K-theory, groupoids and propagation

H. Oyono Oyono

May 22, 2019

Conference for Jean
May 21-24, Orléans

Introduction

Aim : Generalize to (locally compact Hausdorf) groupoids the "cut-and-pasting" strategy for K-computation of the action groupoid $\beta_\Gamma \rtimes \Gamma$ where

Introduction

Aim : Generalize to (locally compact Hausdorff) groupoids the "cut-and-pasting" strategy for K-computation of the action groupoid $\beta_\Gamma \rtimes \Gamma$ where

- Γ is a finitely generated group;

Introduction

Aim : Generalize to (locally compact Hausdorff) groupoids the "cut-and-pasting" strategy for K-computation of the action groupoid $\beta_\Gamma \rtimes \Gamma$ where

- Γ is a finitely generated group;
- β_Γ is the Stone-Cěch compactification of Γ.

Introduction

Aim : Generalize to (locally compact Hausdorf) groupoids the "cut-and-pasting" strategy for K-computation of the action groupoid $\beta_\Gamma \rtimes \Gamma$ where

- Γ is a finitely generated group;
- β_Γ is the Stone-Cěch compactification of Γ.

Let ℓ be a lenght on Γ associated to any finite and symetric generating set and let $d(\gamma, \gamma') = \ell(\gamma'^{-1}\gamma)$ be the associated metric.

Introduction

Aim : Generalize to (locally compact Hausdorf) groupoids the "cut-and-pasting" strategy for K-computation of the action groupoid $\beta_\Gamma \rtimes \Gamma$ where

- Γ is a finitely generated group;
- β_Γ is the Stone-Cěch compactification of Γ.

Let ℓ be a lenght on Γ associated to any finite and symetric generating set and let $d(\gamma,\gamma')=\ell(\gamma'^{-1}\gamma)$ be the associated metric. For r positive, let us set

- $B(e,r)=\{\gamma\in\Gamma;\,\ell(\gamma)\leqslant r\}$;

Introduction

Aim : Generalize to (locally compact Hausdorf) groupoids the "cut-and-pasting" strategy for K-computation of the action groupoid $\beta_\Gamma \rtimes \Gamma$ where

- Γ is a finitely generated group;
- β_Γ is the Stone-Cěch compactification of Γ.

Let ℓ be a lenght on Γ associated to any finite and symetric generating set and let $d(\gamma,\gamma')=\ell(\gamma'^{-1}\gamma)$ be the associated metric. For r positive, let us set

- $B(e,r)=\{\gamma\in\Gamma;\ \ell(\gamma)\leqslant r\}$;
- $C^*_{red}(\beta_\Gamma\rtimes\Gamma)_r$ the set of functions $f:\Gamma\times\Gamma\to\mathbb{C}$ with second variable support in $B(e,r)$

Introduction

Aim : Generalize to (locally compact Hausdorf) groupoids the "cut-and-pasting" strategy for K-computation of the action groupoid $\beta_\Gamma \rtimes \Gamma$ where

- Γ is a finitely generated group;
- β_Γ is the Stone-Cěch compactification of Γ.

Let ℓ be a lenght on Γ associated to any finite and symetric generating set and let $d(\gamma, \gamma') = \ell(\gamma'^{-1}\gamma)$ be the associated metric. For r positive, let us set

- $B(e, r) = \{\gamma \in \Gamma;\ \ell(\gamma) \leqslant r\}$;
- $C^*_{red}(\beta_\Gamma \rtimes \Gamma)_r$ the set of functions $f : \Gamma \times \Gamma \to \mathbb{C}$ with second variable support in $B(e, r)$

Then $C^*_{red}(\beta_\Gamma \rtimes \Gamma)_r \subseteq C^*_{red}(\beta_\Gamma \rtimes \Gamma)$ and $\cup_{r>0} C^*_{red}(\beta_\Gamma \rtimes \Gamma)_r$ is dense in $C^*_{red}(\beta_\Gamma \rtimes \Gamma)$.

Introduction

Let us consider the obvious faithful representation of $C^*_{red}(\beta_\Gamma \rtimes \Gamma)$ on $\ell^2(\Gamma)$.

Introduction

Let us consider the obvious faithful representation of $C^*_{red}(\beta_\Gamma \rtimes \Gamma)$ on $\ell^2(\Gamma)$. If f be an element in $C^*_{red}(\beta_\Gamma \rtimes \Gamma)_r$ (i.e $f(\gamma, \gamma') = 0$ if $\ell(\gamma') > r$)

Let us consider the obvious faithful representation of $C^*_{red}(\beta_\Gamma \rtimes \Gamma)$ on $\ell^2(\Gamma)$. If f be an element in $C^*_{red}(\beta_\Gamma \rtimes \Gamma)_r$ (i.e $f(\gamma,\gamma')=0$ if $\ell(\gamma')>r$) then f acts as kernel operator :

$$f\cdot\xi(\gamma)=\sum_{\gamma'\in\Gamma}\tilde{f}(\gamma,\gamma')\xi(\gamma')$$

with

$$\tilde{f}:\Gamma\times\Gamma\to\mathbb{C}:(\gamma,\gamma')\mapsto f(\gamma,\gamma\gamma'^{-1}).$$

Introduction

Let us consider the obvious faithful representation of $C^*_{red}(\beta_\Gamma \rtimes \Gamma)$ on $\ell^2(\Gamma)$. If f be an element in $C^*_{red}(\beta_\Gamma \rtimes \Gamma)_r$ (i.e $f(\gamma,\gamma')=0$ if $\ell(\gamma')>r$) then f acts as kernel operator :

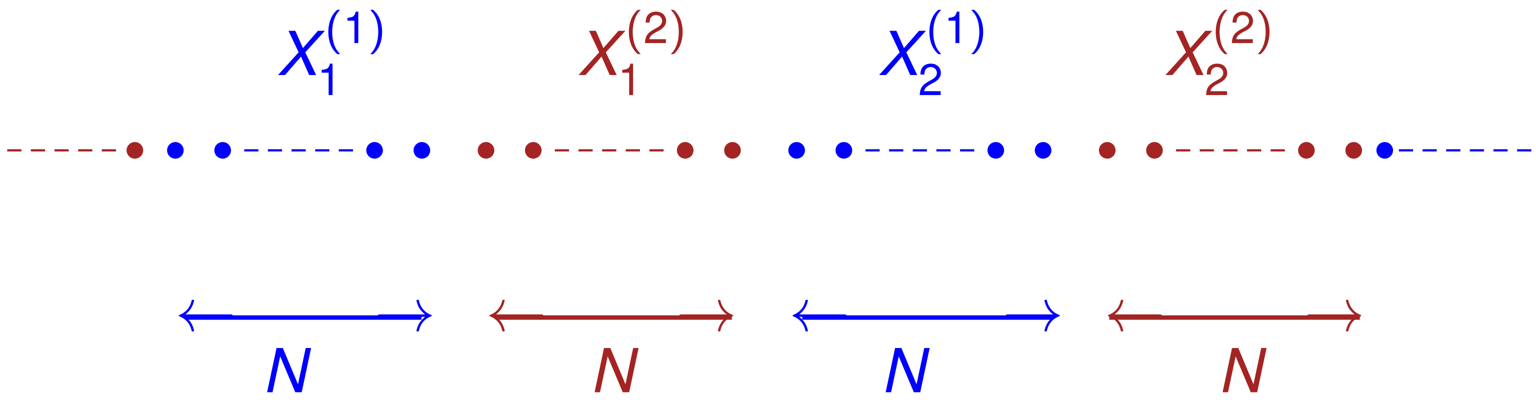
$$f\cdot\xi(\gamma)=\sum_{\gamma'\in\Gamma}\tilde{f}(\gamma,\gamma')\xi(\gamma')$$

with

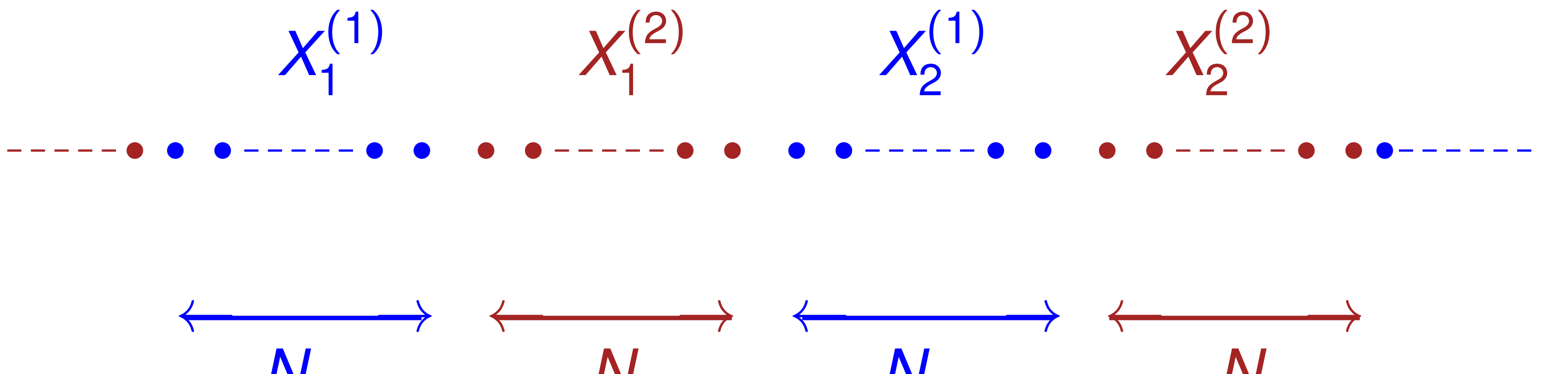
$$\tilde{f}:\Gamma\times\Gamma\to\mathbb{C}:(\gamma,\gamma')\mapsto f(\gamma,\gamma\gamma'^{-1}).$$

Moreover, $\tilde{f}$ has propagation less than r, i.e $\tilde{f}(\gamma,\gamma')=0$ if $d(\gamma,\gamma')>r$.

Example : $\beta_{\mathbb{Z}} \rtimes \mathbb{Z}$.

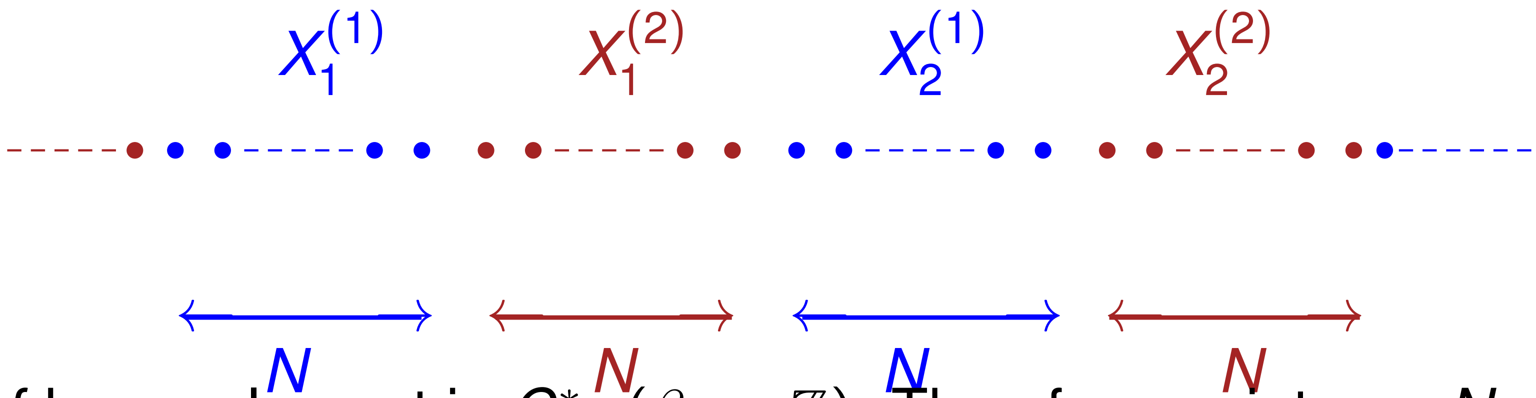


Example : $\beta_{\mathbb{Z}} \rtimes \mathbb{Z}$.



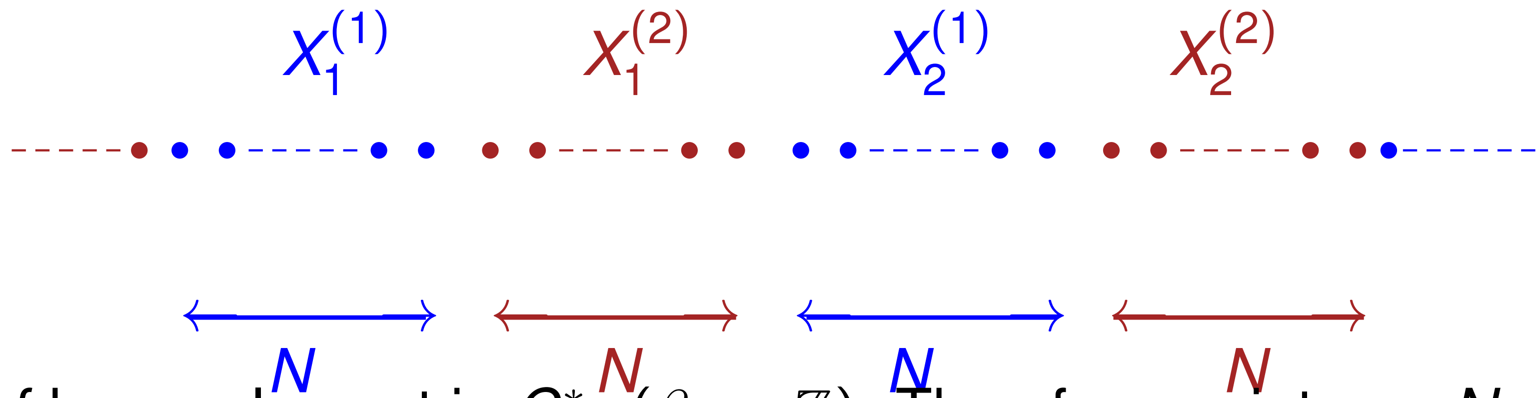
- Let f be an element in $C^*_{red}(\beta_{\mathbb{Z}} \rtimes \mathbb{Z})$. Then for any integer N with $N > 2r$, the we can write $\tilde{f} = \tilde{f}_1 + \tilde{f}_2$ with $\tilde{f}_i : \Gamma \times \Gamma \to \mathbb{C}$ supported in $Z_r^{(i)} = \bigsqcup_{j \in \mathbb{Z}} X_{j,r}^{(i)} \times X_{j,r}^{(i)}$ with $X_{j,r}^{(i)}$ the set of integers at distance to $X_j^{(i)}$ less than r.

Example : $\beta_{\mathbb{Z}} \rtimes \mathbb{Z}$.

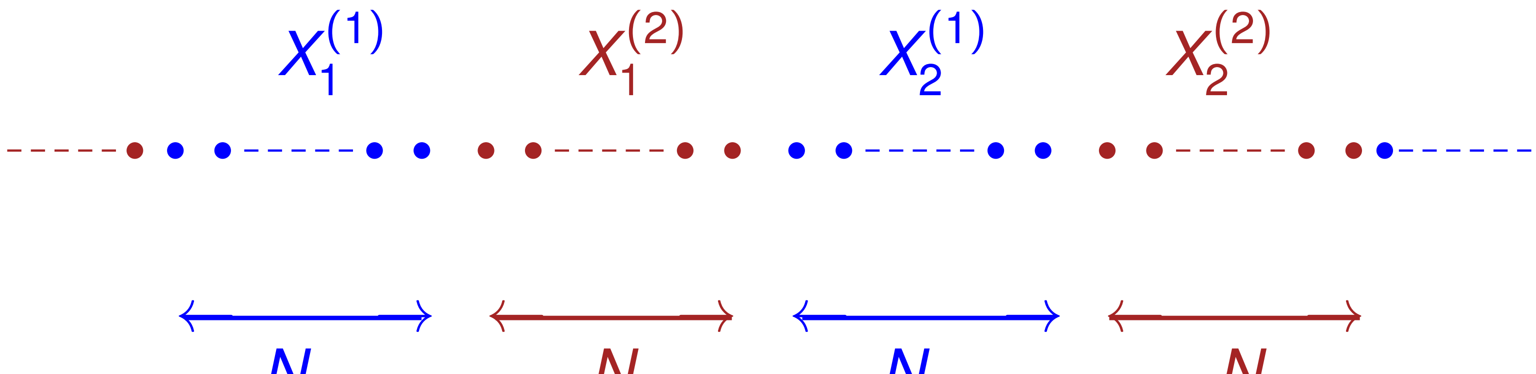


- Let f be an element in $C^*_{red}(\beta_{\mathbb{Z}} \rtimes \mathbb{Z})$. Then for any integer N with $N > 2r$, the we can write $\tilde{f} = \tilde{f}_1 + \tilde{f}_2$ with $\tilde{f}_i : \Gamma \times \Gamma \to \mathbb{C}$ supported in $Z_r^{(i)} = \bigsqcup_{j \in \mathbb{Z}} X_{j,r}^{(i)} \times X_{j,r}^{(i)}$ with $X_{j,r}^{(i)}$ the set of integers at distance to $X_j^{(i)}$ less than r.
- for $i = 1, 2$, set $\chi_i(\gamma, \gamma') = 1$ if $(\gamma, \gamma'^{-1}\gamma) \in Z_r^{(i)}$ and 0 otherwise and view χ_i as a function on $\beta_{\mathbb{Z}} \rtimes \mathbb{Z}$ with finite $\mathbb{Z}$-support.

Example : $\beta_{\mathbb{Z}} \rtimes \mathbb{Z}$.



- Let f be an element in $C^*_{red}(\beta_{\mathbb{Z}} \rtimes \mathbb{Z})$. Then for any integer N with $N > 2r$, the we can write $\tilde{f} = \tilde{f}_1 + \tilde{f}_2$ with $\tilde{f}_i : \Gamma \times \Gamma \to \mathbb{C}$ supported in $Z_r^{(i)} = \bigsqcup_{j \in \mathbb{Z}} X_{j,r}^{(i)} \times X_{j,r}^{(i)}$ with $X_{j,r}^{(i)}$ the set of integers at distance to $X_j^{(i)}$ less than r.
- for $i = 1, 2$, set $\chi_i(\gamma, \gamma') = 1$ if $(\gamma, \gamma'^{-1}\gamma) \in Z_r^{(i)}$ and 0 otherwise and view χ_i as a function on $\beta_{\mathbb{Z}} \rtimes \mathbb{Z}$ with finite $\mathbb{Z}$-support.
- then $\mathcal{H}_i = \chi_i^{-1}(\{1\})$ is a compact-open subgroupoid of $\beta_{\mathbb{Z}} \rtimes \mathbb{Z}$;



- Let f be an element in $C^*_{red}(\beta_{\mathbb{Z}} \rtimes \mathbb{Z})$. Then for any integer N with $N > 2r$, the we can write $\tilde{f} = \tilde{f}_1 + \tilde{f}_2$ with $\tilde{f}_i : \Gamma \times \Gamma \to \mathbb{C}$ supported in $Z_r^{(i)} = \bigsqcup_{j\in\mathbb{Z}} X_{j,r}^{(i)} \times X_{j,r}^{(i)}$ with $X_{j,r}^{(i)}$ the set of integers at distance to $X_j^{(i)}$ less than r.
- for $i = 1, 2$, set $\chi_i(\gamma, \gamma') = 1$ if $(\gamma, \gamma'^{-1}\gamma) \in Z_r^{(i)}$ and 0 otherwise and view χ_i as a function on $\beta_{\mathbb{Z}} \rtimes \mathbb{Z}$ with finite $\mathbb{Z}$-support.
- then $\mathcal{H}_i = \chi_i^{-1}(\{1\})$ is a compact-open subgroupoid of $\beta_{\mathbb{Z}} \rtimes \mathbb{Z}$;
- $\{(x, n) \in \beta_{\mathbb{Z}} \rtimes \mathbb{Z}$ such that $|n| \leqslant r\}$ is contained in $\mathcal{H}_1 \cup \mathcal{H}_2$.

To generalize this decomposition to locally compact groupoids, we have to replace lengths by $\mathcal{G}$-orders.

To generalize this decomposition to locally compact groupoids, we have to replace lengths by $\mathcal{G}$-orders.

Notations

Let $\mathcal{G}$ be locally compact groupoid with unit space X and source and range maps $s, r : \mathcal{G} \to X$ and let Z be a subset of $\mathcal{G}$.

- we set $Z^{-1} = \{\gamma^{-1}; \gamma \in Z\}$;
- for any $Y \subseteq X$, we set $Z_Y = s^{-1}(Y) \cap Z$ and $Z^Y = r^{-1}(Y) \cap Z$;

To generalize this decomposition to locally compact groupoids, we have to replace lengths by $\mathcal{G}$-orders.

Notations

Let $\mathcal{G}$ be locally compact groupoid with unit space X and source and range maps $s, r : \mathcal{G} \to X$ and let Z be a subset of $\mathcal{G}$.

- we set $Z^{-1} = \{\gamma^{-1}; \gamma \in Z\}$;
- for any $Y \subseteq X$, we set $Z_Y = s^{-1}(Y) \cap Z$ and $Z^Y = r^{-1}(Y) \cap Z$;

Definition

A $\mathcal{G}$-order is a subset $\mathcal{R}$ of $\mathcal{G}$ such that

- $X \subseteq \mathcal{R}$;
- $\mathcal{R}^{-1} = \mathcal{R}$ ($\mathcal{R}$ is symetric).
- for every compact subset Y of X, then $\mathcal{R}_Y$ is compact.

$\mathcal{R}$-decomposition of a groupoid

Definition

Let $\mathcal{G}$ be a locally compact groupoid, let $\mathcal{H}$ be a subgroupoid of $\mathcal{G}$ with unit space Y and let $\mathcal{R}$ be a $\mathcal{G}$-order. An $\mathcal{R}$-decomposition of $\mathcal{H}$ is a quadruple $(V_1, V_2, \mathcal{H}_1, \mathcal{H}_2)$ where

$\mathcal{R}$-decomposition of a groupoid

Definition

Let $\mathcal{G}$ be a locally compact groupoid, let $\mathcal{H}$ be a subgroupoid of $\mathcal{G}$ with unit space Y and let $\mathcal{R}$ be a $\mathcal{G}$-order. An $\mathcal{R}$-decomposition of $\mathcal{H}$ is a quadruple $(V_1, V_2, \mathcal{H}_1, \mathcal{H}_2)$ where

- V_1 and V_2 are open subsets of Y with $Y = V_1 \cup V_2$ and such that there exists a partition of the unit subordinated to (V_1, V_2);

$\mathcal{R}$-decomposition of a groupoid

Definition

Let $\mathcal{G}$ be a locally compact groupoid, let $\mathcal{H}$ be a subgroupoid of $\mathcal{G}$ with unit space Y and let $\mathcal{R}$ be a $\mathcal{G}$-order. An $\mathcal{R}$-decomposition of $\mathcal{H}$ is a quadruple $(V_1, V_2, \mathcal{H}_1, \mathcal{H}_2)$ where

- V_1 and V_2 are open subsets of Y with $Y = V_1 \cup V_2$ and such that there exists a partition of the unit subordinated to (V_1, V_2);
- $\mathcal{H}_1$ and $\mathcal{H}_2$ are subgroupoids of $\mathcal{H}$ which are open in $\mathcal{G}$.

$\mathcal{R}$-decomposition of a groupoid

Definition

Let $\mathcal{G}$ be a locally compact groupoid, let $\mathcal{H}$ be a subgroupoid of $\mathcal{G}$ with unit space Y and let $\mathcal{R}$ be a $\mathcal{G}$-order. An $\mathcal{R}$-decomposition of $\mathcal{H}$ is a quadruple $(V_1, V_2, \mathcal{H}_1, \mathcal{H}_2)$ where

- V_1 and V_2 are open subsets of Y with $Y = V_1 \cup V_2$ and such that there exists a partition of the unit subordinated to (V_1, V_2);
- $\mathcal{H}_1$ and $\mathcal{H}_2$ are subgroupoids of $\mathcal{H}$ which are open in $\mathcal{G}$.
- $\mathcal{R}_{V_i}$ is contained in $\mathcal{H}_i$ for $i = 1, 2$.

$\mathcal{R}$-decomposition of a groupoid

Definition

Let $\mathcal{G}$ be a locally compact groupoid, let $\mathcal{H}$ be a subgroupoid of $\mathcal{G}$ with unit space Y and let $\mathcal{R}$ be a $\mathcal{G}$-order. An $\mathcal{R}$-decomposition of $\mathcal{H}$ is a quadruple $(V_1, V_2, \mathcal{H}_1, \mathcal{H}_2)$ where

- V_1 and V_2 are open subsets of Y with $Y = V_1 \cup V_2$ and such that there exists a partition of the unit subordinated to (V_1, V_2);
- $\mathcal{H}_1$ and $\mathcal{H}_2$ are subgroupoids of $\mathcal{H}$ which are open in $\mathcal{G}$.
- $\mathcal{R}_{V_i}$ is contained in $\mathcal{H}_i$ for $i = 1, 2$.

Example

Let $\mathcal{G}$ be the action groupoid $\beta_{\mathbb{Z}} \rtimes \mathbb{Z}$ and consider for $r > 0$ the $\mathcal{G}$-order

$$\mathcal{R} = \{(x, n) \in \mathcal{G};\ |n| \leqslant r\}.$$

$\mathcal{R}$-decomposition of a groupoid

Definition

Let $\mathcal{G}$ be a locally compact groupoid, let $\mathcal{H}$ be a subgroupoid of $\mathcal{G}$ with unit space Y and let $\mathcal{R}$ be a $\mathcal{G}$-order. An $\mathcal{R}$-decomposition of $\mathcal{H}$ is a quadruple $(V_1, V_2, \mathcal{H}_1, \mathcal{H}_2)$ where

- V_1 and V_2 are open subsets of Y with $Y = V_1 \cup V_2$ and such that there exists a partition of the unit subordinated to (V_1, V_2);
- $\mathcal{H}_1$ and $\mathcal{H}_2$ are subgroupoids of $\mathcal{H}$ which are open in $\mathcal{G}$.
- $\mathcal{R}_{V_i}$ is contained in $\mathcal{H}_i$ for $i = 1, 2$.

Example

Let $\mathcal{G}$ be the action groupoid $\beta_{\mathbb{Z}} \rtimes \mathbb{Z}$ and consider for $r > 0$ the $\mathcal{G}$-order

$$\mathcal{R} = \{(x, n) \in \mathcal{G};\ |n| \leqslant r\}.$$

Then $\mathcal{G}$ admits a $\mathcal{R}$-decomposition with $\mathcal{H}_1$ and $\mathcal{H}_2$ compact-open.

Coercive $\mathcal{R}$-decomposition of a groupoid

Definition

Let $\mathcal{G}$ be locally compact groupoid. A relatively clopen sugroupoid of $\mathcal{G}$ is an open subgroupoid $\mathcal{H}$ of $\mathcal{G}$ such that if Y stands for the unit space of $\mathcal{H}$, then $\mathcal{H}$ is closed in $\mathcal{G}_Y$.

Coercive $\mathcal{R}$-decomposition of a groupoid

Definition

Let $\mathcal{G}$ be locally compact groupoid. A relatively clopen sugroupoid of $\mathcal{G}$ is an open subgroupoid $\mathcal{H}$ of $\mathcal{G}$ such that if Y stands for the unit space of $\mathcal{H}$, then $\mathcal{H}$ is closed in $\mathcal{G}_Y$.

Definition

Let $\mathcal{G}$ be a locally compact groupoid, let $\mathcal{H}$ be a subgroupoid of $\mathcal{G}$ and let $\mathcal{R}$ be a $\mathcal{G}$-order. A coercive $\mathcal{R}$-decomposition of $\mathcal{H}$ is a $\mathcal{R}$-decomposition $(V_1, V_2, \mathcal{H}_1, \mathcal{H}_2)$ of $\mathcal{H}$ such that $\mathcal{H}_1$ and $\mathcal{H}_2$ are relatively clopen in $\mathcal{G}$,

Coercive $\mathcal{R}$-decomposition of a groupoid

Definition

Let $\mathcal{G}$ be locally compact groupoid. A relatively clopen sugroupoid of $\mathcal{G}$ is an open subgroupoid $\mathcal{H}$ of $\mathcal{G}$ such that if Y stands for the unit space of $\mathcal{H}$, then $\mathcal{H}$ is closed in $\mathcal{G}_Y$.

Definition

Let $\mathcal{G}$ be a locally compact groupoid, let $\mathcal{H}$ be a subgroupoid of $\mathcal{G}$ and let $\mathcal{R}$ be a $\mathcal{G}$-order. A coercive $\mathcal{R}$-decomposition of $\mathcal{H}$ is a $\mathcal{R}$-decomposition $(V_1, V_2, \mathcal{H}_1, \mathcal{H}_2)$ of $\mathcal{H}$ such that $\mathcal{H}_1$ and $\mathcal{H}_2$ are relatively clopen in $\mathcal{G}$, i.e

- V_1 and V_2 are open subsets of Y with $Y = V_1 \cup V_2$ and such that there exists a partition of the unit subordinated to (V_1, V_2);
- $\mathcal{H}_1$ and $\mathcal{H}_2$ are subgroupoids of $\mathcal{H}$ which are relatively clopen in $\mathcal{G}$.
- $\mathcal{R}_{V_i}$ is contained in $\mathcal{H}_i$ for $i = 1, 2$.

$\mathcal{D}$-decomposable groupoids

Definition

Let $\mathcal{D}$ be a set of open subgroupoids of $\mathcal{G}$. A subgroupoid $\mathcal{H}$ of $\mathcal{G}$ is $\mathcal{D}$-decomposable if for every $\mathcal{G}$-order $\mathcal{R}$, there exists an $\mathcal{R}$-decomposition $(V_1, V_2, \mathcal{H}_1, \mathcal{H}_2)$ with $\mathcal{H}_1$ and $\mathcal{H}_2$ in $\mathcal{D}$.

$\mathcal{D}$-decomposable groupoids

Definition

Let $\mathcal{D}$ be a set of open subgroupoids of $\mathcal{G}$. A subgroupoid $\mathcal{H}$ of $\mathcal{G}$ is $\mathcal{D}$-decomposable if for every $\mathcal{G}$-order $\mathcal{R}$, there exists an $\mathcal{R}$-decomposition $(V_1, V_2, \mathcal{H}_1, \mathcal{H}_2)$ with $\mathcal{H}_1$ and $\mathcal{H}_2$ in $\mathcal{D}$.

Lemma

Let $\mathcal{H}$ be a subgroupoid of $\mathcal{G}$.

1. *if $\mathcal{D}$ is a set of open subgroupoids of $\mathcal{G}$ such that $\mathcal{H}$ is $\mathcal{D}$-decomposable then $\mathcal{H}$ is an open subgroupoid of $\mathcal{G}$.*

$\mathcal{D}$-decomposable groupoids

Definition

Let $\mathcal{D}$ be a set of open subgroupoids of $\mathcal{G}$. A subgroupoid $\mathcal{H}$ of $\mathcal{G}$ is $\mathcal{D}$-decomposable if for every $\mathcal{G}$-order $\mathcal{R}$, there exists an $\mathcal{R}$-decomposition $(V_1, V_2, \mathcal{H}_1, \mathcal{H}_2)$ with $\mathcal{H}_1$ and $\mathcal{H}_2$ in $\mathcal{D}$.

Lemma

Let $\mathcal{H}$ be a subgroupoid of $\mathcal{G}$.

1. *if $\mathcal{D}$ is a set of open subgroupoids of $\mathcal{G}$ such that $\mathcal{H}$ is $\mathcal{D}$-decomposable then $\mathcal{H}$ is an open subgroupoid of $\mathcal{G}$.*
2. *if $\mathcal{D}$ is a set of relatively clopen subgroupoids of $\mathcal{G}$ such that $\mathcal{H}$ is $\mathcal{D}$-decomposable, then $\mathcal{H}$ is a relatively clopen subgroupoid of $\mathcal{G}$.*

$\mathcal{D}$-decomposable groupoids

Definition

Let $\mathcal{D}$ be a set of open subgroupoids of $\mathcal{G}$. A subgroupoid $\mathcal{H}$ of $\mathcal{G}$ is $\mathcal{D}$-decomposable if for every $\mathcal{G}$-order $\mathcal{R}$, there exists an $\mathcal{R}$-decomposition $(V_1, V_2, \mathcal{H}_1, \mathcal{H}_2)$ with $\mathcal{H}_1$ and $\mathcal{H}_2$ in $\mathcal{D}$.

Lemma

Let $\mathcal{H}$ be a subgroupoid of $\mathcal{G}$.

1. *if $\mathcal{D}$ is a set of open subgroupoids of $\mathcal{G}$ such that $\mathcal{H}$ is $\mathcal{D}$-decomposable then $\mathcal{H}$ is an open subgroupoid of $\mathcal{G}$.*
2. *if $\mathcal{D}$ is a set of relatively clopen subgroupoids of $\mathcal{G}$ such that $\mathcal{H}$ is $\mathcal{D}$-decomposable, then $\mathcal{H}$ is a relatively clopen subgroupoid of $\mathcal{G}$.*

Example

$\mathcal{G} = \beta_{\mathbb{Z}} \rtimes \mathbb{Z}$ is $\mathcal{D}$-decomposable with $\mathcal{D}$ the set of its compact-open subgroupoids of $\mathcal{G}$.

$\mathcal{D}$-decomposable groupoids

Definition

Let $\mathcal{D}$ be a set of open subgroupoids of $\mathcal{G}$. A subgroupoid $\mathcal{H}$ of $\mathcal{G}$ is $\mathcal{D}$-decomposable if for every $\mathcal{G}$-order $\mathcal{R}$, there exists an $\mathcal{R}$-decomposition $(V_1, V_2, \mathcal{H}_1, \mathcal{H}_2)$ with $\mathcal{H}_1$ and $\mathcal{H}_2$ in $\mathcal{D}$.

Lemma

Let $\mathcal{H}$ be a subgroupoid of $\mathcal{G}$.

1. *if $\mathcal{D}$ is a set of open subgroupoids of $\mathcal{G}$ such that $\mathcal{H}$ is $\mathcal{D}$-decomposable then $\mathcal{H}$ is an open subgroupoid of $\mathcal{G}$.*
2. *if $\mathcal{D}$ is a set of relatively clopen subgroupoids of $\mathcal{G}$ such that $\mathcal{H}$ is $\mathcal{D}$-decomposable, then $\mathcal{H}$ is a relatively clopen subgroupoid of $\mathcal{G}$.*

Example

$\mathcal{G} = \beta_{\mathbb{Z}} \rtimes \mathbb{Z}$ is $\mathcal{D}$-decomposable with $\mathcal{D}$ the set of its compact-open subgroupoids of $\mathcal{G}$. The same holds for $\mathcal{G} = \beta_{\mathbb{F}_n} \rtimes \mathbb{F}_n$.

Groupoid with finite $\mathcal{D}$-complexity (Guentner/Willet/Yu)

Definition

Let $\mathcal{G}$ be locally compact groupoid.

- A set $\mathcal{D}$ of open subgroupoids of $\mathcal{G}$ is closed under coarse decompositions if every $\mathcal{D}$-decomposable subgroupoid is in $\mathcal{D}$.

Groupoid with finite $\mathcal{D}$-complexity (Guentner/Willet/Yu)

Definition

Let $\mathcal{G}$ be locally compact groupoid.

- A set $\mathcal{D}$ of open subgroupoids of $\mathcal{G}$ is closed under coarse decompositions if every $\mathcal{D}$-decomposable subgroupoid is in $\mathcal{D}$.
- If $\mathcal{D}$ is a set of open subgroupoid of $\mathcal{G}$, let $\widehat{\mathcal{D}}$ be the smallest set of open subgroupoids of $\mathcal{G}$ closed under coarse decompositions.

Groupoid with finite $\mathcal{D}$-complexity (Guentner/Willet/Yu)

Definition

Let $\mathcal{G}$ be locally compact groupoid.

- A set $\mathcal{D}$ of open subgroupoids of $\mathcal{G}$ is closed under coarse decompositions if every $\mathcal{D}$-decomposable subgroupoid is in $\mathcal{D}$.
- If $\mathcal{D}$ is a set of open subgroupoid of $\mathcal{G}$, let $\widehat{\mathcal{D}}$ be the smallest set of open subgroupoids of $\mathcal{G}$ closed under coarse decompositions. A open subgroupoid $\mathcal{H}$ has finite $\mathcal{D}$-complexity if $\mathcal{H}$ belongs to $\widehat{\mathcal{D}}$.

Groupoid with finite $\mathcal{D}$-complexity (Guentner/Willet/Yu)

Definition

Let $\mathcal{G}$ be locally compact groupoid.

- A set $\mathcal{D}$ of open subgroupoids of $\mathcal{G}$ is closed under coarse decompositions if every $\mathcal{D}$-decomposable subgroupoid is in $\mathcal{D}$.
- If $\mathcal{D}$ is a set of open subgroupoid of $\mathcal{G}$, let $\widehat{\mathcal{D}}$ be the smallest set of open subgroupoids of $\mathcal{G}$ closed under coarse decompositions. A open subgroupoid $\mathcal{H}$ has finite $\mathcal{D}$-complexity if $\mathcal{H}$ belongs to $\widehat{\mathcal{D}}$.

Lemma

Let $\mathcal{D}$ be a set of relatively clopen subgroupoids of $\mathcal{G}$.

1. *if $\mathcal{H}$ is in $\widehat{\mathcal{D}}$, then $\mathcal{H}$ is locally clopen;*

Groupoid with finite $\mathcal{D}$-complexity (Guentner/Willet/Yu)

Definition

Let $\mathcal{G}$ be locally compact groupoid.

- A set $\mathcal{D}$ of open subgroupoids of $\mathcal{G}$ is closed under coarse decompositions if every $\mathcal{D}$-decomposable subgroupoid is in $\mathcal{D}$.
- If $\mathcal{D}$ is a set of open subgroupoid of $\mathcal{G}$, let $\widehat{\mathcal{D}}$ be the smallest set of open subgroupoids of $\mathcal{G}$ closed under coarse decompositions. A open subgroupoid $\mathcal{H}$ has finite $\mathcal{D}$-complexity if $\mathcal{H}$ belongs to $\widehat{\mathcal{D}}$.

Lemma

Let $\mathcal{D}$ be a set of relatively clopen subgroupoids of $\mathcal{G}$.

1. *if $\mathcal{H}$ is in $\widehat{\mathcal{D}}$, then $\mathcal{H}$ is locally clopen;*
2. *$\widehat{\mathcal{D}}$ is the smallest set of relatively clopen subgroupoids of $\mathcal{G}$ closed under coercive coarse decomposition;*

Groupoid with finite $\mathcal{D}$-complexity (Guentner/Willet/Yu)

Definition

Let $\mathcal{G}$ be locally compact groupoid.

- A set $\mathcal{D}$ of open subgroupoids of $\mathcal{G}$ is closed under coarse decompositions if every $\mathcal{D}$-decomposable subgroupoid is in $\mathcal{D}$.
- If $\mathcal{D}$ is a set of open subgroupoid of $\mathcal{G}$, let $\widehat{\mathcal{D}}$ be the smallest set of open subgroupoids of $\mathcal{G}$ closed under coarse decompositions. A open subgroupoid $\mathcal{H}$ has finite $\mathcal{D}$-complexity if $\mathcal{H}$ belongs to $\widehat{\mathcal{D}}$.

Lemma

Let $\mathcal{D}$ be a set of relatively clopen subgroupoids of $\mathcal{G}$.

1. *if $\mathcal{H}$ is in $\widehat{\mathcal{D}}$, then $\mathcal{H}$ is locally clopen;*
2. *$\widehat{\mathcal{D}}$ is the smallest set of relatively clopen subgroupoids of $\mathcal{G}$ closed under coercive coarse decomposition;*
3. *If $\mathcal{D}$ is closed under taking relatively clopen subgroupoids, so is $\widehat{\mathcal{D}}$.*

Metric spaces with finite asymptotic dimension

Definition

Let X be a proper discrete metric space. Then X has asymptotic dimension m if for every $r > 0$ there exist $m + 1$ subsets $X^{(1)}, \dots X^{(m+1)}$ of X such that

Metric spaces with finite asymptotic dimension

Definition

Let X be a proper discrete metric space. Then X has asymptotic dimension m if for every $r > 0$ there exist $m + 1$ subsets $X^{(1)}, \ldots X^{(m+1)}$ of X such that

- $X = \cup_{i=1}^{m+1} X^{(i)}$;

Metric spaces with finite asymptotic dimension

Definition

Let X be a proper discrete metric space. Then X has asymptotic dimension m if for every $r > 0$ there exist $m + 1$ subsets $X^{(1)}, \dots X^{(m+1)}$ of X such that

- $X = \cup_{i=1}^{m+1} X^{(i)}$;
- $X^{(i)}$ is a r-disjoint union of a uniformly bounded family of subsets,

Metric spaces with finite asymptotic dimension

Definition

Let X be a proper discrete metric space. Then X has asymptotic dimension m if for every $r > 0$ there exist $m+1$ subsets $X^{(1)}, \ldots X^{(m+1)}$ of X such that

- $X = \cup_{i=1}^{m+1} X^{(i)}$;
- $X^{(i)}$ is a r-disjoint union of a uniformly bounded family of subsets, i.e $X^{(i)} = \bigsqcup_{k\in\mathbb{N}} X_k^{(i)}$ with $(diam\, X_k^{(i)})_{k\in\mathbb{N}}$ bounded and $d(X_k^{(i)}, X_l^{(i)}) > r$ if $k \neq l$;

Metric spaces with finite asymptotic dimension

Definition

Let X be a proper discrete metric space. Then X has asymptotic dimension m if for every $r > 0$ there exist $m+1$ subsets $X^{(1)}, \ldots X^{(m+1)}$ of X such that

- $X = \cup_{i=1}^{m+1} X^{(i)}$;
- $X^{(i)}$ is a r-disjoint union of a uniformly bounded family of subsets, i.e $X^{(i)} = \bigsqcup_{k\in\mathbb{N}} X_k^{(i)}$ with $(diam\, X_k^{(i)})_{k\in\mathbb{N}}$ bounded and $d(X_k^{(i)}, X_l^{(i)}) > r$ if $k \neq l$;

Example

1. $\mathbb{Z}^n$ has asymptotic dimension n;

Metric spaces with finite asymptotic dimension

Definition

Let X be a proper discrete metric space. Then X has asymptotic dimension m if for every $r > 0$ there exist $m + 1$ subsets $X^{(1)}, \ldots X^{(m+1)}$ of X such that

- $X = \cup_{i=1}^{m+1} X^{(i)}$;
- $X^{(i)}$ is a r-disjoint union of a uniformly bounded family of subsets, i.e $X^{(i)} = \bigsqcup_{k \in \mathbb{N}} X_k^{(i)}$ with $(diam\, X_k^{(i)})_{k \in \mathbb{N}}$ bounded and $d(X_k^{(i)}, X_l^{(i)}) > r$ if $k \neq l$;

Example

1. $\mathbb{Z}^n$ has asymptotic dimension n;
2. Gromov hyperbolic spaces have finite asymptotic dimension;

Metric spaces with finite asymptotic dimension

Definition

Let X be a proper discrete metric space. Then X has asymptotic dimension m if for every $r > 0$ there exist $m + 1$ subsets $X^{(1)}, \dots X^{(m+1)}$ of X such that

- $X = \cup_{i=1}^{m+1} X^{(i)}$;
- $X^{(i)}$ is a r-disjoint union of a uniformly bounded family of subsets, i.e $X^{(i)} = \bigsqcup_{k\in\mathbb{N}} X_k^{(i)}$ with $(diam\, X_k^{(i)})_{k\in\mathbb{N}}$ bounded and $d(X_k^{(i)}, X_l^{(i)}) > r$ if $k \neq l$;

Example

1. $\mathbb{Z}^n$ has asymptotic dimension n;
2. Gromov hyperbolic spaces have finite asymptotic dimension;
3. Discrete subgroups in Lie groups have finite asymptotic dimension.

Theorem (Dranishnikov-Zarichnyi)

Let Σ be a proper discrete metric space with bounded geometry. If X has finite asymptotic dimension, then Σ admit a coarse embedding into a product of trees $T_1 \times \cdots \times T_n$

Theorem (Dranishnikov-Zarichnyi)

Let Σ be a proper discrete metric space with bounded geometry. If X has finite asymptotic dimension, then Σ admit a coarse embedding into a product of trees $T_1 \times \cdots \times T_n$

(i.e there exists proper maps $\rho_\pm : \mathbb{R}^+ \to \mathbb{R}^+$ and a map $f : \Sigma \to T_1 \times \cdots \times T_n$ such that $\rho_-(d(f(x), f(y))) \leqslant d(x, y) \leqslant \rho_+(d(f(x), f(y)))$ for all x, y in Σ).

Theorem (Dranishnikov-Zarichnyi)

Let Σ be a proper discrete metric space with bounded geometry. If X has finite asymptotic dimension, then Σ admit a coarse embedding into a product of trees $T_1 \times \cdots \times T_n$

(i.e there exists proper maps $\rho_\pm : \mathbb{R}^+ \to \mathbb{R}^+$ and a map $f : \Sigma \to T_1 \times \cdots \times T_n$ such that $\rho_-(d(f(x), f(y))) \leqslant d(x, y) \leqslant \rho_+(d(f(x), f(y)))$ for all x, y in Σ).

Corollary

Let Γ be a finitely generated group viewed as a metric space for a metric arising from a word metric length associated to a finite symmetric generating set.

Theorem (Dranishnikov-Zarichnyi)

Let Σ be a proper discrete metric space with bounded geometry. If X has finite asymptotic dimension, then Σ admit a coarse embedding into a product of trees $T_1 \times \cdots \times T_n$

(i.e there exists proper maps $\rho_\pm : \mathbb{R}^+ \to \mathbb{R}^+$ and a map $f : \Sigma \to T_1 \times \cdots \times T_n$ such that $\rho_-(d(f(x), f(y))) \leqslant d(x, y) \leqslant \rho_+(d(f(x), f(y)))$ for all x, y in Σ).

Corollary

Let Γ be a finitely generated group viewed as a metric space for a metric arising from a word metric length associated to a finite symmetric generating set.

If Γ has finite asymptotic dimension, then $\mathcal{G} = \beta_\Gamma \rtimes \Gamma$ is $\mathcal{D}$-decomposable with respect the set $\mathcal{D}$ of compact-open subgroupoids of $\mathcal{G}$.

K-theory computations for groupoid crossed product algebras

Let $\mathcal{G}$ be a locally compact groupoid provided with a Haar System and let A be a $\mathcal{G}$-algebra. For any $\mathcal{G}$-order $\mathcal{R}$, let $(V_1, V_2, \mathcal{G}_1, \mathcal{G}_2)$ be a $\mathcal{R}$-decomposition for $\mathcal{G}$.

K-theory computations for groupoid crossed product algebras

Let $\mathcal{G}$ be a locally compact groupoid provided with a Haar System and let A be a $\mathcal{G}$-algebra. For any $\mathcal{G}$-order $\mathcal{R}$, let $(V_1, V_2, \mathcal{G}_1, \mathcal{G}_2)$ be a $\mathcal{R}$-decomposition for $\mathcal{G}$.Then we have inclusions

$A \rtimes_r (\mathcal{G}_1 \cap \mathcal{G}_2) \hookrightarrow A \rtimes_r \mathcal{G}_1,$

$A \rtimes_r (\mathcal{G}_1 \cap \mathcal{G}_2) \hookrightarrow A \rtimes_r \mathcal{G}_2,$

$A \rtimes_r \mathcal{G}_1 \hookrightarrow A \rtimes_r \mathcal{G}$

$A \rtimes_r \mathcal{G}_1 \hookrightarrow A \rtimes_r \mathcal{G}$

K-theory computations for groupoid crossed product algebras

Let $\mathcal{G}$ be a locally compact groupoid provided with a Haar System and let A be a $\mathcal{G}$-algebra. For any $\mathcal{G}$-order $\mathcal{R}$, let $(V_1, V_2, \mathcal{G}_1, \mathcal{G}_2)$ be a $\mathcal{R}$-decomposition for $\mathcal{G}$.Then we have inclusions
$A \rtimes_r (\mathcal{G}_1 \cap \mathcal{G}_2) \hookrightarrow A \rtimes_r \mathcal{G}_1,$
$A \rtimes_r (\mathcal{G}_1 \cap \mathcal{G}_2) \hookrightarrow A \rtimes_r \mathcal{G}_2,$
$A \rtimes_r \mathcal{G}_1 \hookrightarrow A \rtimes_r \mathcal{G}$
$A \rtimes_r \mathcal{G}_1 \hookrightarrow A \rtimes_r \mathcal{G}$
We deduce a sequence

$$K_*(A \rtimes_r (\mathcal{G}_1 \cap \mathcal{G}_2)) \longrightarrow K_*(A \rtimes_r \mathcal{G}_1) \oplus K_*(A \rtimes_r \mathcal{G}_2) \longrightarrow K_*(A \rtimes_r \mathcal{G}).$$

K-theory computations for groupoid crossed product algebras

Let $\mathcal{G}$ be a locally compact groupoid provided with a Haar System and let A be a $\mathcal{G}$-algebra. For any $\mathcal{G}$-order $\mathcal{R}$, let $(V_1, V_2, \mathcal{G}_1, \mathcal{G}_2)$ be a $\mathcal{R}$-decomposition for $\mathcal{G}$.Then we have inclusions
$A \rtimes_r (\mathcal{G}_1 \cap \mathcal{G}_2) \hookrightarrow A \rtimes_r \mathcal{G}_1$,
$A \rtimes_r (\mathcal{G}_1 \cap \mathcal{G}_2) \hookrightarrow A \rtimes_r \mathcal{G}_2$,
$A \rtimes_r \mathcal{G}_1 \hookrightarrow A \rtimes_r \mathcal{G}$
$A \rtimes_r \mathcal{G}_1 \hookrightarrow A \rtimes_r \mathcal{G}$
We deduce a sequence

$$K_*(A \rtimes_r (\mathcal{G}_1 \cap \mathcal{G}_2)) \longrightarrow K_*(A \rtimes_r \mathcal{G}_1) \oplus K_*(A \rtimes_r \mathcal{G}_2) \longrightarrow K_*(A \rtimes_r \mathcal{G}).$$

If we take into account propagation control by $\mathcal{G}$-orders, this sequence becomes exact up to rescaling for $\mathcal{G}$-order $\mathcal{R}'$ with $\mathcal{R}' \ll \mathcal{R}$.

K-theory computations for groupoid crossed product algebras

Let $\mathcal{G}$ be a locally compact groupoid provided with a Haar System and let A be a $\mathcal{G}$-algebra. For any $\mathcal{G}$-order $\mathcal{R}$, let $(V_1, V_2, \mathcal{G}_1, \mathcal{G}_2)$ be a $\mathcal{R}$-decomposition for $\mathcal{G}$.Then we have inclusions

$A \rtimes_r (\mathcal{G}_1 \cap \mathcal{G}_2) \hookrightarrow A \rtimes_r \mathcal{G}_1,$

$A \rtimes_r (\mathcal{G}_1 \cap \mathcal{G}_2) \hookrightarrow A \rtimes_r \mathcal{G}_2,$

$A \rtimes_r \mathcal{G}_1 \hookrightarrow A \rtimes_r \mathcal{G}$

$A \rtimes_r \mathcal{G}_1 \hookrightarrow A \rtimes_r \mathcal{G}$

We deduce a sequence

$$K_*(A \rtimes_r (\mathcal{G}_1 \cap \mathcal{G}_2)) \longrightarrow K_*(A \rtimes_r \mathcal{G}_1) \oplus K_*(A \rtimes_r \mathcal{G}_2) \longrightarrow K_*(A \rtimes_r \mathcal{G}).$$

If we take into account propagation control by $\mathcal{G}$-orders, this sequence becomes exact up to rescaling for $\mathcal{G}$-order $\mathcal{R}'$ with $\mathcal{R}' \ll \mathcal{R}$. Moreover, we can constructs boundary maps at order $\mathcal{R}'$ which turns the above sequence into a six-term exact sequence (up to rescaling) at order $\mathcal{R}'$.

QUANTITATIVE K-THEORY FOR GROUPOIDS CROSSED PRODUCT ALGEBRAS
(Dell'aiera)

- Aim : Generalize quantitative K-theory (O-Yu) to groupoids crossed product algebras when there is no length arising.

QUANTITATIVE K-THEORY FOR GROUPOIDS CROSSED PRODUCT ALGEBRAS
(Dell'aiera)

- Aim : Generalize quantitative K-theory (O-Yu) to groupoids crossed product algebras when there is no length arising.
- We replace the length by the lattice of $\mathcal{G}$-orders.

The lattice of $\mathcal{G}$-orders

Let $\mathcal{G}$ be a locally compact groupoid with unit space X. Recall that a $\mathcal{G}$-order is a subset $\mathcal{R}$ of $\mathcal{G}$ such that $X \subseteq \mathcal{R}$, $\mathcal{R}^{-1} = \mathcal{R}$ and $\mathcal{R}_Y$ is compact for every compact subset Y of X.

The lattice of $\mathcal{G}$-orders

Let $\mathcal{G}$ be a locally compact groupoid with unit space X. Recall that a $\mathcal{G}$-order is a subset $\mathcal{R}$ of $\mathcal{G}$ such that $X \subseteq \mathcal{R}$, $\mathcal{R}^{-1} = \mathcal{R}$ and $\mathcal{R}_Y$ is compact for every compact subset Y of X.
Let $\mathcal{E}_\mathcal{G}$ be the set of $\mathcal{G}$-order of $\mathcal{G}$.

The lattice of $\mathcal{G}$-orders

Let $\mathcal{G}$ be a locally compact groupoid with unit space X. Recall that a $\mathcal{G}$-order is a subset $\mathcal{R}$ of $\mathcal{G}$ such that $X \subseteq \mathcal{R}$, $\mathcal{R}^{-1} = \mathcal{R}$ and $\mathcal{R}_Y$ is compact for every compact subset Y of X.
Let $\mathcal{E}_\mathcal{G}$ be the set of $\mathcal{G}$-order of $\mathcal{G}$.

- $\mathcal{E}_\mathcal{G}$ is a poset for the inclusion;

The lattice of $\mathcal{G}$-orders

Let $\mathcal{G}$ be a locally compact groupoid with unit space X. Recall that a $\mathcal{G}$-order is a subset $\mathcal{R}$ of $\mathcal{G}$ such that $X \subseteq \mathcal{R}$, $\mathcal{R}^{-1} = \mathcal{R}$ and $\mathcal{R}_Y$ is compact for every compact subset Y of X.

Let $\mathcal{E}_\mathcal{G}$ be the set of $\mathcal{G}$-order of $\mathcal{G}$.

- $\mathcal{E}_\mathcal{G}$ is a poset for the inclusion;
- if $\mathcal{R}$ and $\mathcal{R}'$ are in $\mathcal{E}_\mathcal{G}$ the $\mathcal{R} \cap \mathcal{R}'$ and $\mathcal{R} \cup \mathcal{R}'$ are in $\mathcal{E}_\mathcal{G}$.

The lattice of $\mathcal{G}$-orders

Let $\mathcal{G}$ be a locally compact groupoid with unit space X. Recall that a $\mathcal{G}$-order is a subset $\mathcal{R}$ of $\mathcal{G}$ such that $X \subseteq \mathcal{R}$, $\mathcal{R}^{-1} = \mathcal{R}$ and $\mathcal{R}_Y$ is compact for every compact subset Y of X.
Let $\mathcal{E}_\mathcal{G}$ be the set of $\mathcal{G}$-order of $\mathcal{G}$.

- $\mathcal{E}_\mathcal{G}$ is a poset for the inclusion;
- if $\mathcal{R}$ and $\mathcal{R}'$ are in $\mathcal{E}_\mathcal{G}$ the $\mathcal{R} \cap \mathcal{R}'$ and $\mathcal{R} \cup \mathcal{R}'$ are in $\mathcal{E}_\mathcal{G}$.
- for any compact K in $\mathcal{G}$, there exists $\mathcal{R}$ in $\mathcal{E}_\mathcal{G}$ such that $K \subseteq \mathcal{R}$ (take $\mathcal{R} = K \cup K^{-1} \cup X$).

The lattice of $\mathcal{G}$-orders

Let $\mathcal{G}$ be a locally compact groupoid with unit space X. Recall that a $\mathcal{G}$-order is a subset $\mathcal{R}$ of $\mathcal{G}$ such that $X \subseteq \mathcal{R}$, $\mathcal{R}^{-1} = \mathcal{R}$ and $\mathcal{R}_Y$ is compact for every compact subset Y of X.
Let $\mathcal{E}_\mathcal{G}$ be the set of $\mathcal{G}$-order of $\mathcal{G}$.

- $\mathcal{E}_\mathcal{G}$ is a poset for the inclusion;
- if $\mathcal{R}$ and $\mathcal{R}'$ are in $\mathcal{E}_\mathcal{G}$ the $\mathcal{R} \cap \mathcal{R}'$ and $\mathcal{R} \cup \mathcal{R}'$ are in $\mathcal{E}_\mathcal{G}$.
- for any compact K in $\mathcal{G}$, there exists $\mathcal{R}$ in $\mathcal{E}_\mathcal{G}$ such that $K \subseteq \mathcal{R}$ (take $\mathcal{R} = K \cup K^{-1} \cup X$).
- if $\mathcal{R}$ and $\mathcal{R}'$ are in $\mathcal{G}$ then

$$\mathcal{R} * \mathcal{R}' = \mathcal{R} \cdot \mathcal{R}' \cup \mathcal{R}' \cdot \mathcal{R}$$

  is in $\mathcal{E}_\mathcal{G}$, with

$$\mathcal{R} \cdot \mathcal{R}' = \{\gamma\gamma'; \gamma \in \mathcal{R}, \gamma \in \mathcal{R}', s(\gamma) = r(\gamma')\}.$$

The lattice of $\mathcal{G}$-orders

Let $\mathcal{G}$ be a locally compact groupoid with unit space X. Recall that a $\mathcal{G}$-order is a subset $\mathcal{R}$ of $\mathcal{G}$ such that $X \subseteq \mathcal{R}$, $\mathcal{R}^{-1} = \mathcal{R}$ and $\mathcal{R}_Y$ is compact for every compact subset Y of X.
Let $\mathcal{E}_\mathcal{G}$ be the set of $\mathcal{G}$-order of $\mathcal{G}$.

- $\mathcal{E}_\mathcal{G}$ is a poset for the inclusion;
- if $\mathcal{R}$ and $\mathcal{R}'$ are in $\mathcal{E}_\mathcal{G}$ the $\mathcal{R} \cap \mathcal{R}'$ and $\mathcal{R} \cup \mathcal{R}'$ are in $\mathcal{E}_\mathcal{G}$.
- for any compact K in $\mathcal{G}$, there exists $\mathcal{R}$ in $\mathcal{E}_\mathcal{G}$ such that $K \subseteq \mathcal{R}$ (take $\mathcal{R} = K \cup K^{-1} \cup X$).
- if $\mathcal{R}$ and $\mathcal{R}'$ are in $\mathcal{G}$ then

$$\mathcal{R} * \mathcal{R}' = \mathcal{R} \cdot \mathcal{R}' \cup \mathcal{R}' \cdot \mathcal{R}$$

  is in $\mathcal{E}_\mathcal{G}$, with

$$\mathcal{R} \cdot \mathcal{R}' = \{\gamma\gamma'; \gamma \in \mathcal{R}, \gamma \in \mathcal{R}', s(\gamma) = r(\gamma')\}.$$

- for any integer n, we set $\mathcal{R}^{*n} = \mathcal{R} * \cdots * \mathcal{R}$ (n products).

The framework : $\mathcal{G}$-filtered algebras

Definition

A $\mathcal{G}$-filtered C^*-algebra B is a C^*-algebra equipped with a family $(B_{\mathcal{R}})_{\mathcal{R}\in\mathcal{E}_{\mathcal{G}}}$ of closed linear subspaces such that

- $B_{\mathcal{R}} \subseteq B_{\mathcal{R}'}$ if $\mathcal{R} \subseteq \mathcal{R}'$;
- $B_{\mathcal{R}}$ is closed under involution;
- $B_{\mathcal{R}} \cdot B_{\mathcal{R}'} \subseteq B_{\mathcal{R}*\mathcal{R}'}$;
- the subalgebra $\bigcup_{\mathcal{R}\in\mathcal{E}_{\mathcal{G}}} B_{\mathcal{R}}$ is dense in B.

The framework : $\mathcal{G}$-filtered algebras

Definition

A $\mathcal{G}$-filtered C^*-algebra B is a C^*-algebra equipped with a family $(B_\mathcal{R})_{\mathcal{R}\in\mathcal{E}_\mathcal{G}}$ of closed linear subspaces such that

- $B_\mathcal{R} \subseteq B_{\mathcal{R}'}$ if $\mathcal{R} \subseteq \mathcal{R}'$;
- $B_\mathcal{R}$ is closed under involution;
- $B_\mathcal{R} \cdot B_{\mathcal{R}'} \subseteq B_{\mathcal{R}*\mathcal{R}'}$;
- the subalgebra $\bigcup_{\mathcal{R}\in\mathcal{E}_\mathcal{G}} B_\mathcal{R}$ is dense in B.

Example

Let A be a $\mathcal{G}$-algebra. Then A can be viewed as the algebra of continuous sections of a bundle algebra $\mathfrak{A}$.

The framework : $\mathcal{G}$-filtered algebras

Definition

A $\mathcal{G}$-filtered C^*-algebra B is a C^*-algebra equipped with a family $(B_\mathcal{R})_{\mathcal{R}\in\mathcal{E}_\mathcal{G}}$ of closed linear subspaces such that

- $B_\mathcal{R} \subseteq B_{\mathcal{R}'}$ if $\mathcal{R} \subseteq \mathcal{R}'$;
- $B_\mathcal{R}$ is closed under involution;
- $B_\mathcal{R} \cdot B_{\mathcal{R}'} \subseteq B_{\mathcal{R}*\mathcal{R}'}$;
- the subalgebra $\bigcup_{\mathcal{R}\in\mathcal{E}_\mathcal{G}} B_\mathcal{R}$ is dense in B.

Example

Let A be a $\mathcal{G}$-algebra. Then A can be viewed as the algebra of continuous sections of a bundle algebra $\mathfrak{A}$. For any $\mathcal{G}$-order $\mathcal{R}$, let $A \rtimes_r \mathcal{R}$ be the closure of the set of continuous sections $f : \mathcal{G} \to s^*\mathfrak{A}$ compactly supported in $\mathcal{R}$.

The framework : $\mathcal{G}$-filtered algebras

Definition

A $\mathcal{G}$-filtered C^*-algebra B is a C^*-algebra equipped with a family $(B_\mathcal{R})_{\mathcal{R}\in\mathcal{E}_\mathcal{G}}$ of closed linear subspaces such that

- $B_\mathcal{R} \subseteq B_{\mathcal{R}'}$ if $\mathcal{R} \subseteq \mathcal{R}'$;
- $B_\mathcal{R}$ is closed under involution;
- $B_\mathcal{R} \cdot B_{\mathcal{R}'} \subseteq B_{\mathcal{R}*\mathcal{R}'}$;
- the subalgebra $\bigcup_{\mathcal{R}\in\mathcal{E}_\mathcal{G}} B_\mathcal{R}$ is dense in B.

Example

Let A be a $\mathcal{G}$-algebra. Then A can be viewed as the algebra of continuous sections of a bundle algebra $\mathfrak{A}$. For any $\mathcal{G}$-order $\mathcal{R}$, let $A \rtimes_r \mathcal{R}$ be the closure of the set of continuous sections $f : \mathcal{G} \to s^*\mathfrak{A}$ compactly supported in $\mathcal{R}$. Then $(A \rtimes_r \mathcal{R})_{\mathcal{R}\in\mathcal{E}_\mathcal{G}}$ provides $A \rtimes_r \mathcal{G}$ with a $\mathcal{G}$-filtered C^*-algebra structure.

Almost projections and almost unitaries

Let A, $(A_{\mathcal{R}})_{\mathcal{R}\in\mathcal{E}_{\mathcal{G}}}$ be a unital $\mathcal{G}$-filtered C^*-algebra. Let us fix $\mathcal{R} \in \mathcal{E}_{\mathcal{G}}$ (propagation) and $0 < \varepsilon < 1/4$ (control):

Almost projections and almost unitaries

Let A, $(A_{\mathcal{R}})_{\mathcal{R}\in\mathcal{E}_{\mathcal{G}}}$ be a unital $\mathcal{G}$-filtered C^*-algebra. Let us fix $\mathcal{R}\in\mathcal{E}_{\mathcal{G}}$ (propagation) and $0<\varepsilon<1/4$ (control):

- $p\in A$ is an ε-$\mathcal{R}$-projection if $p\in A_{\mathcal{R}}$, $p=p^*$ and $\|p^2-p\|<\varepsilon$.

Almost projections and almost unitaries

Let A, $(A_{\mathcal{R}})_{\mathcal{R}\in\mathcal{E}_{\mathcal{G}}}$ be a unital $\mathcal{G}$-filtered C^*-algebra. Let us fix $\mathcal{R} \in \mathcal{E}_{\mathcal{G}}$ (propagation) and $0 < \varepsilon < 1/4$ (control):

- $p \in A$ is an ε-$\mathcal{R}$-projection if $p \in A_{\mathcal{R}}$, $p = p^*$ and $\|p^2 - p\| < \varepsilon$.
- an ε-$\mathcal{R}$ projection p has a spectral gap around $1/2$ and hence gives rise by functional calculus to a projection $\kappa(p)$ s.t $\|p - \kappa(p)\| < 2\varepsilon$.

Almost projections and almost unitaries

Let A, $(A_\mathcal{R})_{\mathcal{R}\in\mathcal{E}_\mathcal{G}}$ be a unital $\mathcal{G}$-filtered C^*-algebra. Let us fix $\mathcal{R}\in\mathcal{E}_\mathcal{G}$ (propagation) and $0<\varepsilon<1/4$ (control):

- $p\in A$ is an ε-$\mathcal{R}$-projection if $p\in A_\mathcal{R}$, $p=p^*$ and $\|p^2-p\|<\varepsilon$.
- an ε-$\mathcal{R}$ projection p has a spectral gap around $1/2$ and hence gives rise by functional calculus to a projection $\kappa(p)$ s.t $\|p-\kappa(p)\|<2\varepsilon$.
- $u\in A$ is an ε-$\mathcal{R}$-unitary if $u\in A_\mathcal{R}$, $\|u^*\cdot u-1\|<\varepsilon$ and $\|u\cdot u^*-1\|<\varepsilon$.

Almost projections and almost unitaries

Let A, $(A_{\mathcal{R}})_{\mathcal{R}\in\mathcal{E}_{\mathcal{G}}}$ be a unital $\mathcal{G}$-filtered C^*-algebra. Let us fix $\mathcal{R} \in \mathcal{E}_{\mathcal{G}}$ (propagation) and $0 < \varepsilon < 1/4$ (control):

- $p \in A$ is an ε-$\mathcal{R}$-projection if $p \in A_{\mathcal{R}}$, $p = p^*$ and $\|p^2 - p\| < \varepsilon$.
- an ε-$\mathcal{R}$ projection p has a spectral gap around $1/2$ and hence gives rise by functional calculus to a projection $\kappa(p)$ s.t $\|p - \kappa(p)\| < 2\varepsilon$.
- $u \in A$ is an ε-$\mathcal{R}$-unitary if $u \in A_{\mathcal{R}}$, $\|u^* \cdot u - 1\| < \varepsilon$ and $\|u \cdot u^* - 1\| < \varepsilon$. (in particular, ε-$\mathcal{R}$-unitaries are invertible).

Almost projections and almost unitaries

Let A, $(A_{\mathcal{R}})_{\mathcal{R}\in\mathcal{E}_{\mathcal{G}}}$ be a unital $\mathcal{G}$-filtered C^*-algebra. Let us fix $\mathcal{R} \in \mathcal{E}_{\mathcal{G}}$ (propagation) and $0 < \varepsilon < 1/4$ (control):

- $p \in A$ is an ε-$\mathcal{R}$-projection if $p \in A_{\mathcal{R}}$, $p = p^*$ and $\|p^2 - p\| < \varepsilon$.
- an ε-$\mathcal{R}$ projection p has a spectral gap around $1/2$ and hence gives rise by functional calculus to a projection $\kappa(p)$ s.t $\|p - \kappa(p)\| < 2\varepsilon$.
- $u \in A$ is an ε-$\mathcal{R}$-unitary if $u \in A_{\mathcal{R}}$, $\|u^* \cdot u - 1\| < \varepsilon$ and $\|u \cdot u^* - 1\| < \varepsilon$. (in particular, ε-$\mathcal{R}$-unitaries are invertible).

Remark

- if q and q' are ε-$\mathcal{R}$-projections of A, then $\mathrm{diag}(q, q')$ and $\mathrm{diag}(q', q)$ are homotopic ε-$\mathcal{R}$-projections in $M_2(A)$;

Almost projections and almost unitaries

Let A, $(A_{\mathcal{R}})_{\mathcal{R}\in\mathcal{E}_{\mathcal{G}}}$ be a unital $\mathcal{G}$-filtered C^*-algebra. Let us fix $\mathcal{R} \in \mathcal{E}_{\mathcal{G}}$ (propagation) and $0 < \varepsilon < 1/4$ (control):

- $p \in A$ is an ε-$\mathcal{R}$-projection if $p \in A_{\mathcal{R}}$, $p = p^*$ and $\|p^2 - p\| < \varepsilon$.
- an ε-$\mathcal{R}$ projection p has a spectral gap around $1/2$ and hence gives rise by functional calculus to a projection $\kappa(p)$ s.t $\|p - \kappa(p)\| < 2\varepsilon$.
- $u \in A$ is an ε-$\mathcal{R}$-unitary if $u \in A_{\mathcal{R}}$, $\|u^* \cdot u - 1\| < \varepsilon$ and $\|u \cdot u^* - 1\| < \varepsilon$. (in particular, ε-$\mathcal{R}$-unitaries are invertible).

Remark

- if q and q' are ε-$\mathcal{R}$-projections of A, then $\mathrm{diag}(q, q')$ and $\mathrm{diag}(q', q)$ are homotopic ε-$\mathcal{R}$-projections in $M_2(A)$;
- if u and v are ε-$\mathcal{R}$-unitaries in A, then $\mathrm{diag}(u, v)$, $\mathrm{diag}(v, u)$ and $\mathrm{diag}(uv, 1)$ are homotopic as **3ε-$2\mathcal{R}$-unitaries** in $M_2(A)$;

Almost projections and almost unitaries

Let A, $(A_\mathcal{R})_{\mathcal{R}\in\mathcal{E}_\mathcal{G}}$ be a unital $\mathcal{G}$-filtered C^*-algebra. Let us fix $\mathcal{R}\in\mathcal{E}_\mathcal{G}$ (propagation) and $0<\varepsilon<1/4$ (control):

- $p\in A$ is an ε-$\mathcal{R}$-projection if $p\in A_\mathcal{R}$, $p=p^*$ and $\|p^2-p\|<\varepsilon$.
- an ε-$\mathcal{R}$ projection p has a spectral gap around $1/2$ and hence gives rise by functional calculus to a projection $\kappa(p)$ s.t $\|p-\kappa(p)\|<2\varepsilon$.
- $u\in A$ is an ε-$\mathcal{R}$-unitary if $u\in A_\mathcal{R}$, $\|u^*\cdot u-1\|<\varepsilon$ and $\|u\cdot u^*-1\|<\varepsilon$. (in particular, ε-$\mathcal{R}$-unitaries are invertible).

Remark

- if q and q' are ε-$\mathcal{R}$-projections of A, then $\mathrm{diag}(q,q')$ and $\mathrm{diag}(q',q)$ are homotopic ε-$\mathcal{R}$-projections in $M_2(A)$;
- if u and v are ε-$\mathcal{R}$-unitaries in A, then $\mathrm{diag}(u,v)$, $\mathrm{diag}(v,u)$ and $\mathrm{diag}(uv,1)$ are homotopic as **3ε-$2\mathcal{R}$-unitaries** in $M_2(A)$;
- If u is an ε-$\mathcal{R}$-unitary in A, then $\mathrm{diag}(u,u^*)$ and I_2 are homotopic as 3ε-$2\mathcal{R}$-unitaries in $M_2(A)$.

Notations

- $\mathrm{P}^{\varepsilon,\mathcal{R}}(A)$ is the set of ε-$\mathcal{R}$-projections of A.

Notations

- $\mathrm{P}^{\varepsilon,\mathcal{R}}(A)$ is the set of ε-$\mathcal{R}$-projections of A.
- $\mathrm{U}^{\varepsilon,\mathcal{R}}(A)$ is the set of ε-$\mathcal{R}$-unitaries of A.

Notations

- $P^{\varepsilon,\mathcal{R}}(A)$ is the set of ε-$\mathcal{R}$-projections of A.
- $U^{\varepsilon,\mathcal{R}}(A)$ is the set of ε-$\mathcal{R}$-unitaries of A.
- $P_\infty^{\varepsilon,\mathcal{R}}(A) = \bigcup_{n\in\mathbb{N}} P^{\varepsilon,\mathcal{R}}(M_n(A))$ for $P^{\varepsilon,\mathcal{R}}(M_n(A)) \hookrightarrow P^{\varepsilon,\mathcal{R}}(M_{n+1}(A));\ x \mapsto \operatorname{diag}(x,0)$.

Notations

- $P^{\varepsilon,\mathcal{R}}(A)$ is the set of ε-$\mathcal{R}$-projections of A.
- $U^{\varepsilon,\mathcal{R}}(A)$ is the set of ε-$\mathcal{R}$-unitaries of A.
- $P_\infty^{\varepsilon,\mathcal{R}}(A) = \bigcup_{n\in\mathbb{N}} P^{\varepsilon,\mathcal{R}}(M_n(A))$ for $P^{\varepsilon,\mathcal{R}}(M_n(A)) \hookrightarrow P^{\varepsilon,\mathcal{R}}(M_{n+1}(A));\ x \mapsto \operatorname{diag}(x,0)$.
- $U_\infty^{\varepsilon,\mathcal{R}}(A) = \bigcup_{n\in\mathbb{N}} U^{\varepsilon,\mathcal{R}}(M_n(A))$ for $U^{\varepsilon,\mathcal{R}}(M_n(A)) \hookrightarrow U^{\varepsilon,\mathcal{R}}(M_{n+1}(A));\ x \mapsto \operatorname{diag}(x,1)$.

Quantitative K-groups

Define for a unital $\mathcal{G}$-filtred C^*-algebra A, $\mathcal{R} \in \mathcal{E}_\mathcal{G}$ and $0 < \varepsilon < 1/4$ the (stably)-homotopy equivalence relations on $P_\infty^{\varepsilon,\mathcal{R}}(A) \times \mathbb{N}$ and $U_\infty^{\varepsilon,\mathcal{R}}(A)$ (with $P_\infty^{\varepsilon,\mathcal{R}}(A) = \bigcup_{n\in\mathbb{N}} P^{\varepsilon,\mathcal{R}}(M_n(A))$ and $U_\infty^{\varepsilon,\mathcal{R}}(A) = \bigcup_{n\in\mathbb{N}} U^{\varepsilon,\mathcal{R}}(M_n(A))$)

Quantitative K-groups

Define for a unital $\mathcal{G}$-filtred C^*-algebra A, $\mathcal{R} \in \mathcal{E}_{\mathcal{G}}$ and $0 < \varepsilon < 1/4$ the (stably)-homotopy equivalence relations on $\mathrm{P}_\infty^{\varepsilon,\mathcal{R}}(A) \times \mathbb{N}$ and $\mathrm{U}_\infty^{\varepsilon,\mathcal{R}}(A)$ (with $\mathrm{P}_\infty^{\varepsilon,\mathcal{R}}(A) = \bigcup_{n\in\mathbb{N}} \mathrm{P}^{\varepsilon,\mathcal{R}}(M_n(A))$ and $\mathrm{U}_\infty^{\varepsilon,\mathcal{R}}(A) = \bigcup_{n\in\mathbb{N}} \mathrm{U}^{\varepsilon,\mathcal{R}}(M_n(A))$):

- $(p, l) \sim (q, l')$ if there exists $k \in \mathbb{N}$ such that $\mathrm{diag}(p, I_{k+l'})$ and $\mathrm{diag}(q, I_{k+l})$ are homotopic as ε-$\mathcal{R}$-projections.

Quantitative K-groups

Define for a unital $\mathcal{G}$-filtred C^*-algebra A, $\mathcal{R} \in \mathcal{E}_{\mathcal{G}}$ and $0 < \varepsilon < 1/4$ the (stably)-homotopy equivalence relations on $\mathrm{P}_\infty^{\varepsilon,\mathcal{R}}(A) \times \mathbb{N}$ and $\mathrm{U}_\infty^{\varepsilon,\mathcal{R}}(A)$ (with $\mathrm{P}_\infty^{\varepsilon,\mathcal{R}}(A) = \bigcup_{n\in\mathbb{N}} \mathrm{P}^{\varepsilon,\mathcal{R}}(M_n(A))$ and $\mathrm{U}_\infty^{\varepsilon,\mathcal{R}}(A) = \bigcup_{n\in\mathbb{N}} \mathrm{U}^{\varepsilon,\mathcal{R}}(M_n(A))$):

- $(p,l) \sim (q,l')$ if there exists $k \in \mathbb{N}$ such that $\mathrm{diag}(p, I_{k+l'})$ and $\mathrm{diag}(q, I_{k+l})$ are homotopic as ε-$\mathcal{R}$-projections.
- $u \sim v$ if u and v are homotopic as 3ε-$2\mathcal{R}$-unitaries.

Quantitative K-groups

Define for a unital $\mathcal{G}$-filtred C^*-algebra A, $\mathcal{R} \in \mathcal{E}_{\mathcal{G}}$ and $0 < \varepsilon < 1/4$ the (stably)-homotopy equivalence relations on $\mathrm{P}_{\infty}^{\varepsilon,\mathcal{R}}(A) \times \mathbb{N}$ and $\mathrm{U}_{\infty}^{\varepsilon,\mathcal{R}}(A)$ (with $\mathrm{P}_{\infty}^{\varepsilon,\mathcal{R}}(A) = \bigcup_{n\in\mathbb{N}} \mathrm{P}^{\varepsilon,\mathcal{R}}(M_n(A))$ and $\mathrm{U}_{\infty}^{\varepsilon,\mathcal{R}}(A) = \bigcup_{n\in\mathbb{N}} \mathrm{U}^{\varepsilon,\mathcal{R}}(M_n(A))$):

- $(p,l) \sim (q,l')$ if there exists $k \in \mathbb{N}$ such that $\operatorname{diag}(p, I_{k+l'})$ and $\operatorname{diag}(q, I_{k+l})$ are homotopic as ε-$\mathcal{R}$-projections.
- $u \sim v$ if u and v are homotopic as 3ε-$2\mathcal{R}$-unitaries.

Definition

1. $K_0^{\varepsilon,\mathcal{R}}(A) = \mathrm{P}^{\varepsilon,\mathcal{R}}(A)/\sim$ *and* $[p,l]_{\varepsilon,\mathcal{R}}$ *is the class of* (p,l) *mod.* $\sim$;
2. $K_1^{\varepsilon,\mathcal{R}}(A) = \mathrm{U}^{\varepsilon,r}(A)/\sim$ *and* $[u]_{\varepsilon,\mathcal{R}}$ *is the class of* u *mod.* $\sim$.

Quantitative K-groups

Define for a unital $\mathcal{G}$-filtred C^*-algebra A, $\mathcal{R} \in \mathcal{E}_{\mathcal{G}}$ and $0 < \varepsilon < 1/4$ the (stably)-homotopy equivalence relations on $\mathrm{P}_\infty^{\varepsilon,\mathcal{R}}(A) \times \mathbb{N}$ and $\mathrm{U}_\infty^{\varepsilon,\mathcal{R}}(A)$ (with $\mathrm{P}_\infty^{\varepsilon,\mathcal{R}}(A) = \bigcup_{n\in\mathbb{N}} \mathrm{P}^{\varepsilon,\mathcal{R}}(M_n(A))$ and $\mathrm{U}_\infty^{\varepsilon,\mathcal{R}}(A) = \bigcup_{n\in\mathbb{N}} \mathrm{U}^{\varepsilon,\mathcal{R}}(M_n(A))$):

- $(p,l) \sim (q,l')$ if there exists $k \in \mathbb{N}$ such that $\mathrm{diag}(p, I_{k+l'})$ and $\mathrm{diag}(q, I_{k+l})$ are homotopic as ε-$\mathcal{R}$-projections.
- $u \sim v$ if u and v are homotopic as 3ε-$2\mathcal{R}$-unitaries.

Definition

1. $K_0^{\varepsilon,\mathcal{R}}(A) = \mathrm{P}^{\varepsilon,\mathcal{R}}(A)/\sim$ *and* $[p,l]_{\varepsilon,\mathcal{R}}$ *is the class of* (p,l) *mod.* $\sim$;
2. $K_1^{\varepsilon,\mathcal{R}}(A) = \mathrm{U}^{\varepsilon,r}(A)/\sim$ *and* $[u]_{\varepsilon,\mathcal{R}}$ *is the class of* u *mod.* $\sim$.

- $K_0^{\varepsilon,\mathcal{R}}(A)$ is an abelian group for $[p,l]_{\varepsilon,\mathcal{R}} + [p',l']_{\varepsilon,\mathcal{R}} = [\mathrm{diag}(p,p'), l+l']_{\varepsilon,\mathcal{R}}$;

Quantitative K-groups

Define for a unital $\mathcal{G}$-filtred C^*-algebra A, $\mathcal{R} \in \mathcal{E}_\mathcal{G}$ and $0 < \varepsilon < 1/4$ the (stably)-homotopy equivalence relations on $\mathrm{P}_\infty^{\varepsilon,\mathcal{R}}(A) \times \mathbb{N}$ and $\mathrm{U}_\infty^{\varepsilon,\mathcal{R}}(A)$ (with $\mathrm{P}_\infty^{\varepsilon,\mathcal{R}}(A) = \bigcup_{n\in\mathbb{N}} \mathrm{P}^{\varepsilon,\mathcal{R}}(M_n(A))$ and $\mathrm{U}_\infty^{\varepsilon,\mathcal{R}}(A) = \bigcup_{n\in\mathbb{N}} \mathrm{U}^{\varepsilon,\mathcal{R}}(M_n(A))$):

- $(p, l) \sim (q, l')$ if there exists $k \in \mathbb{N}$ such that $\mathrm{diag}(p, I_{k+l'})$ and $\mathrm{diag}(q, I_{k+l})$ are homotopic as ε-$\mathcal{R}$-projections.
- $u \sim v$ if u and v are homotopic as 3ε-$2\mathcal{R}$-unitaries.

Definition

1. $K_0^{\varepsilon,\mathcal{R}}(A) = \mathrm{P}^{\varepsilon,\mathcal{R}}(A)/\sim$ *and* $[p, l]_{\varepsilon,\mathcal{R}}$ *is the class of* (p, l) *mod.* $\sim$;
2. $K_1^{\varepsilon,\mathcal{R}}(A) = \mathrm{U}^{\varepsilon,r}(A)/\sim$ *and* $[u]_{\varepsilon,\mathcal{R}}$ *is the class of* u *mod.* $\sim$.

- $K_0^{\varepsilon,\mathcal{R}}(A)$ is an abelian group for $[p, l]_{\varepsilon,\mathcal{R}} + [p', l']_{\varepsilon,\mathcal{R}} = [\mathrm{diag}(p, p'), l + l']_{\varepsilon,\mathcal{R}}$;
- $K_1^{\varepsilon,\mathcal{R}}(A)$ is an abelian group for $[u]_{\varepsilon,\mathcal{R}} + [v]_{\varepsilon,\mathcal{R}} = [\mathrm{diag}(u, v)]_{\varepsilon,\mathcal{R}}$.

Quantitative K-groups

Define for a unital $\mathcal{G}$-filtred C^*-algebra A, $\mathcal{R} \in \mathcal{E}_{\mathcal{G}}$ and $0 < \varepsilon < 1/4$ the (stably)-homotopy equivalence relations on $\mathrm{P}_\infty^{\varepsilon,\mathcal{R}}(A) \times \mathbb{N}$ and $\mathrm{U}_\infty^{\varepsilon,\mathcal{R}}(A)$ (with $\mathrm{P}_\infty^{\varepsilon,\mathcal{R}}(A) = \bigcup_{n\in\mathbb{N}} \mathrm{P}^{\varepsilon,\mathcal{R}}(M_n(A))$ and $\mathrm{U}_\infty^{\varepsilon,\mathcal{R}}(A) = \bigcup_{n\in\mathbb{N}} \mathrm{U}^{\varepsilon,\mathcal{R}}(M_n(A))$):

- $(p,l) \sim (q,l')$ if there exists $k \in \mathbb{N}$ such that $\mathrm{diag}(p, I_{k+l'})$ and $\mathrm{diag}(q, I_{k+l})$ are homotopic as ε-$\mathcal{R}$-projections.
- $u \sim v$ if u and v are homotopic as 3ε-$2\mathcal{R}$-unitaries.

Definition

1. $K_0^{\varepsilon,\mathcal{R}}(A) = \mathrm{P}^{\varepsilon,\mathcal{R}}(A)/\sim$ *and* $[p,l]_{\varepsilon,\mathcal{R}}$ *is the class of* (p,l) *mod.* $\sim$;
2. $K_1^{\varepsilon,\mathcal{R}}(A) = \mathrm{U}^{\varepsilon,r}(A)/\sim$ *and* $[u]_{\varepsilon,\mathcal{R}}$ *is the class of* u *mod.* $\sim$.

- $K_0^{\varepsilon,\mathcal{R}}(A)$ is an abelian group for $[p,l]_{\varepsilon,\mathcal{R}} + [p',l']_{\varepsilon,\mathcal{R}} = [\mathrm{diag}(p,p'), l+l']_{\varepsilon,\mathcal{R}}$;
- $K_1^{\varepsilon,\mathcal{R}}(A)$ is an abelian group for $[u]_{\varepsilon,\mathcal{R}} + [v]_{\varepsilon,\mathcal{R}} = [\mathrm{diag}(u,v)]_{\varepsilon,\mathcal{R}}$.
- if A is not unital, we use its unitarization to define $K_0^{\varepsilon,\mathcal{R}}$ and $K_1^{\varepsilon,\mathcal{R}}$.

Approximation of K-theory

For any $\mathcal{G}$-filtered C^*-algebra A, any $0 < \varepsilon < 1/4$ and $\mathcal{R} \in \mathcal{E}_{\mathcal{G}}$, we have natural homomorphisms

- $K_0^{\varepsilon,\mathcal{R}}(A) \longrightarrow K_0(A)$; $[p, l]_{\varepsilon,\mathcal{R}} \mapsto [\kappa(p)] - [I_l]$; (with $\kappa(p)$ the spectral projection)
- $K_1^{\varepsilon,r}(A) \longrightarrow K_1(A)$; $[u]_{\varepsilon,\mathcal{R}} \mapsto [u]$; ($\varepsilon$-$\mathcal{R}$-unitaries are invertible);

Approximation of K-theory

For any $\mathcal{G}$-filtered C^*-algebra A, any $0 < \varepsilon < 1/4$ and $\mathcal{R} \in \mathcal{E}_{\mathcal{G}}$, we have natural homomorphisms

- $K_0^{\varepsilon,\mathcal{R}}(A) \longrightarrow K_0(A)$; $[p, l]_{\varepsilon,\mathcal{R}} \mapsto [\kappa(p)] - [I_l]$; (with $\kappa(p)$ the spectral projection)
- $K_1^{\varepsilon,r}(A) \longrightarrow K_1(A)$; $[u]_{\varepsilon,\mathcal{R}} \mapsto [u]$; ($\varepsilon$-$\mathcal{R}$-unitaries are invertible);

For any $\varepsilon \in (0, 1/4)$ and any projection p in A, there exists a $\mathcal{G}$-order $\mathcal{R}$ and q an ε-$\mathcal{R}$-projection of A such that $\kappa(q)$ and p are closed and hence homotopic projections.

Approximation of K-theory

For any $\mathcal{G}$-filtered C^*-algebra A, any $0 < \varepsilon < 1/4$ and $\mathcal{R} \in \mathcal{E}_{\mathcal{G}}$, we have natural homomorphisms

- $K_0^{\varepsilon,\mathcal{R}}(A) \longrightarrow K_0(A)$; $[p, l]_{\varepsilon,\mathcal{R}} \mapsto [\kappa(p)] - [I_l]$; (with $\kappa(p)$ the spectral projection)
- $K_1^{\varepsilon,r}(A) \longrightarrow K_1(A)$; $[u]_{\varepsilon,\mathcal{R}} \mapsto [u]$; ($\varepsilon$-$\mathcal{R}$-unitaries are invertible);

For any $\varepsilon \in (0, 1/4)$ and any projection p in A, there exists a $\mathcal{G}$-order $\mathcal{R}$ and q an ε-$\mathcal{R}$-projection of A such that $\kappa(q)$ and p are closed and hence homotopic projections. We have a similar result for unitaries

Approximation of K-theory

For any $\mathcal{G}$-filtered C^*-algebra A, any $0 < \varepsilon < 1/4$ and $\mathcal{R} \in \mathcal{E}_{\mathcal{G}}$, we have natural homomorphisms

- $K_0^{\varepsilon,\mathcal{R}}(A) \longrightarrow K_0(A)$; $[p,l]_{\varepsilon,\mathcal{R}} \mapsto [\kappa(p)] - [I_l]$; (with $\kappa(p)$ the spectral projection)
- $K_1^{\varepsilon,r}(A) \longrightarrow K_1(A)$; $[u]_{\varepsilon,\mathcal{R}} \mapsto [u]$; ($\varepsilon$-$\mathcal{R}$-unitaries are invertible);

For any $\varepsilon \in (0, 1/4)$ and any projection p in A, there exists a $\mathcal{G}$-order $\mathcal{R}$ and q an ε-$\mathcal{R}$-projection of A such that $\kappa(q)$ and p are closed and hence homotopic projections. We have a similar result for unitaries

Consequence

If we fix ε in $(0, 1/4)$, then

$$K_*(A) = \lim_{\mathcal{R} \in \mathcal{E}_{\mathcal{G}}} K_*^{\varepsilon,\mathcal{R}}(A).$$

The controlled Mayer-Vietoris exact sequence associated to a $\mathcal{R}$-decomposition (O-Yu/Dell'Aiera)

There exists a integer valued non increasing function $\varepsilon \mapsto n_\varepsilon$ such that the following holds :

The controlled Mayer-Vietoris exact sequence associated to a $\mathcal{R}$-decomposition (O-Yu/Dell'Aiera)

There exists a integer valued non increasing function $\varepsilon \mapsto n_\varepsilon$ such that the following holds :

Let $\mathcal{G}$ be a locally compact groupoid and let A be a $\mathcal{G}$-algebra. Let $\mathcal{R}'$ be a $\mathcal{G}$-order and let $(V_1, V_2, \mathcal{G}_1, \mathcal{G}_2)$ be a $\mathcal{R}'$-decomposition.

The controlled Mayer-Vietoris exact sequence associated to a $\mathcal{R}$-decomposition (O-Yu/Dell'Aiera)

There exists a integer valued non increasing function $\varepsilon \mapsto n_\varepsilon$ such that the following holds :

Let $\mathcal{G}$ be a locally compact groupoid and let A be a $\mathcal{G}$-algebra. Let $\mathcal{R}'$ be a $\mathcal{G}$-order and let $(V_1, V_2, \mathcal{G}_1, \mathcal{G}_2)$ be a $\mathcal{R}'$-decomposition. For any $\varepsilon \in (0, 1/4)$, the sequence

$$K_*^{\varepsilon,\mathcal{R}}(A \rtimes_r (\mathcal{G}_1 \cap \mathcal{G}_2)) \longrightarrow K_*^{\varepsilon,\mathcal{R}}(A \rtimes_r \mathcal{G}_1) \oplus K_*^{\varepsilon,\mathcal{R}}(A \rtimes_r \mathcal{G}_2) \longrightarrow K_*^{\varepsilon,\mathcal{R}}(A \rtimes_r \mathcal{G})$$

is exact up to rescaling in the middle for any $\mathcal{G}$-order $\mathcal{R}$ with $\mathcal{R}^{*n_\varepsilon} \subseteq \mathcal{R}'$, up to rescaling,

The controlled Mayer-Vietoris exact sequence associated to a $\mathcal{R}$-decomposition (O-Yu/Dell'Aiera)

There exists a integer valued non increasing function $\varepsilon \mapsto n_\varepsilon$ such that the following holds :

Let $\mathcal{G}$ be a locally compact groupoid and let A be a $\mathcal{G}$-algebra. Let $\mathcal{R}'$ be a $\mathcal{G}$-order and let $(V_1, V_2, \mathcal{G}_1, \mathcal{G}_2)$ be a $\mathcal{R}'$-decomposition. For any $\varepsilon \in (0, 1/4)$, the sequence

$$K_*^{\varepsilon,\mathcal{R}}(A \rtimes_r (\mathcal{G}_1 \cap \mathcal{G}_2)) \longrightarrow K_*^{\varepsilon,\mathcal{R}}(A \rtimes_r \mathcal{G}_1) \oplus K_*^{\varepsilon,\mathcal{R}}(A \rtimes_r \mathcal{G}_2) \longrightarrow K_*^{\varepsilon,\mathcal{R}}(A \rtimes_r \mathcal{G})$$

is exact up to rescaling in the middle for any $\mathcal{G}$-order $\mathcal{R}$ with $\mathcal{R}^{*n_\varepsilon} \subseteq \mathcal{R}'$, up to rescaling, i.e kernel elements at order $\mathcal{R}$ are in the image at order $\mathcal{R}^{*l_\varepsilon}$ for some (universal) integer valued non increasing function $\varepsilon \mapsto l_\varepsilon$ with $l_\varepsilon \ll n_\varepsilon$.

The controled Mayer-Vietoris boundary

Let $\mathcal{G}$ be a locally compact groupoid and let A be a $\mathcal{G}$-algebra. Let $\mathcal{R}'$ be a $\mathcal{G}$-order and let $(V_1, V_2, \mathcal{G}_1, \mathcal{G}_2)$ be a $\mathcal{R}'$-decomposition.

The controled Mayer-Vietoris boundary

Let $\mathcal{G}$ be a locally compact groupoid and let A be a $\mathcal{G}$-algebra. Let $\mathcal{R}'$ be a $\mathcal{G}$-order and let $(V_1, V_2, \mathcal{G}_1, \mathcal{G}_2)$ be a $\mathcal{R}'$-decomposition. For some (universal) integer valued non increasing function $\varepsilon \mapsto m_\varepsilon$, there exists for any $\mathcal{G}$-order $\mathcal{R}$ with $\mathcal{R}^{*n_\varepsilon m_\varepsilon} \subseteq \mathcal{R}'$ a morphisms

$$\partial^{\varepsilon,\mathcal{R}} : K_*^{\varepsilon,\mathcal{R}}(A \rtimes_r \mathcal{G}) \longrightarrow K_{*+1}^{\varepsilon,\mathcal{R}^{*m_\varepsilon}}(A \rtimes_r (\mathcal{G}_1 \cap \mathcal{G}_2))$$

such that the sequence

$$K_*^{\varepsilon,\mathcal{R}}(A \rtimes_r \mathcal{G}_1) \oplus K_*^{\varepsilon,\mathcal{R}}(A \rtimes_r \mathcal{G}_2) \longrightarrow K_*^{\varepsilon,\mathcal{R}}(A \rtimes_r \mathcal{G}) \stackrel{\partial^{\varepsilon,\mathcal{R}}}{\longrightarrow}$$
$$K_{*+1}^{\varepsilon,\mathcal{R}^{*m_\varepsilon}}(A \rtimes_r (\mathcal{G}_1 \cap \mathcal{G}_2)) \longrightarrow K_{*+1}^{\varepsilon,\mathcal{R}^{*m_\varepsilon}}(A \rtimes_r \mathcal{G}_1) \oplus K_{*+1}^{\varepsilon,\mathcal{R}^{*m_\varepsilon}}(A \rtimes_r \mathcal{G}_2)$$

is exact up to rescaling.

The controled Mayer-Vietoris boundary

Let $\mathcal{G}$ be a locally compact groupoid and let A be a $\mathcal{G}$-algebra. Let $\mathcal{R}'$ be a $\mathcal{G}$-order and let $(V_1, V_2, \mathcal{G}_1, \mathcal{G}_2)$ be a $\mathcal{R}'$-decomposition. For some (universal) integer valued non increasing function $\varepsilon \mapsto m_\varepsilon$, there exists for any $\mathcal{G}$-order $\mathcal{R}$ with $\mathcal{R}^{*n_\varepsilon m_\varepsilon} \subseteq \mathcal{R}'$ a morphisms

$$\partial^{\varepsilon,\mathcal{R}} : K_*^{\varepsilon,\mathcal{R}}(A \rtimes_r \mathcal{G}) \longrightarrow K_{*+1}^{\varepsilon,\mathcal{R}^{*m_\varepsilon}}(A \rtimes_r (\mathcal{G}_1 \cap \mathcal{G}_2))$$

such that the sequence

$$K_*^{\varepsilon,\mathcal{R}}(A \rtimes_r \mathcal{G}_1) \oplus K_*^{\varepsilon,\mathcal{R}}(A \rtimes_r \mathcal{G}_2) \longrightarrow K_*^{\varepsilon,\mathcal{R}}(A \rtimes_r \mathcal{G}) \stackrel{\partial^{\varepsilon,\mathcal{R}}}{\longrightarrow}$$
$$K_{*+1}^{\varepsilon,\mathcal{R}^{*m_\varepsilon}}(A \rtimes_r (\mathcal{G}_1 \cap \mathcal{G}_2)) \longrightarrow K_{*+1}^{\varepsilon,\mathcal{R}^{*m_\varepsilon}}(A \rtimes_r \mathcal{G}_1) \oplus K_{*+1}^{\varepsilon,\mathcal{R}^{*m_\varepsilon}}(A \rtimes_r \mathcal{G}_2)$$

is exact up to rescaling. Moreover, if the $\mathcal{R}'$-decomposition is coercive, the exactness is persistent at any order.

Application to the Künneth formula in K-theory

Definition

Let A be a C^*-algebra. We say that A is of class $\mathcal{N}$ if for every C^*-algebra B with free abelian K-theory, then the K-theory external product

$$\alpha : K_*(A) \otimes K_*(B) \longrightarrow K_*(A \otimes B)$$

is an isomorphism.

Application to the Künneth formula in K-theory

Definition

Let A be a C^*-algebra. We say that A is of class $\mathcal{N}$ if for every C^*-algebra B with free abelian K-theory, then the K-theory external product

$$\alpha : K_*(A) \otimes K_*(B) \longrightarrow K_*(A \otimes B)$$

is an isomorphism.

Theorem (Schochet)

If A is a C^*-algebra of class $\mathcal{N}$, then the Künneth formula holds for any C^*-algebras B :

Application to the Künneth formula in K-theory

Definition

Let A be a C^*-algebra. We say that A is of class $\mathcal{N}$ if for every C^*-algebra B with free abelian K-theory, then the K-theory external product

$$\alpha : K_*(A) \otimes K_*(B) \longrightarrow K_*(A \otimes B)$$

is an isomorphism.

Theorem (Schochet)

If A is a C^*-algebra of class $\mathcal{N}$, then the Künneth formula holds for any C^*-algebras B : there exist a natural exact sequence

$$0 \longrightarrow K_*(A) \otimes K_*(B) \stackrel{\alpha}{\longrightarrow} K_*(A \otimes B) \stackrel{\beta}{\longrightarrow} \mathrm{Tor}(K_*(A), K_*(B)) \longrightarrow 0.$$

Quantitative Künneth formula

- the Künneth formula admits a quantitative version;

Quantitative Künneth formula

- the Künneth formula admits a quantitative version;
- the quantitative Künneth formula implies the genuine one.

Quantitative Künneth formula

- the Künneth formula admits a quantitative version;
- the quantitative Künneth formula implies the genuine one.
- many examples are provided by crossed products algebra by groupoids satisfying the Baum-Connes conjecture with coefficients (Bönicke-Dell'Aiera)

Quantitative Künneth formula

- the Künneth formula admits a quantitative version;
- the quantitative Künneth formula implies the genuine one.
- many examples are provided by crossed products algebra by groupoids satisfying the Baum-Connes conjecture with coefficients (Bönicke-Dell'Aiera)

Theorem

Let $\mathcal{G}$ be a locally compact groupoid and let A be $\mathcal{G}$-algebra. Assume that for any $\mathcal{G}$-order $\mathcal{R}$, there exists a coercice $\mathcal{R}$-decomposition $(V_1, V_2, \mathcal{G}_1, \mathcal{G}_2)$ such that $A \rtimes_r \mathcal{G}_1$, $A \rtimes_r \mathcal{G}_2$ and $A \rtimes_r (\mathcal{G}_1 \cap \mathcal{G}_2)$ satisfy the quantitative Künneth formula. Then $A \rtimes_r \mathcal{G}$ satisfies the quantitative Künneth formula.

Theorem

Let $\mathcal{G}$ be a locally compact groupoid and let A be $\mathcal{G}$-algebra. Assume that for any $\mathcal{G}$-order $\mathcal{R}$, there exists a coercice $\mathcal{R}$-decomposition $(V_1, V_2, \mathcal{G}_1, \mathcal{G}_2)$ such that $A \rtimes_r \mathcal{G}_1$, $A \rtimes_r \mathcal{G}_2$ and $A \rtimes_r (\mathcal{G}_1 \cap \mathcal{G}_2)$ satisfy the quantitative Künneth formula. Then $A \rtimes_r \mathcal{G}$ satisfies the quantitative Künneth formula.

Theorem

Let $\mathcal{G}$ be a locally compact groupoid and let A be $\mathcal{G}$-algebra. Assume that for any $\mathcal{G}$-order $\mathcal{R}$, there exists a coercice $\mathcal{R}$-decomposition $(V_1, V_2, \mathcal{G}_1, \mathcal{G}_2)$ such that $A \rtimes_r \mathcal{G}_1$, $A \rtimes_r \mathcal{G}_2$ and $A \rtimes_r (\mathcal{G}_1 \cap \mathcal{G}_2)$ satisfy the quantitative Künneth formula. Then $A \rtimes_r \mathcal{G}$ satisfies the quantitative Künneth formula.

Corollary

Let $\mathcal{G}$ be a locally compact groupoid and let A be $\mathcal{G}$-algebra.

Theorem

Let $\mathcal{G}$ be a locally compact groupoid and let A be $\mathcal{G}$-algebra. Assume that for any $\mathcal{G}$-order $\mathcal{R}$, there exists a coercice $\mathcal{R}$-decomposition $(V_1, V_2, \mathcal{G}_1, \mathcal{G}_2)$ such that $A \rtimes_r \mathcal{G}_1$, $A \rtimes_r \mathcal{G}_2$ and $A \rtimes_r (\mathcal{G}_1 \cap \mathcal{G}_2)$ satisfy the quantitative Künneth formula. Then $A \rtimes_r \mathcal{G}$ satisfies the quantitative Künneth formula.

Corollary

Let $\mathcal{G}$ be a locally compact groupoid and let A be $\mathcal{G}$-algebra. Assume that $\mathcal{G}$ has finite $\mathcal{D}$-complexity with respect to a set $\mathcal{D}$ of relatively clopen subgroupoids of $\mathcal{G}$ such that

Theorem

Let $\mathcal{G}$ be a locally compact groupoid and let A be $\mathcal{G}$-algebra. Assume that for any $\mathcal{G}$-order $\mathcal{R}$, there exists a coercice $\mathcal{R}$-decomposition $(V_1, V_2, \mathcal{G}_1, \mathcal{G}_2)$ such that $A \rtimes_r \mathcal{G}_1$, $A \rtimes_r \mathcal{G}_2$ and $A \rtimes_r (\mathcal{G}_1 \cap \mathcal{G}_2)$ satisfy the quantitative Künneth formula. Then $A \rtimes_r \mathcal{G}$ satisfies the quantitative Künneth formula.

Corollary

Let $\mathcal{G}$ be a locally compact groupoid and let A be $\mathcal{G}$-algebra. Assume that $\mathcal{G}$ has finite $\mathcal{D}$-complexity with respect to a set $\mathcal{D}$ of relatively clopen subgroupoids of $\mathcal{G}$ such that

- *$\mathcal{D}$ is stable under taking relatively clopen subgroupoids*

Theorem

Let $\mathcal{G}$ be a locally compact groupoid and let A be $\mathcal{G}$-algebra. Assume that for any $\mathcal{G}$-order $\mathcal{R}$, there exists a coercice $\mathcal{R}$-decomposition $(V_1, V_2, \mathcal{G}_1, \mathcal{G}_2)$ such that $A \rtimes_r \mathcal{G}_1$, $A \rtimes_r \mathcal{G}_2$ and $A \rtimes_r (\mathcal{G}_1 \cap \mathcal{G}_2)$ satisfy the quantitative Künneth formula. Then $A \rtimes_r \mathcal{G}$ satisfies the quantitative Künneth formula.

Corollary

Let $\mathcal{G}$ be a locally compact groupoid and let A be $\mathcal{G}$-algebra. Assume that $\mathcal{G}$ has finite $\mathcal{D}$-complexity with respect to a set $\mathcal{D}$ of relatively clopen subgroupoids of $\mathcal{G}$ such that

- *$\mathcal{D}$ is stable under taking relatively clopen subgroupoids*
- *$A \rtimes_r \mathcal{H}$ satisfies the quantitative Künneth formula for any $\mathcal{H}$ in $\mathcal{D}$.*

Theorem

Let $\mathcal{G}$ be a locally compact groupoid and let A be $\mathcal{G}$-algebra. Assume that for any $\mathcal{G}$-order $\mathcal{R}$, there exists a coercice $\mathcal{R}$-decomposition $(V_1, V_2, \mathcal{G}_1, \mathcal{G}_2)$ such that $A \rtimes_r \mathcal{G}_1$, $A \rtimes_r \mathcal{G}_2$ and $A \rtimes_r (\mathcal{G}_1 \cap \mathcal{G}_2)$ satisfy the quantitative Künneth formula. Then $A \rtimes_r \mathcal{G}$ satisfies the quantitative Künneth formula.

Corollary

Let $\mathcal{G}$ be a locally compact groupoid and let A be $\mathcal{G}$-algebra. Assume that $\mathcal{G}$ has finite $\mathcal{D}$-complexity with respect to a set $\mathcal{D}$ of relatively clopen subgroupoids of $\mathcal{G}$ such that

- *$\mathcal{D}$ is stable under taking relatively clopen subgroupoids*
- *$A \rtimes_r \mathcal{H}$ satisfies the quantitative Künneth formula for any $\mathcal{H}$ in $\mathcal{D}$.*

then $A \rtimes_r \mathcal{G}$ satisfies the quantitative Künneth formula.

Baum-Connes conjecture for group action with finite dynamic complexity (Guentner-Willet-Yu)

Definition

Let Γ be a finitely generated group acting on a second countable compact space X.

Baum-Connes conjecture for group action with finite dynamic complexity (Guentner-Willet-Yu)

Definition

Let Γ be a finitely generated group acting on a second countable compact space X. Then the action has finite dynamic complexity if the groupoid $X \rtimes \Gamma$ has finite $\mathcal{D}$-complexity with respect to the set $\mathcal{D}$ of open and relatively compact subgroupoids.

Baum-Connes conjecture for group action with finite dynamic complexity (Guentner-Willet-Yu)

Definition

Let Γ be a finitely generated group acting on a second countable compact space X. Then the action has finite dynamic complexity if the groupoid $X \rtimes \Gamma$ has finite $\mathcal{D}$-complexity with respect to the set $\mathcal{D}$ of open and relatively compact subgroupoids.

Remark

If the has finite dynamic complexity, then $X \rtimes \Gamma$ is amenable.

Baum-Connes conjecture for group action with finite dynamic complexity (Guentner-Willet-Yu)

Definition

Let Γ be a finitely generated group acting on a second countable compact space X. Then the action has finite dynamic complexity if the groupoid $X \rtimes \Gamma$ has finite $\mathcal{D}$-complexity with respect to the set $\mathcal{D}$ of open and relatively compact subgroupoids.

Remark

If the has finite dynamic complexity, then $X \rtimes \Gamma$ is amenable.

Theorem (Guentner-Willet-Yu)

If an action of a finitely generated group Γ on a compact space X has finite dynamic complexity, then $X \rtimes \Gamma$ satisfies the Baum-Connes conjecture.

THANK YOU FOR YOUR ATTENTION

MERCI JEAN!!!