A Groupoid Approach to Fourier Integral Operators

Groupoids and Operator algebras
Orléans

Joint works with J.M. Lescure and D. Manchon (UCA)

May 22, 2019

Some examples of Lie groupoids

- Spaces $X \rightrightarrows X$ and Lie groups $G \rightrightarrows \{e\}$.

Some examples of Lie groupoids

- Spaces $X \rightrightarrows X$ and Lie groups $G \rightrightarrows \{e\}$.
- Pair groupoids: $X \times X \rightrightarrows X$ and fibred pair groupoids : $X \underset{B}{\times} X \rightrightarrows X$.

Some examples of Lie groupoids

- Spaces $X \rightrightarrows X$ and Lie groups $G \rightrightarrows \{e\}$.
- Pair groupoids: $X \times X \rightrightarrows X$ and fibred pair groupoids : $X \underset{B}{\times} X \rightrightarrows X$.
- Vector bundles $E \rightrightarrows X$.

Some examples of Lie groupoids

- Spaces $X \rightrightarrows X$ and Lie groups $G \rightrightarrows \{e\}$.
- Pair groupoids: $X \times X \rightrightarrows X$ and fibred pair groupoids : $X \underset{B}{\times} X \rightrightarrows X$.
- Vector bundles $E \rightrightarrows X$.
- Transformation groupoids: $G \times X \rightrightarrows X$:

$$s(g,x) = x,\ r(g,x) = g.x,\ (g,hx).(h,x) = (gh,x),\ \iota(g,x) = (g^{-1},gx).$$

Some examples of Lie groupoids

- Spaces $X \rightrightarrows X$ and Lie groups $G \rightrightarrows \{e\}$.
- Pair groupoids: $X \times X \rightrightarrows X$ and fibred pair groupoids : $X \underset{B}{\times} X \rightrightarrows X$.
- Vector bundles $E \rightrightarrows X$.
- Transformation groupoids: $G \times X \rightrightarrows X$:

$$s(g,x) = x,\ r(g,x) = g.x,\ (g,hx).(h,x) = (gh,x),\ \iota(g,x) = (g^{-1},gx).$$

- The tangent groupoid of a manifold:

$$\mathcal{T}M = (TM \times \{0\}) \sqcup (0,1] \times M \times M \rightrightarrows [0,1] \times M.$$

Some examples of Lie groupoids

- Spaces $X \rightrightarrows X$ and Lie groups $G \rightrightarrows \{e\}$.
- Pair groupoids: $X \times X \rightrightarrows X$ and fibred pair groupoids : $X \underset{B}{\times} X \rightrightarrows X$.
- Vector bundles $E \rightrightarrows X$.
- Transformation groupoids: $G \times X \rightrightarrows X$:

$$s(g,x) = x,\ r(g,x) = g.x,\ (g,hx).(h,x) = (gh,x),\ \iota(g,x) = (g^{-1},gx).$$

- The tangent groupoid of a manifold:

$$\mathcal{T}M = (TM \times \{0\}) \sqcup (0,1] \times M \times M \rightrightarrows [0,1] \times M.$$

- The b-groupoid of a manifold with boundary:

$$(M \setminus \partial M) \times (M \setminus \partial M) \cup \partial M \times \partial M \times \mathbb{R} \rightrightarrows M.$$

Some examples of Lie groupoids

- Spaces $X \rightrightarrows X$ and Lie groups $G \rightrightarrows \{e\}$.
- Pair groupoids: $X \times X \rightrightarrows X$ and fibred pair groupoids : $X \underset{B}{\times} X \rightrightarrows X$.
- Vector bundles $E \rightrightarrows X$.
- Transformation groupoids: $G \times X \rightrightarrows X$:

$$s(g,x) = x,\ r(g,x) = g.x,\ (g,hx).(h,x) = (gh,x),\ \iota(g,x) = (g^{-1},gx).$$

- The tangent groupoid of a manifold:

$$\mathcal{T}M = (TM \times \{0\}) \sqcup (0,1] \times M \times M \rightrightarrows [0,1] \times M.$$

- The b-groupoid of a manifold with boundary:

$$(M \setminus \partial M) \times (M \setminus \partial M) \cup \partial M \times \partial M \times \mathbb{R} \rightrightarrows M.$$

- Analogous groupoids for stratified spaces and (singular) foliations...

G-PDOs : Examples

G	G-PDOs
Lie groups	Right invariant PDOs
$X \times X \rightrightarrows X$	PDOs on X
$X \times_B X \rightrightarrows X$	Families $(P_b)_{b \in B}$ of PDOs in the fibers
Vector bundles	Families of translation invariant PDOs
$\mathcal{T}M \rightrightarrows [0,1] \times M$	Asymptotic PDOs
b-groupoid	b-PDOs (+ support conditions)
Stratified spaces	PDOs on manifolds with fibred corners

- Can we find a notion of G-FIOs yielding a similar table ?

FIOs are useful in the theory of linear PDE on compact manifolds: strictly hyperbolic problems, asymptotics of spectra, singularities of $\mathrm{Tr}(e^{-itP}) \in \mathcal{D}'(\mathbb{R})$, Egorov theorem ...

1. Lie groups: Nielsen-Stetkær (1974).

FIOs are useful in the theory of linear PDE on compact manifolds: strictly hyperbolic problems, asymptotics of spectra, singularities of $\mathrm{Tr}(e^{-itP}) \in \mathcal{D}'(\mathbb{R})$, Egorov theorem ...

1. Lie groups: Nielsen-Stetkær (1974).
2. Mfds with boundary, b-geometry framework: Melrose (1981).

FIOs are useful in the theory of linear PDE on compact manifolds: strictly hyperbolic problems, asymptotics of spectra, singularities of $\mathrm{Tr}(e^{-itP}) \in \mathcal{D}'(\mathbb{R})$, Egorov theorem ...

1. Lie groups: Nielsen-Stetkær (1974).
2. Mfds with boundary, b-geometry framework: Melrose (1981).
3. Foliations: Y. Kordyukov (1994).

FIOs are useful in the theory of linear PDE on compact manifolds: strictly hyperbolic problems, asymptotics of spectra, singularities of $\mathrm{Tr}(e^{-itP}) \in \mathcal{D}'(\mathbb{R})$, Egorov theorem ...

1. Lie groups: Nielsen-Stetkær (1974).
2. Mfds with boundary, b-geometry framework: Melrose (1981).
3. Foliations: Y. Kordyukov (1994).
4. Mfds with conical sing.: Nazaikinskii-Schulze-Sternin (2001), Nazaikinskii-Savin-Schulze-Sternin (2005).

FIOs are useful in the theory of linear PDE on compact manifolds: strictly hyperbolic problems, asymptotics of spectra, singularities of $\mathrm{Tr}(e^{-itP}) \in \mathcal{D}'(\mathbb{R})$, Egorov theorem ...

1. Lie groups: Nielsen-Stetkær (1974).
2. Mfds with boundary, b-geometry framework: Melrose (1981).
3. Foliations: Y. Kordyukov (1994).
4. Mfds with conical sing.: Nazaikinskii-Schulze-Sternin (2001), Nazaikinskii-Savin-Schulze-Sternin (2005).
5. Mfds with boundary, Boutet de Monvel's framework: Battisti-Coriasco-Schrohe (2014,2015); Bohlen (2015).

Lagrangian distributions (Hörmander)

Let X be a C^∞ manifold of dimension n and $\Lambda \subset T^*X \setminus 0$ a conic lagrangian sub-mfd.
The set $I^m(X,\Lambda)$, $m \in \mathbb{R}$, consists of distributions $u \in \mathcal{D}'(X)$ of the form:

$$u = \sum_{j \in J} \int e^{i\phi_j(x,\theta_j)} a_j(x,\theta_j) d\theta_j \mod C^\infty(X)$$

where

- $(x,\theta_j) \in \mathcal{V}_j \subset U_j \times \mathbb{R}^{N_j}$ with U_j a coord. patch, $\mathcal{V}_j$ open and homogeneous;
- $\phi_j : \mathcal{V}_j \to \mathbb{R}$ is a non-degenerate phase parametrizing Λ;
- $a_j(x,\theta_j) \in S^{m+(n_X-2N_j)/4}(U_j \times \mathbb{R}^{N_j})$ and $\mathrm{supp}(a_j) \subset \mathcal{V}_j \setminus 0$.

☞ $I^m(X,\Lambda)$ = *Lagrangian distributions on X subordinated to Λ.*

Convolution and G-ops

Convolution in $C_c^\infty(G)$

$$f * g(\gamma) = \int_{\gamma_1\gamma_2=\gamma} f(\gamma_1)g(\gamma_2) = m_*(f \otimes g|_{G^{(2)}}), \quad f, g \in C_c^\infty(G)$$

G-operators

Continuous linear maps $P : C_c^\infty(G) \to C^\infty(G)$ such that

$$P(f * g) = P(f) * g \qquad \text{for any } f, g \in C_c^\infty(G),$$

Equivalently, $P = (P_x)_{x\in M}$ with $P_x : C_c^\infty(G_x) \to C^\infty(G_x)$, $P_{r(\gamma)}R_\gamma = R_\gamma P_{s(\gamma)}$ and

$$(Pf)|_{G_x} = P_x(f|_{G_x}), \qquad f \in C_c^\infty(G).$$

(Notations: $G_x = s^{-1}(x)$, $G^x = r^{-1}(x)$).

G-PDOs = G-Ops P with P_x a pseudodifferential operator on G_x for any x. Actually:

$$G - \text{PDOs} = I(G, M)$$

Convolution of distributions

FIOs are lagrangian distributions. To get a satisfactory approach on groupoids (calculus, continuity, Egorov, evolution equations. . .), we need:

- to understand the convolution of distributions;
- to select a suitable set of Lagrangian submanifolds.

There are two ways of extending convolution product of functions to distributions:

Under transversality conditions

Under wave front sets sonditions

Schwartz kernel theorem and transversal distributions

Let $G \rightrightarrows M$ be a Lie groupoid.
• For any $u \in \mathcal{D}'(G)$ and $f \in C_c^\infty(G)$, define $s_*(uf) \in \mathcal{D}'(M)$ by:

$$g \in C^\infty(M), \quad \langle s_*(uf), g\rangle = \langle u, f.s^*g\rangle.$$

This provides an isomorphism:

$$s_* : \mathcal{D}'(G) \xrightarrow{\simeq} \mathcal{L}_{C^\infty(M)}(C_c^\infty(G), \mathcal{D}'(M)). \qquad (1)$$

• Consider the subspace of s-transversal distributions on G:

$$\mathcal{D}'_s(G) = \{u \in \mathcal{D}'(G)\ ;\ \mathrm{Im}(s_*(u)) \subset C^\infty(M)\}. \qquad (2)$$

Then

$$s_* : \mathcal{D}'_s(G) \xrightarrow{\simeq} \mathcal{L}_{C^\infty(M)}(C_c^\infty(G), C^\infty(M)). \qquad (3)$$

appropriate densities are understood, similar statements hold with r instead of s.

Convolution in $\mathcal{D}'$

Theorem (LMV)

Convolution of functions extends (separately) continuously to:

- $\mathcal{D}'_s(G) \times \mathcal{E}'(G) \xrightarrow{*} \mathcal{D}'(G)$,
- $\mathcal{D}'_r(G) \times C^\infty_c(G) \xrightarrow{*} C^\infty(G)$,
- $\mathcal{D}'_s(G) \times \mathcal{E}'_s(G) \xrightarrow{*} \mathcal{D}'_s(G)$
- ...

Corollary

- $\mathcal{E}'_s(G)$, $\mathcal{E}'_r(G)$ are unital algebras.
- $\mathcal{E}'_{r,s}(G) = \mathcal{E}'_s(G) \cap \mathcal{E}'_r(G)$ is a unital involutive algebra.

unit: $\langle \delta, f \rangle = \int_M f$, involution: $u^\star = \overline{\iota^*(u)}$.

G-operators are transversal distributions

This also proves that G-ops are operators given by convolution with distributions.
The map

$$\mathcal{D}'_r(G) \longrightarrow \mathrm{Op}_G,\ u \longmapsto (f \mapsto u * f)$$

is well defined and gives:

$$\mathcal{D}'_r(G) \simeq \mathrm{Op}_G \qquad \text{(space of G-operators)}$$
$$\mathcal{D}'_{r,s}(G) \simeq \mathrm{Op}^*_G \qquad \text{(subspace of G-operators with adjoints)}$$

Coste-Dazord-Weinstein groupoid T^*G

Let $G \rightrightarrows M$ be a Lie groupoid.

Understanding under which conditions on the wave front sets of $u, v \in \mathcal{D}'(G)$, the convolution product $u * v$ is defined leads to the algebraic structure of the cotangent space T^*G.

There exists a natural symplectic Lie groupoid structure on T^*G with unit space $A^*G = N^*M$.

$$\Gamma = (T^*G \rightrightarrows A^*G = N^*M)$$

A groupoid $\Gamma \rightrightarrows \Gamma^{(0)}$ is sympletic if Γ is a symplectic manifold and if the graph of the multiplication map is lagrangian in $(-\Gamma) \times \Gamma \times \Gamma$

CDW groupoid: $\Gamma = T^*G \rightrightarrows A^*G$.

The product $(\gamma_1, \xi_1).(\gamma_2, \xi_2) = (\gamma, \xi)$ is defined by

$$\gamma = \gamma_1 \gamma_2$$

and the equality

$$\xi(t) = \xi_1(t_1) + \xi_2(t_2)$$

for any $t \in T_\gamma G$ and $(t_1, t_2) \in T_{(\gamma_1, \gamma_2)} G^{(2)}$ such that

$$t = dm(t_1, t_2)$$

Well defined iff it does not depend on the choice of such t_1, t_2.

CDW groupoid: $\Gamma = T^*G \rightrightarrows A^*G$.

The product $(\gamma_1, \xi_1).(\gamma_2, \xi_2) = (\gamma, \xi)$ is defined by

$$\gamma = \gamma_1\gamma_2$$

and the equality

$$\xi(t) = \xi_1(t_1) + \xi_2(t_2)$$

for any $t \in T_\gamma G$ and $(t_1, t_2) \in T_{(\gamma_1,\gamma_2)}G^{(2)}$ such that

$$t = dm(t_1, t_2)$$

Well defined iff it does not depend on the choice of such t_1, t_2.

(Source) $\quad s_\Gamma(\gamma, \xi) = (s(\gamma), L_\gamma^*(\xi)) \in A^*_{s(\gamma)}G,$

(Range) $\quad r_\Gamma(\gamma, \xi) = (r(\gamma), R_\gamma^*(\xi)) \in A^*_{r(\gamma)}G,$

$s_\Gamma(\gamma, \xi)$ is obtained by applying the codifferential of $L_\gamma : G^x \to G^{r(\gamma)}$ to the restriction $\xi : T_\gamma G^{r(\gamma)} \to \mathbb{R}$. The result is a linear form on T_xG vanishing on $T_xG^{(0)}$, thus an element of A_x^*G.

CDW groupoid: $\Gamma = T^*G \rightrightarrows A^*G$.

The product $(\gamma_1, \xi_1).(\gamma_2, \xi_2) = (\gamma, \xi)$ is defined by

$$\gamma = \gamma_1\gamma_2$$

and the equality

$$\xi(t) = \xi_1(t_1) + \xi_2(t_2)$$

for any $t \in T_\gamma G$ and $(t_1, t_2) \in T_{(\gamma_1,\gamma_2)}G^{(2)}$ such that

$$t = dm(t_1, t_2)$$

Well defined iff it does not depend on the choice of such t_1, t_2.

(Source) $\quad s_\Gamma(\gamma, \xi) = (s(\gamma), L^*_\gamma(\xi)) \in A^*_{s(\gamma)}G,$

(Range) $\quad r_\Gamma(\gamma, \xi) = (r(\gamma), R^*_\gamma(\xi)) \in A^*_{r(\gamma)}G,$

$s_\Gamma(\gamma, \xi)$ is obtained by applying the codifferential of $L_\gamma : G^x \to G^{r(\gamma)}$ to the restriction $\xi : T_\gamma G^{r(\gamma)} \to \mathbb{R}$. The result is a linear form on T_xG vanishing on $T_xG^{(0)}$, thus an element of A^*_xG.

(Inversion) $\quad (\gamma, \xi)^{-1} = (\gamma^{-1}, -{}^t(d\iota_\gamma)(\xi)).$

Wave front and convolution

Let $W_1, W_2 \subset T^*G \setminus 0$ be closed, conic subsets such that

$$(W_1 \times W_2) \cap \ker m_\Gamma = \emptyset.$$

Then convolution extends continuously to

$$\mathcal{E}'_{W_1}(G) \times \mathcal{E}'_{W_2}(G) \longrightarrow \mathcal{E}'_W(G) \tag{4}$$

where W is the product of the sets $W_1 \cup 0$ and $W_2 \cup 0$ in T^*G.

Distributions and G-ops

Set :
$\mathcal{E}'_a(G) := \{u \in \mathcal{E}'(G) \ ; \ \mathrm{WF}(u) \cap \ker s_\Gamma = \emptyset$ and $\mathrm{WF}(u) \cap \ker r_\Gamma = \emptyset\}$.

- $(\mathcal{E}'_a(G), *)$ is a unital involutive algebra,

Distributions and G-ops

Set :
$\mathcal{E}'_a(G) := \{u \in \mathcal{E}'(G) \ ; \ \mathrm{WF}(u) \cap \ker s_\Gamma = \emptyset$ and $\mathrm{WF}(u) \cap \ker r_\Gamma = \emptyset\}$.

- $(\mathcal{E}'_a(G), *)$ is a unital involutive algebra,
- { compactly supported G-PDOs } $\subset \mathcal{E}'_a(G) \subset \mathcal{E}'_{r,s}(G)$;

Distributions and G-ops

Set :
$\mathcal{E}'_a(G) := \{u \in \mathcal{E}'(G) \; ; \; \mathrm{WF}(u) \cap \ker s_\Gamma = \emptyset$ and $\mathrm{WF}(u) \cap \ker r_\Gamma = \emptyset\}$.

- $(\mathcal{E}'_a(G), *)$ is a unital involutive algebra,
- { compactly supported G-PDOs } $\subset \mathcal{E}'_a(G) \subset \mathcal{E}'_{r,s}(G)$;
- for any $u_1, u_2 \in \mathcal{E}'_a(G)$ we have

$$\mathrm{WF}(u_1 * u_2) \subset \mathrm{WF}(u_1) \, . \, \mathrm{WF}(u_2) \subset T^*G \setminus (\ker r_\Gamma \cup \ker s_\Gamma).$$

Distributions and G-ops

Set :
$\mathcal{E}'_a(G) := \{u \in \mathcal{E}'(G)\ ;\ \mathrm{WF}(u) \cap \ker s_\Gamma = \emptyset$ and $\mathrm{WF}(u) \cap \ker r_\Gamma = \emptyset\}$.

- $(\mathcal{E}'_a(G), *)$ is a unital involutive algebra,
- $\{$ compactly supported G-PDOs $\} \subset \mathcal{E}'_a(G) \subset \mathcal{E}'_{r,s}(G)$;
- for any $u_1, u_2 \in \mathcal{E}'_a(G)$ we have

$$\mathrm{WF}(u_1 * u_2) \subset \mathrm{WF}(u_1)\,.\,\mathrm{WF}(u_2) \subset T^*G \setminus (\ker r_\Gamma \cup \ker s_\Gamma).$$

- Elements of $\mathcal{E}'_a(G)$ and even $\mathcal{D}'_a(G)$ provide G-operators :

$$u \in \mathcal{D}'_a(G), \quad \begin{pmatrix} C_c^\infty(G) & \longrightarrow & C^\infty(G) \\ f & \longmapsto & u * f \end{pmatrix} \in \mathrm{Op}_G.$$

Examples of the rule for convolution

Let X be a manifold and $X \rightrightarrows X$ be the trivial groupoid.

- Convolution = pointwise multiplication;

Examples of the rule for convolution

Let X be a manifold and $X \rightrightarrows X$ be the trivial groupoid.

- Convolution = pointwise multiplication;
- $\Gamma = T^*X \rightrightarrows N^*X = X$ is the ordinary vector bundle structure;

Examples of the rule for convolution

Let X be a manifold and $X \rightrightarrows X$ be the trivial groupoid.

- Convolution = pointwise multiplication;
- $\Gamma = T^*X \rightrightarrows N^*X = X$ is the ordinary vector bundle structure;
- If $\mathrm{WF}(u_1) \times \mathrm{WF}(u_2) \cap \{(x, \xi, x, -\xi)\} = \emptyset$ then $u_1 u_2$ is well defined and

$$\mathrm{WF}(u_1 u_2) \subset (\mathrm{WF}(u_1) + \mathrm{WF}(u_2)) \cup \mathrm{WF}(u_1) \cup \mathrm{WF}(u_2)\,.$$

Examples of the rule for convolution

Let $G = X \times X \rightrightarrows X$ be the pair groupoid. Then

- Convolution = composition of integral kernels;

Examples of the rule for convolution

Let $G = X \times X \rightrightarrows X$ be the pair groupoid. Then

- Convolution = composition of integral kernels;
- $\Gamma = T^*X \times T^*X \rightrightarrows N^*\Delta_X \simeq T^*X$ is the pair$'$ groupoid:

$$(x, \xi, y, \eta) \circ' (y, -\eta, z, \zeta) = (x, \xi, z, \zeta).$$

Examples of the rule for convolution

Let $G = X \times X \rightrightarrows X$ be the pair groupoid. Then

- Convolution = composition of integral kernels;
- $\Gamma = T^*X \times T^*X \rightrightarrows N^*\Delta_X \simeq T^*X$ is the pair$'$ groupoid:

$$(x, \xi, y, \eta) \circ' (y, -\eta, z, \zeta) = (x, \xi, z, \zeta).$$

- If $\mathrm{WF}(u_1) \times \mathrm{WF}(u_2) \cap \{(x, 0, y, \eta, y, -\eta, z, 0)\} = \emptyset$ then $u_1 \circ u_2 \in \mathcal{D}'(X \times X)$ is well defined and

$$\mathrm{WF}(u_1 \circ u_2) \subset (\mathrm{WF}(u_1) \cup 0) \circ' (\mathrm{WF}(u_2) \cup 0).$$

Examples of the rule for convolution

Let $G \rightrightarrows \{e\}$ be a Lie group. Then

- Convolution = ordinary convolution in G;

Examples of the rule for convolution

Let $G \rightrightarrows \{e\}$ be a Lie group. Then

- Convolution = ordinary convolution in G;
- $(T^*G \rightrightarrows \mathfrak{g}^*) \simeq (\mathfrak{g}^* \rtimes G)$ $((g,\xi) \mapsto (g, R_g^*\xi))$

Examples of the rule for convolution

Let $G \rightrightarrows \{e\}$ be a Lie group. Then

- Convolution = ordinary convolution in G;
- $(T^*G \rightrightarrows \mathfrak{g}^*) \simeq (\mathfrak{g}^* \rtimes G)$ $((g,\xi) \mapsto (g, R_g^*\xi))$
- Here $\ker m_\Gamma = G^2 \times \{0\}$, therefore $u_1 * u_2$ is always defined and

$$\mathrm{WF}(u_1 * u_2) \subset \{(\xi, g)\,;\, (\xi, h, \mathrm{Ad}_h^*.\xi, h^{-1}g) \in \mathrm{WF}(u_1)\times\mathrm{WF}(u_2), h \in G\} \subset \mathfrak{g}^*\times G$$

Calculus of Lagrangian submanifolds in T^*G

Set $T^*_a G = T^*G \setminus (\ker r_\Gamma \cup \ker s_\Gamma)$.

G-relations

Conic Lagrangian submanifolds of T^*G contained in $T^*_a G$.

If $G = X \times X$, these are the conic Lagrangian submanifolds contained in $T^*X \setminus 0 \times T^*X \setminus 0$.

1. Let Λ_1, Λ_2 be two G-relations. If $\Lambda_1 \times \Lambda_2 \cap \Gamma^{(2)}$ is clean, then $\Lambda_1.\Lambda_2 \subset \Gamma$ is a (immersed) G-relation.

Calculus of Lagrangian submanifolds in T^*G

Set $T_a^*G = T^*G \setminus (\ker r_\Gamma \cup \ker s_\Gamma)$.

G-relations

Conic Lagrangian submanifolds of T^*G contained in T_a^*G.

If $G = X \times X$, these are the conic Lagrangian submanifolds contained in $T^*X \setminus 0 \times T^*X \setminus 0$.

1. Let Λ_1, Λ_2 be two G-relations. If $\Lambda_1 \times \Lambda_2 \cap \Gamma^{(2)}$ is clean, then $\Lambda_1.\Lambda_2 \subset \Gamma$ is a (immersed) G-relation.
2. Let Λ be a G-relation. Then $\Lambda^\star := \iota_\Gamma(\Lambda)$ is a G-relation.

Calculus of Lagrangian submanifolds in T^*G

Set $T^*_a G = T^*G \setminus (\ker r_\Gamma \cup \ker s_\Gamma)$.

G-relations

Conic Lagrangian submanifolds of T^*G contained in $T^*_a G$.

If $G = X \times X$, these are the conic Lagrangian submanifolds contained in $T^*X \setminus 0 \times T^*X \setminus 0$.

1. Let Λ_1, Λ_2 be two G-relations. If $\Lambda_1 \times \Lambda_2 \cap \Gamma^{(2)}$ is clean, then $\Lambda_1.\Lambda_2 \subset \Gamma$ is a (immersed) G-relation.
2. Let Λ be a G-relation. Then $\Lambda^\star := \iota_\Gamma(\Lambda)$ is a G-relation.
3. Let Λ be a G-relation. Then

$$r_\Gamma : \Lambda \to A^*G \text{ and } s_\Gamma : \Lambda \to A^*G$$

are diffeomorphisms if and only if

$$\Lambda\Lambda^\star = A^*G \quad \text{and} \quad \Lambda^\star\Lambda = A^*G.$$

Such G-relations are called invertible.

G-FIOs: definition, composition

Definition

G-FIOs = Elements of $I(G, \Lambda)$, for any G-relation Λ.

$$G - \text{PDOs} = I(G, A^*G) \subset G - \text{FIOs}$$

$$G - \text{FIOs} \subset \mathcal{D}'_a(G) \subset \mathrm{Op}_G^\star$$

G-FIOs: definition, composition

Definition

G-FIOs = Elements of $I(G,\Lambda)$, for any G-relation Λ.

$$G - \text{PDOs} = I(G, A^*G) \subset G - \text{FIOs}$$

$$G - \text{FIOs} \subset \mathcal{D}'_a(G) \subset \text{Op}_G^\star$$

If Λ is a G-relation and $A \in I^m(G,\Lambda)$, then $A^\star \in I^m(G,\Lambda^\star)$.

G-FIOs: definition, composition

Definition

G-FIOs = Elements of $I(G, \Lambda)$, for any G-relation Λ.

$$G - \text{PDOs} = I(G, A^*G) \subset G - \text{FIOs}$$

$$G - \text{FIOs} \subset \mathcal{D}'_a(G) \subset \mathrm{Op}^{\star}_G$$

If Λ is a G-relation and $A \in I^m(G, \Lambda)$, then $A^\star \in I^m(G, \Lambda^\star)$.

Assume that Λ_1, Λ_2 are closed G-relations, with a clean intersection $\Lambda_1 \times \Lambda_2 \cap (T^*G)^{(2)}$ of excess e. Then convolution gives a map:

$$I_c^{m_1}(G, \Lambda_1) \times I^{m_2}(G, \Lambda_2) \longrightarrow I^{m_1+m_2+e/2-(n-2n^{(0)})/4}(G, \Lambda_1\Lambda_2).$$

$(n = \dim(G),\ n^{(0)} = \dim(G^{(0)}))$

$$\Psi^m(G) := I^{m+(n-2n^{(0)})/4}(G, A^*G)$$

Since $A^*G.A^*G = A^*G$, the previous composition result recovers

$$\Psi_c^{m_1}(G).\Psi^{m_2}(G) \subset \Psi^{m_1+m_2}(G).$$

Since $A^*G \times \Lambda \cap (T^*G)^{(2)}$ is always transversal, we get:

$$\Psi_c^{m_1}(G) * I^{m_2}(G, \Lambda) \subset I^{m_1+m_2}(G, \Lambda)$$

$$\Psi^m(G) := I^{m+(n-2n^{(0)})/4}(G, A^*G)$$

Since $A^*G.A^*G = A^*G$, the previous composition result recovers

$$\Psi_c^{m_1}(G).\Psi^{m_2}(G) \subset \Psi^{m_1+m_2}(G).$$

Since $A^*G \times \Lambda \cap (T^*G)^{(2)}$ is always transversal, we get:

$$\Psi_c^{m_1}(G) * I^{m_2}(G, \Lambda) \subset I^{m_1+m_2}(G, \Lambda)$$

Egorov thm, C^*-continuity

1. If $\Lambda_1 \times \Lambda_2 \cap (T^*G)^{(2)}$ is clean and $\Lambda_1.\Lambda_2 \subset A^*G$, Then

$$I_c^{m_1}(G, \Lambda_1) * \Psi^{m_2}(G) * I_c^{m_3}(G, \Lambda_2) \subset \Psi^{m+e/2-(n-2n^{(0)})/4}(G).$$

2. Let Λ be an invertible closed G-relation. Then

$$I^{(n-2n^{(0)})/4}(G, \Lambda) \subset \mathcal{M}(C^*(G)),$$
$$I^{<(n-2n^{(0)})/4}(G, \Lambda) \subset C^*(G).$$

Product of symbols

Remember the underlying densities bundles:

$$I(G,\Lambda) \subset \mathcal{D}'(G,\Omega^{1/2}) \quad \text{with } \Omega^{1/2} = \Omega^{1/2}(\ker ds) \otimes \Omega^{1/2}(\ker dr).$$

Hörmander's principal symbol map reads here:

$$\sigma : I^m(G,\Lambda) \longrightarrow S^{[m+n/4]}(\Lambda, M_\Lambda \otimes \Omega^{1/2}_\Lambda \otimes \Omega^{1/2}(\ker ds_\Gamma)),$$

Let $A_j \in I^{m_j}(G,\Lambda_j)$, $j=1,2$ be as in the composition theorem.
A principal symbol a of $A_1.A_2$ is given by:

$$\forall \delta \in \Lambda_1.\Lambda_2, \quad a(\delta) = \int_{\delta_1\delta_2=\delta} a_1(\delta_1)a_2(\delta_2).$$

When $\Lambda = A^*G$, the Maslov bundle M_Λ is trivial and

$$\Omega^{1/2}_{A^*G} \otimes \Omega^{1/2}(\ker ds_\Gamma) = (\Omega^{1/2}_{T^*G})|_{A^*G} \simeq A^*G \times \mathbb{C}.$$

The last trivialization decreases by $(n-n^{(0)})/2$ the degree of symbols, thus :

$$\sigma : \Psi^m(G) = I^{m+(n-2n^{(0)})/4}(G,A^*G) \longrightarrow S^{[m]}(A^*G).$$

Representations of G-FIOs

Any $P \in I_c(G, \Lambda)$ gives:

- continuous linear operators $P_x \in \mathcal{L}(C^\infty(G_x), C^\infty(G_x))$, $x \in M$.
- a continuous linear operator $r_M(P) : C^\infty(M) \longrightarrow C^\infty(M)$:

$$r_M(P)(f) = P(r^* f)|_M, \quad f \in C^\infty(M), \qquad \text{(vector rep.)}$$

- continuous linear operators $r_O(P) : C^\infty(O) \longrightarrow C^\infty(O)$, for any orbit $O = r(s^{-1}(x)) \subset M$:

$$r_O(P)(f) = (P|_{r^{-1}(O)} r^* f)|_O, \quad f \in C^\infty(O).$$

G_x, O are manifolds (C^∞, without boundary) therefore on may ask whether P_x and $r_O(P)$ are ordinary FIOs. Actually, they are given by oscillatory integrals with possibly degenerated phases.

Representations of G-FIOs

Let $P \in I^m(G, \Lambda)$ with Λ satisfying:

For any orbit $U \subset G$ the intersection $T_U^* G \cap \Lambda$ is transversal

then (Λ is called a *family* G-relation and)

$$P_x \in I^{m-(n-2n^{(0)})/4}(G_x \times G_x, \Lambda_x), \quad \forall x.$$

where $\Lambda_x \subset (T^* G_x \setminus 0) \times (T^* G_x \setminus 0)$ is a family of canonical relations induced by Λ.

Representations of G-FIOs

There are converse statements:

Theorem

Let $(\Lambda_x)_{x\in G^{(0)}}$ be an equivariant C^∞ family of Lagrangians $\subset T^*G_x\setminus 0\times T^*G_x\setminus 0$. Then there exists a unique (family) G-relation Λ "gluing" the family in the sense that

$$d_x^*(\Lambda)=\Lambda_x \quad \forall x\in G^{(0)}.$$

Here d_x is the map: $(\gamma_1,\gamma_2)\rightarrow\gamma_1\gamma_2^{-1}$.

Proposition

G-FFIOs are in one-to-one correspondence with G-op P such that for all x, the operator P_x is a FIO on G_x.

G-FFIO = G-FIO associated with a family G-rel.

Evolution equations on groupoids

Let $P \in \Psi_c^1(G)$ be elliptic, symmetric. Then

$e^{-itP} \in M(C^*(G))$ is strongly differentiable and

$$(D_t + P)e^{-itP} = 0.$$

Evolution equations on groupoids

Let $P \in \Psi_c^1(G)$ be elliptic, symmetric. Then

$e^{-itP} \in M(C^*(G))$ is strongly differentiable and

$$(D_t + P)e^{-itP} = 0.$$

Consider the Hamilton flow χ of $p : T_a^*G \xrightarrow{r_\Gamma} A^*G \setminus 0 \xrightarrow{\sigma_{pr}(P)} \mathbb{R}$.

- χ is complete and homogeneous,
- $s_\Gamma \circ \chi_t = s_\Gamma$ for all t,
- $\chi_t(\alpha\beta) = \chi_t(\alpha)\beta$.

This uses again the full structure of the symplectic groupoid T^*G.

Evolution of A^*G under Hamilton flows

Consider the G-relations

$$\Lambda_t = \chi_t(A^*G \setminus 0), \quad t \in \mathbb{R}$$

and the $(\mathbb{R} \times G)$-relation

$$\Lambda = \{(t, \tau, \delta) \in T^*(\mathbb{R} \times G) \mid \tau = -p(\delta),\ \delta \in \Lambda_t\}.$$

Evolution of A^*G under Hamilton flows

Consider the G-relations

$$\Lambda_t = \chi_t(A^*G \setminus 0), \quad t \in \mathbb{R}$$

and the $(\mathbb{R} \times G)$-relation

$$\Lambda = \{(t, \tau, \delta) \in T^*(\mathbb{R} \times G) \mid \tau = -p(\delta),\ \delta \in \Lambda_t\}.$$

Theorem

There exists a C^∞ family $U(t) \in I^{(n-2n^{(0)})/4}(G, \Lambda_t)$ such that

$$e^{-itP} - U(t) \text{ is smoothing}$$

Main steps in the proof:

- Compute the principal symbol of QA when $Q \in \Psi_c(G)$, $A \in I(G, \Lambda)$ and $\sigma_{pr}(Q)$ vanishes on $r_\Gamma(\Lambda)$.
- Solve the resulting transport equations and construct recursively G-FIOs approximations of e^{-itP}.

Illustration: manifold with boundary

Let X be a manifold with boundary Y and defining function x.

$$G = \{(p, q, t) \in X^2 \times \mathbb{R}_+ \; ; \; x(q) = tx(p)\}.$$

$$\partial G \simeq Y^2 \times \mathbb{R}_+ \text{ and } \overset{\circ}{G} \simeq \overset{\circ}{X} \times \overset{\circ}{X},$$

If X_b^2 denotes the b-stretched product, then: $G \simeq X_b^2 \setminus (\mathrm{lb} \cup \mathrm{rb})$.

There are exactly two orbits in $G^{(0)} = X$, namely $\overset{\circ}{X}$ and Y.

The corresponding two orbits in G are ∂G and $\overset{\circ}{X} \times \overset{\circ}{X}$.
The symplectic groupoid T^*G splits into two satured subgroupoids:

$$T^*G = T^*_{(\overset{\circ}{X} \times \overset{\circ}{X})} G \bigcup T^*_{\partial G} G.$$

The first one is the cotangent groupoid of the pair groupoid $\overset{\circ}{X} \times \overset{\circ}{X}$
The second one is (isomorphic to) the restriction of
$T^*(Y^2 \times [0, +\infty) \rtimes \mathbb{R}_+)$ over $Y^2 \times \{0\} \times \mathbb{R}_+$.

Illustration: manifold with boundary

$\Lambda \subset T^*G$ is a G-relation if and only if

$$\Lambda \cap T^*(\mathring{X} \times \mathring{X}) \subset (T^*\mathring{X} \setminus 0) \times (T^*\mathring{X} \setminus 0), \text{ and}$$

$$\Lambda \cap T^*_{Y^2 \times \mathbb{R}^*_+} G \subset [T^*Y^2 \times (T^*\mathbb{R}_+ \setminus 0)] \bigcup [(T^*Y \setminus 0) \times (T^*Y \setminus 0) \times (\mathbb{R}_+ \times \{0\})].$$

Then Λ is a family G-relation iff

$$\Lambda \cap T^*_{Y^2 \times \mathbb{R}_+} G \text{ is transversal.}$$

It is equivalent to the fact that

$$\Lambda \xrightarrow{\pi} G \longrightarrow [0, +\infty); \; \delta \mapsto (p, q, t) \mapsto x(p)$$

is a submersion near $x = 0$. It then follows that near $x = 0$, we can parametrize Λ by phase functions $\phi(x, t, p, q, \theta)$ such that

$$\phi_x(t, p, q, \theta) = \phi(x, t, p, q, \theta)$$

is a again a non-degenerate phase parametrizing $\Lambda_x = i_x^*(\Lambda)$.

Illustration: manifold with boundary

Next, we get a an *indicial operator map*:

$$I^m(G,\Lambda) \ni P \longmapsto I(P) = i^*(P) \in I^{m+1/4}(Y^2 \times \mathbb{R}_+, \Lambda_0). \qquad (5)$$

If P is given by the oscillatory integral

$$P = \int e^{i\phi_x(t,p,p',\theta)} a(x,t,p,p',\theta) d\theta$$

then the indicial operator of P is given by

$$I(P) = \int e^{i\phi_0(t,p,p',\theta)} a(0,t,p,p',\theta) d\theta.$$

Moreover, we get $P_p = P_q$, for any $p, q \in \overset{\circ}{X}$, and this common operator lies in

$$P_\circ \in I^m(\overset{\circ}{X} \times \overset{\circ}{X}, \overset{\circ}{\Lambda}).$$