M)). \tag{3}
\]

appropriate densities are understood, similar statements hold with $r$ instead of $s$. 
Convolution in \( \mathcal{D}' \)

**Theorem (LMV)**

Convolution of functions extends (separately) continuously to:
- \( \mathcal{D}'_s(G) \times \mathcal{E}'(G) \rightarrow^* \mathcal{D}'(G) \),
- \( \mathcal{D}'_r(G) \times C^\infty_c(G) \rightarrow^* C^\infty(G) \),
- \( \mathcal{D}'_s(G) \times \mathcal{E}'_r(G) \rightarrow^* \mathcal{D}'_s(G) \)
- \( \ldots \)

**Corollary**

- \( \mathcal{E}'_s(G), \mathcal{E}'_r(G) \) are unital algebras.
- \( \mathcal{E}'_{r,s}(G) = \mathcal{E}'_s(G) \cap \mathcal{E}'_r(G) \) is a unital involutive algebra.

Unit: \( \langle \delta, f \rangle = \int_M f \), involution: \( u^* = \bar{v}^*(u) \).
This also proves that $G$-ops are operators given by convolution with distributions.
The map

$$
\mathcal{D}_r'(G) \longrightarrow \text{Op}_G, \ u \longmapsto (f \mapsto u * f)
$$

is well defined and gives:

$$
\mathcal{D}_r'(G) \cong \text{Op}_G \quad \text{(space of G-operators)}
$$

and

$$
\mathcal{D}_{r,s}'(G) \cong \text{Op}_G^* \quad \text{(subspace of G-operators with adjoints)}
$$
Coste-Dazord-Weinstein groupoid $T^*G$

Let $G \xrightarrow{} M$ be a Lie groupoid.

Understanding under which conditions on the wave front sets of $u, v \in \mathcal{D}'(G)$, the convolution product $u \ast v$ is defined leads to the algebraic structure of the cotangent space $T^*G$.

There exists a natural symplectic Lie groupoid structure on $T^*G$ with unit space $A^*G = N^*M$.

$$\Gamma = (T^*G \xrightarrow{} A^*G = N^*M)$$

A groupoid $\Gamma \xrightarrow{} \Gamma^{(0)}$ is symplectic if $\Gamma$ is a symplectic manifold and if the graph of the multiplication map is lagrangian in $(-\Gamma) \times \Gamma \times \Gamma$. 
CDW groupoid: $\Gamma = T^*G \rightrightarrows A^*G$.

The product $(\gamma_1, \xi_1).(\gamma_2, \xi_2) = (\gamma, \xi)$ is defined by

$$\gamma = \gamma_1 \gamma_2$$

and the equality

$$\xi(t) = \xi_1(t_1) + \xi_2(t_2)$$

for any $t \in T_{\gamma}G$ and $(t_1, t_2) \in T_{(\gamma_1, \gamma_2)}G^{(2)}$ such that

$$t = dm(t_1, t_2)$$

Well defined iff it does not depend on the choice of such $t_1, t_2$. 
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(Source) $s_{\Gamma}(\gamma, \xi) = (s(\gamma), L^*_\gamma(\xi)) \in A_{s(\gamma)}^*G$,

(Range) $r_{\Gamma}(\gamma, \xi) = (r(\gamma), R^*_\gamma(\xi)) \in A_{r(\gamma)}^*G$,

$s_{\Gamma}(\gamma, \xi)$ is obtained by applying the codifferential of $L_{\gamma} : G^x \to G^r(\gamma)$ to the restriction

$\xi : T_{\gamma}G^r(\gamma) \to \mathbb{R}$. The result is a linear form on $T_xG$ vanishing on $T_xG^{(0)}$, thus an element of $A_x^*G$. 
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(Source) \hspace{1cm} s_{\Gamma}(\gamma, \xi) = (s(\gamma), L^*_\gamma(\xi)) \in A^*_s(\gamma)G,

(Range) \hspace{1cm} r_{\Gamma}(\gamma, \xi) = (r(\gamma), R^*_\gamma(\xi)) \in A^*_r(\gamma)G,

$s_{\Gamma}(\gamma, \xi)$ is obtained by applying the codifferential of $L_\gamma : G^x \to G^r(\gamma)$ to the restriction $\xi : T_{\gamma}G^r(\gamma) \to \mathbb{R}$. The result is a linear form on $T_xG$ vanishing on $T_xG^{(0)}$, thus an element of $A^*_xG$.

(Inversion) \hspace{1cm} (\gamma, \xi)^{-1} = (\gamma^{-1}, -^t(d\iota_\gamma)(\xi)).
Wave front and convolution

Let $W_1, W_2 \subset T^*G \setminus 0$ be closed, conic subsets such that

$$(W_1 \times W_2) \cap \ker m_\Gamma = \emptyset.$$

Then convolution extends continuously to

$$\mathcal{E}'_{W_1}(G) \times \mathcal{E}'_{W_2}(G) \longrightarrow \mathcal{E}'_W(G)$$

(4)

where $W$ is the product of the sets $W_1 \cup 0$ and $W_2 \cup 0$ in $T^*G$. 
Distributions and $G$-ops

Set :
$\mathcal{E}_a'(G) := \{ u \in \mathcal{E}'(G) ; \text{WF}(u) \cap \ker s_\Gamma = \emptyset \text{ and } \text{WF}(u) \cap \ker r_\Gamma = \emptyset \}$.

- $(\mathcal{E}_a'(G), \ast)$ is a unital involutive algebra,
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\[ \mathcal{E}_a'(G) := \{ u \in \mathcal{E}'(G) ; \ WF(u) \cap \ker s_{\Gamma} = \emptyset \text{ and } WF(u) \cap \ker r_{\Gamma} = \emptyset \}. \]

- $(\mathcal{E}_a'(G), \ast)$ is a unital involutive algebra,
- \{ compactly supported $G$-PDOs \} $\subset \mathcal{E}_a'(G) \subset \mathcal{E}_{r,s}'(G)$;
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- \((\mathcal{E}'_a(G), *)\) is a unital involutive algebra,
- \{ compactly supported \(G\)-PDOs \} \(\subset \mathcal{E}'_a(G) \subset \mathcal{E}'_{r,s}(G) ; \)
- for any \(u_1, u_2 \in \mathcal{E}'_a(G)\) we have
  \[ WF(u_1 * u_2) \subset WF(u_1) \cdot WF(u_2) \subset T^*G \setminus (\ker r_\Gamma \cup \ker s_\Gamma) . \]
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- $(\mathcal{E}_a'(G), \ast)$ is a unital involutive algebra,
- $\{ \text{compactly supported } G\text{-PDOs} \} \subset \mathcal{E}_a'(G) \subset \mathcal{E}_{r,s}'(G)$;
- for any $u_1, u_2 \in \mathcal{E}_a'(G)$ we have
  \[ WF(u_1 \ast u_2) \subset WF(u_1) \cdot WF(u_2) \subset T^*G \setminus (\ker r_\Gamma \cup \ker s_\Gamma). \]

- Elements of $\mathcal{E}_a'(G)$ and even $\mathcal{D}_a'(G)$ provide $G$-operators:
  \[ u \in \mathcal{D}_a'(G), \quad \left( \begin{array}{c} \mathcal{C}^\infty_c(G) \\ f \\ \mapsto \quad \mathcal{C}^\infty(G) \\ u \ast f \end{array} \right) \in \text{Op}_G. \]
Examples of the rule for convolution

Let $X$ be a manifold and $X \Rightarrow X$ be the trivial groupoid.

○ Convolution = pointwise multiplication;
Examples of the rule for convolution

Let \( X \) be a manifold and \( X \rightrightarrows X \) be the trivial groupoid.
- Convolution = pointwise multiplication;
- \( \Gamma = T^*X \rightrightarrows N^*X = X \) is the ordinary vector bundle structure;
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Let $X$ be a manifold and $X \xrightarrow{\sim} X$ be the trivial groupoid.

- Convolution = pointwise multiplication;
- $\Gamma = T^*X \xrightarrow{\sim} N^*X = X$ is the ordinary vector bundle structure;
- If $WF(u_1) \times WF(u_2) \cap \{(x, \xi, x, -\xi)\} = \emptyset$ then $u_1 u_2$ is well defined and

$$WF(u_1 u_2) \subset (WF(u_1) + WF(u_2)) \cup WF(u_1) \cup WF(u_2).$$
Examples of the rule for convolution

Let $G = X \times X \Rightarrow X$ be the pair groupoid. Then

- Convolution = composition of integral kernels;

\[
\Gamma = T^* X \times T^* X \Rightarrow N^* \Delta X \simeq T^* X \text{' groupoid:}
\]

\[
(x, \xi, y, \eta) \circ' (y, -\eta, z, \zeta) = (x, \xi, z, \zeta).
\]

If $WF(u_1) \times WF(u_2) \cap \{(x, 0, y, \eta, y, -\eta, z, 0)\} = \emptyset$ then $u_1 \circ u_2 \in D'(X \times X)$ is well defined and $WF(u_1 \circ u_2) \subset (WF(u_1) \cup 0) \circ' (WF(u_2) \cup 0)$. 

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Examples of the rule for convolution

Let $G \ni \{e\}$ be a Lie group. Then

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Examples of the rule for convolution

Let $G \supseteq \{e\}$ be a Lie group. Then

- Convolution = ordinary convolution in $G$;
- $(T^* G \supseteq g^*) \cong (g^* \ltimes G) \ ( (g, \xi) \mapsto (g, R_g^*\xi) )$
Examples of the rule for convolution

Let $G \supseteq \{e\}$ be a Lie group. Then

- Convolution = ordinary convolution in $G$;
- $(T^*G \supseteq g^*) \simeq (g^* \ltimes G) \ ((g, \xi) \mapsto (g, R_g^*\xi))$
- Here $\ker m_G = G^2 \times \{0\}$, therefore $u_1 \ast u_2$ is always defined and

$$WF(u_1 \ast u_2) \subset \{(\xi, g) ; (\xi, h, \text{Ad}_h^*\xi, h^{-1}g) \in WF(u_1) \times WF(u_2), h \in G\} \subset g^* \ltimes G$$
Calculus of Lagrangian submanifolds in $T^*G$

Set $T^*_aG = T^*G \setminus (\ker r_\Gamma \cup \ker s_\Gamma)$.

**$G$-relations**

Conic Lagrangian submanifolds of $T^*G$ contained in $T^*_aG$.

If $G = X \times X$, these are the conic Lagrangian submanifolds contained in $T^*X \setminus 0 \times T^*X \setminus 0$.

1. Let $\Lambda_1, \Lambda_2$ be two $G$-relations. If $\Lambda_1 \times \Lambda_2 \cap \Gamma^{(2)}$ is clean, then $\Lambda_1.\Lambda_2 \subset \Gamma$ is a (immersed) $G$-relation.
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2. Let $\Lambda$ be a $G$-relation. Then $\Lambda^* := \iota_\Gamma(\Lambda)$ is a $G$-relation.
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Set $T_a^*G = T^*G \setminus (\ker r_\Gamma \cup \ker s_\Gamma)$.

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2. Let $\Lambda$ be a $G$-relation. Then $\Lambda^* := \iota_\Gamma(\Lambda)$ is a $G$-relation.

3. Let $\Lambda$ be a $G$-relation. Then

$$r_\Gamma : \Lambda \rightarrow A^*G \text{ and } s_\Gamma : \Lambda \rightarrow A^*G$$

are diffeomorphisms if and only if

$$\Lambda \Lambda^* = A^*G \text{ and } \Lambda^* \Lambda = A^*G.$$ 

Such $G$-relations are called invertible.
$G$-FIOs: definition, composition

**Definition**

$G$-FIOs = Elements of $I(G, \Lambda)$, for any $G$-relation $\Lambda$.

$$G - \text{PDOs} = I(G, A^*G) \subset G - \text{FIOs}$$

$$G - \text{FIOs} \subset D'_a(G) \subset \text{Op}^*_G$$
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\]

If \( \Lambda \) is a \( G \)-relation and \( A \in \text{Im}(G, \Lambda) \), then \( A^* \in \text{Im}(G, \Lambda^*) \).
**$G$-FIOs: definition, composition**

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G - \text{FIOs} \subset D'_a(G) \subset \text{Op}^*_G
\]

If $\Lambda$ is a $G$-relation and $A \in I^m(G, \Lambda)$, then $A^* \in I^m(G, \Lambda^*)$.

Assume that $\Lambda_1, \Lambda_2$ are closed $G$-relations, with a clean intersection $\Lambda_1 \times \Lambda_2 \cap (T^*G)^{(2)}$ of excess $e$. Then convolution gives a map:

\[
I^m_{c1}(G, \Lambda_1) \times I^m_{c2}(G, \Lambda_2) \longrightarrow I^{m_1+m_2+e/2-(n-2n^{(0)})/4}(G, \Lambda_1\Lambda_2).
\]

$n = \dim(G)$, $n^{(0)} = \dim(G^{(0)})$
\[ \Psi^m(G) := I^{m+(n-2n^{(0)})/4}(G, A^* G) \]

Since \( A^* G \times \Lambda \cap (T^* G)^{(2)} \) is always transversal, we get:

\[ \Psi^m_c(G) * I^m(G, \Lambda) \subset I^{m_1+m_2}(G, \Lambda) \]
\[ \Psi^m(G) := I^{m+(n-2n(0))/4}(G, A^*G) \]

Since \( A^*G.A^*G = A^*G \), the previous composition result recovers

\[ \Psi^{m_1}_c(G).\Psi^{m_2}(G) \subset \Psi^{m_1+m_2}(G). \]

Since \( A^*G \times \Lambda \cap (T^*G)^{(2)} \) is always transversal, we get:

\[ \Psi^{m_1}_c(G) \ast I^{m_2}(G, \Lambda) \subset I^{m_1+m_2}(G, \Lambda) \]

---

**Egorov thm, \( C^* \)-continuity**

1. If \( \Lambda_1 \times \Lambda_2 \cap (T^*G)^{(2)} \) is clean and \( \Lambda_1.\Lambda_2 \subset A^*G \), Then

\[ I^{m_1}_c(G, \Lambda_1) \ast \Psi^{m_2}(G) \ast I^{m_3}_c(G, \Lambda_2) \subset \Psi^{m_e/2-(n-2n(0))/4}(G). \]

2. Let \( \Lambda \) be an invertible closed \( G \)-relation. Then

\[ I^{(n-2n(0))/4}(G, \Lambda) \subset \mathcal{M}(C^*(G)), \]

\[ I^{<(n-2n(0))/4}(G, \Lambda) \subset C^*(G). \]
Product of symbols

Remember the underlying densities bundles:

\[ I(G, \Lambda) \subset \mathcal{D}'(G, \Omega^{1/2}) \quad \text{with} \quad \Omega^{1/2} = \Omega^{1/2}(\ker ds) \otimes \Omega^{1/2}(\ker dr). \]

Hörmander’s principal symbol map reads here:

\[ \sigma : I^m(G, \Lambda) \longrightarrow S^{[m+n/4]}(\Lambda, M_\Lambda \otimes \Omega^{1/2}_\Lambda \otimes \Omega^{1/2}(\ker ds_\Gamma)). \]

Let \( A_j \in I^{m_j}(G, \Lambda_j), j = 1, 2 \) be as in the composition theorem. A principal symbol \( a \) of \( A_1.A_2 \) is given by:

\[ \forall \delta \in \Lambda_1.\Lambda_2, \quad a(\delta) = \int_{\delta_1 \delta_2 = \delta} a_1(\delta_1)a_2(\delta_2). \]

When \( \Lambda = A^*G \), the Maslov bundle \( M_\Lambda \) is trivial and

\[ \Omega^{1/2}_{A^*G} \otimes \Omega^{1/2}(\ker ds_\Gamma) = (\Omega^{1/2}_{T^*G})|_{A^*G} \simeq A^*G \times \mathbb{C}. \]

The last trivialization decreases by \((n - n^{(0)}/2\) the degree of symbols, thus:

\[ \sigma : \Psi^m(G) = I^{m+(n-2n^{(0)})/4}(G, A^*G) \longrightarrow S^{[m]}(A^*G). \]
Representations of $G$-FIOs

Any $P \in I_c(G, \Lambda)$ gives:

- continuous linear operators $P_x \in \mathcal{L}(C^\infty(G_x), C^\infty(G_x))$, $x \in M$.
- a continuous linear operator $r_M(P) : C^\infty(M) \to C^\infty(M)$:

$$r_M(P)(f) = P(r^*f)|_M, \quad f \in C^\infty(M),$$

(vector rep.)

- continuous linear operators $r_O(P) : C^\infty(O) \to C^\infty(O)$, for any orbit $O = r(s^{-1}(x)) \subset M$:

$$r_O(P)(f) = (P|_{r^{-1}(O)}r^*f)|_O, \quad f \in C^\infty(O).$$

$G_x, O$ are manifolds ($C^\infty$, without boundary) therefore on may ask whether $P_x$ and $r_O(P)$ are ordinary FIOs. Actually, they are given by oscillatory integrals with possibly degenerated phases.
Representations of $G$-FIOs

Let $P \in I^m(G, \Lambda)$ with $\Lambda$ satisfying:

For any orbit $U \subset G$ the intersection $T^*_U G \cap \Lambda$ is transversal
then ($\Lambda$ is called a *family* $G$-relation and)

$$P_x \in I^{m-(n-2n^{(0)})/4}(G_x \times G_x, \Lambda_x), \quad \forall x.$$ 

where $\Lambda_x \subset (T^*G_x \setminus 0) \times (T^*G_x \setminus 0)$ is a family of canonical relations
induced by $\Lambda$. 
Representations of $G$-FIOs

There are converse statements:

**Theorem**

Let $(\Lambda_x)_{x \in G^{(0)}}$ be an equivariant $C^\infty$ family of Lagrangians $\subset T^*G_x \setminus 0 \times T^*G_x \setminus 0$. Then there exists a unique (family) $G$-relation $\Lambda$ “gluing” the family in the sense that

$$d^*_x(\Lambda) = \Lambda_x \quad \forall x \in G^{(0)}.$$ 

Here $d_x$ is the map: $(\gamma_1, \gamma_2) \to \gamma_1\gamma_2^{-1}$.

**Proposition**

$G$-FFIOs are in one-to-one correspondence with $G$-op $P$ such that for all $x$, the operator $P_x$ is a FIO on $G_x$.

$G$-FFIO = $G$-FIO associated with a family $G$-rel.
Evolution equations on groupoids

Let $P \in \Psi^1_c(G)$ be elliptic, symmetric. Then

$e^{-itP} \in M(C^*(G))$ is strongly differentiable and

$$(D_t + P)e^{-itP} = 0.$$
Evolution equations on groupoids

Let $P \in \Psi^1_c(G)$ be elliptic, symmetric. Then

$$e^{-itP} \in M(C^*(G))$$

is strongly differentiable and

$$(D_t + P)e^{-itP} = 0.$$ 

Consider the Hamilton flow $\chi$ of $p : T^*_aG \xrightarrow{r_{\Gamma}} A^*G \setminus 0 \xrightarrow{\sigma_{\text{pr}(P)}} \mathbb{R}$. 

- $\chi$ is complete and homogeneous,
- $s_{\Gamma} \circ \chi_t = s_{\Gamma}$ for all $t$,
- $\chi_t(\alpha \beta) = \chi_t(\alpha) \beta$.

This uses again the full structure of the symplectic groupoid $T^*G$. 
Evolution of $A^*G$ under Hamilton flows

Consider the $G$-relations

$$\Lambda_t = \chi_t(A^*G \setminus 0), \quad t \in \mathbb{R}$$

and the $(\mathbb{R} \times G)$-relation

$$\Lambda = \{(t, \tau, \delta) \in T^*(\mathbb{R} \times G) \mid \tau = -p(\delta), \ \delta \in \Lambda_t\}.$$
Evolution of \( A^*G \) under Hamilton flows

Consider the \( G \)-relations

\[
\Lambda_t = \chi_t(A^*G \setminus 0), \quad t \in \mathbb{R}
\]

and the \((\mathbb{R} \times G)\)-relation

\[
\Lambda = \{(t, \tau, \delta) \in T^*(\mathbb{R} \times G) \mid \tau = -p(\delta), \ \delta \in \Lambda_t\}.
\]

**Theorem**

There exists a \( C^\infty \) family \( U(t) \in I^{(n-2n(0))/4}(G, \Lambda_t) \) such that

\[
e^{-itP} - U(t) \text{ is smoothing}
\]

Main steps in the proof:

- Compute the principal symbol of \( QA \) when \( Q \in \Psi_c(G), A \in I(G, \Lambda) \) and \( \sigma_{pr}(Q) \) vanishes on \( r_T(\Lambda) \).
- Solve the resulting transport equations and construct recursively \( G \)-FIOs approximations of \( e^{-itP} \).
Let $X$ be a manifold with boundary $Y$ and defining function $x$.

$$G = \{(p, q, t) \in X^2 \times \mathbb{R}_+ ; x(q) = tx(p)\}.$$  

$$\partial G \simeq Y^2 \times \mathbb{R}_+ \text{ and } \overset{\circ}{G} \simeq \overset{\circ}{X} \times \overset{\circ}{X},$$

If $X_b^2$ denotes the $b$-stretched product, then: $G \simeq X_b^2 \setminus (lb \cup rb)$.

There are exactly two orbits in $G^{(0)} = X$, namely $\overset{\circ}{X}$ and $Y$.

The corresponding two orbits in $G$ are $\partial G$ and $\overset{\circ}{X} \times \overset{\circ}{X}$.

The symplectic groupoid $T^*G$ splits into two saturated subgroupoids:

$$T^*G = T^*_{\overset{\circ}{X} \times \overset{\circ}{X}} G \bigcup T^*_{\partial G} G.$$ 

The first one is the cotangent groupoid of the pair groupoid $\overset{\circ}{X} \times \overset{\circ}{X}$

The second one is (isomorphic to) the restriction of $T^* (Y^2 \times [0, +\infty) \times \mathbb{R}_+) \text{ over } Y^2 \times \{0\} \times \mathbb{R}_+$. 
Illustration: manifold with boundary

\( \Lambda \subset T^* G \) is a \( G \)-relation if and only if

\[
\Lambda \cap T^* (\dot{X} \times \dot{X}) \subset (T^* \dot{X} \setminus 0) \times (T^* \dot{X} \setminus 0), \quad \text{and}
\]

\[
\Lambda \cap T^* Y^2 \times \mathbb{R}_+ \supset [T^* Y^2 \times (T^* \mathbb{R}_+ \setminus 0)] \cup [(T^* Y \setminus 0) \times (T^* Y \setminus 0) \times (\mathbb{R}_+ \times \{0\})].
\]

Then \( \Lambda \) is a family \( G \)-relation iff

\[
\Lambda \cap T^* Y^2 \times \mathbb{R}_+ G \text{ is transversal}.
\]

It is equivalent to the fact that

\[
\Lambda \xrightarrow{\pi} G \longrightarrow [0, +\infty); \delta \mapsto (p, q, t) \mapsto x(p)
\]

is a submersion near \( x = 0 \). It then follows that near \( x = 0 \), we can parametrize \( \Lambda \) by phase functions \( \phi(x, t, p, q, \theta) \) such that

\[
\phi_x(t, p, q, \theta) = \phi(x, t, p, q, \theta)
\]

is a again a non-degenerate phase parametrizing \( \Lambda_x = i_x^*(\Lambda) \).
Next, we get an indicial operator map:

\[ I^m(G, \Lambda) \ni P \mapsto I(P) = i^*(P) \in I^{m+1/4}(Y^2 \times \mathbb{R}_+, \Lambda_0). \]  

(5)

If \( P \) is given by the oscillatory integral

\[ P = \int e^{i\phi_x(t,p,p',\theta)} a(x, t, p, p', \theta) d\theta \]

then the indicial operator of \( P \) is given by

\[ I(P) = \int e^{i\phi_0(t,p,p',\theta)} a(0, t, p, p', \theta) d\theta. \]

Moreover, we get \( P_p = P_q \), for any \( p, q \in \hat{X} \), and this common operator lies in

\[ P_\circ \in I^m(\hat{X} \times \hat{X}, \hat{\Lambda}). \]