Diffeological groupoids and their C^*-algebras

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Motivation for diffeology...

Steenrod, Chen, Souriau, etc: $\mathcal{O}, M, N$ smooth manifolds, $\dim < \infty$. Usual $C^\infty(M, N)$ (functions with continuous derivaties at all orders).

- $C^\infty(M, N)$ is an infinite dimensional manifold...
- Is there a topology making this map a homeomorphism?

  $$C^\infty(\mathcal{O} \times M, N) \to C^\infty(\mathcal{O}, C^\infty(M, N)), \quad f \mapsto \widetilde{f}(x)(m) := f(x, m)$$

- Quotients? (And limits, colimits?)

Diffeology 1: Souriau's definition

Let X be a set and $k \in \mathbb{N}$.

- **k-plots**: Collection $P^k(X)$ of $\chi : \mathcal{O}_\chi \to X$, $\mathcal{O}_\chi \subseteq \mathbb{R}^k$ open.
- Example: $\tilde{\chi} : \mathcal{O}_\chi \to C^\infty(M, N)$ such that $\chi : \mathcal{O}_\chi \times M \to N, \chi(x, m) = \tilde{\chi}(x)(m)$ is smooth.

Definition

$P(X) = \bigcup_{k \in \mathbb{N}} P^k(X)$ is a diffeology of X if:

- contains constant maps $x : \mathcal{O} \to X$;
- closed by "gluing" families of k-plots $\{\chi_i\}_{i \in I}$, for all $k \in \mathbb{N}$;
- For any open $\mathcal{O} \subset \mathbb{R}^n$, the composition of $\chi \in P^k(X)$ with some smooth[1] $h : \mathcal{O} \to \mathcal{O}_\chi$ is in $P^n(X)$.

[1] "smooth" here means h has continuous derivatives of all orders.

Diffeology 2: D-topology

Definition

1. D-topology: $W \subseteq X$ open iff $\chi^{-1}(W) \subseteq \mathcal{O}_\chi$ open for all $\chi \in P(X)$.
2. $f : X \to Y$ smooth if $f \circ \chi \in P(Y)$ for all $f \in P(X)$.

Examples

- M usual smooth manifold M (finite dimensional): D-topology is usual topology.
- Any subset X' of a diffeological space $(X, P(X))$.
- $C^\infty(\mathcal{O} \times M, N) \to C^\infty(\mathcal{O}, C^\infty(M, N))$ homeomorphism.

Diffeology 3: Constructions

Let $(X, P(X)), (Y, P(Y))$ diffeological spaces, $\{(X_i, P(X_i))\}_{i \in I}$ family of diffeological spaces, Z a set.

- Maps $f_i : X_i \to Z$: "final" diffeology on Z generated by $f_i \circ \chi$
- Disjoint union: $\iota_i : X_i \to \coprod_{i \in I} X_i$
- Quotients $X/\sim$ are diffeological spaces.

Diffeological groupoids

Definition

Groupoid $G \rightrightarrows M$, where G, M diffeological spaces and all structure maps are smooth.

Example: $(M, \mathcal{F})$ singular foliation, $\mathcal{U} = \{(U_i, t_i, s_i)\}$ atlas of bisubmersions by exponentiation ("path-holonomy"...)

$X = \coprod_{i \in I} U_i$: *path-holonomy diffeology* by inclusions $\iota_i : U_i \to X$

$H(\mathcal{F}) = X/\sim$ diffeological groupoid.

Proposition

$H(\mathcal{F})$ is "submersive": $q \circ \iota_i : U_i \to H(\mathcal{F})$ is an open map for the D-topology of $H(\mathcal{F})$.

Singular Subalgebroids (with M. Zambon)

$A \to M$ integrable Lie algebroid, anchor $\rho : A \to TM$.

Choose s-connected integrating Lie groupoid $\mathscr{G} \rightrightarrows M$.

Definition: Singular subalgebroid

$\mathcal{B}$ a $C^\infty(M)$-submodule of $\Gamma_c(A)$

- locally finitely generated.
- involutive

Correspondence with singular foliation on $\mathscr{G}$

$$\mathcal{B} \leftrightarrow \overrightarrow{\mathcal{B}}, \quad \boldsymbol{\alpha} \mapsto \overrightarrow{\boldsymbol{\alpha}} \in \mathfrak{X}(\mathscr{G}) \text{ vertical + left-invariant}$$

Associated singular foliation

$$\mathcal{B} \mapsto \mathcal{F}_{\mathcal{B}}, \quad \boldsymbol{\alpha} \mapsto \rho \circ \boldsymbol{\alpha} \in \mathfrak{X}(M)$$

Examples from Poisson geometry

- $B \to M$ Lie subalgebroid of A, $\mathcal{B} = \Gamma_c(B)$ (Moerdijk-Mrcun)
  e.g. (M, π) Poisson, $N \subseteq M$ coisotropic, $B = TN^0 \leqslant T^*_\pi M$.

- $\Psi : A' \to A$ Lie algebroid morphism, $\mathcal{B} = \Psi(\Gamma(A'))$
  e.g. (M^{2n}, ω) symplectic, $J^* : \mathfrak{g} \to C^\infty(M)$ co-moment map.
  $$\mathfrak{g} \to \Gamma(TM \oplus_\omega (M \times \mathbb{R})), \quad v \mapsto (X_{J^* v}, \pm J^* v)$$

- (M^{2n}, ω) symplectic, $\mathcal{S} \subset C^\infty(M)$ Poisson subalgebra (fin. gen.).
  $$\mathcal{B} = \langle df : f \in \mathcal{S} \rangle \text{ sing subalgd of } T^*_\omega M$$
  e.g. $M = \mathbb{R}^2$, $\omega = dx \wedge dy$, $f = xy$.
  $\mathcal{B} = \{\phi d(xy) : \phi \in C^\infty_c(\mathbb{R}^2)\}$
  $\mathcal{F}_\mathcal{B} = \langle -x\partial_x + y\partial_y \rangle$. Leaves: Connected components of level sets. On $f^{-1}(0)$, 4 open half-axes and origin.

Central example

N^n closed embedded in M^m, $B \to N$ (honest) Lie subalgd, $\mathrm{rank}(B) = b$.

$$\mathcal{B} = C_c^\infty(M)\text{-module generated by } \{\boldsymbol{\alpha} \in \Gamma A : \boldsymbol{\alpha}|_N \in \Gamma B\}$$

- $x^1, \ldots, x^m$ coordinates of M around $p \in N$ such that $x^1|_N, \ldots, x^n|_N$ coordinates of N and $x^{n+1}, \ldots, x^m$ vanish on N.
- $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_k \in \Gamma A$ ($k = \mathrm{rank}(A)$) such that $\{\boldsymbol{\alpha}_j|_N\}_{1 \leqslant j \leqslant b} \subset \Gamma B$

Near p, $\mathcal{B}$ is generated by

$$\{\boldsymbol{\alpha}_j\}_{1 \leqslant j \leqslant b} \cup \{x^i \cdot \boldsymbol{\alpha}_j\}_{i > n, j > b}$$

(away from N $\mathcal{B}$ generated by some elements of $\Gamma_c A$)

- If $\mathrm{codim}(N) \geqslant 2$ and $B \neq A|_N$ then $\mathcal{B}$ not projective, since the number of generators of $\mathcal{B}$ is *strictly* larger than $\mathrm{rank}(A)$.
- If N hypersurface then $\mathcal{B}$ is projective.

Construction of $\mathscr{G}$-bisubmersions

Let $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n \in \mathcal{B}$ forming basis of $\mathcal{B}_x = \mathcal{B}/I_x\mathcal{B}$.

$$\varphi : M \times \mathbb{R}^n \to \mathscr{G}, \quad \varphi(x, \lambda_1, \ldots, \lambda_n) = \exp_x\left(\textstyle\sum_{i=1}^n \lambda_i \vec{\boldsymbol{\alpha}_i}\right)$$

defined in small neighborhood U of $(x, 0) \in M \times \mathbb{R}^n$.

bisections → local diffeos. So $(U, \varphi, \mathscr{G})$ carries local diffeos near the identity.

- Composition:

$$U_1 \times_{s_1,t_2} U_2 \xrightarrow{(\varphi_1,\varphi_2)} \mathscr{G} \times_{s,t} \mathscr{G} \xrightarrow{m} \mathscr{G}$$

- Inverse:

$$U \xrightarrow{\varphi} \mathscr{G} \xrightarrow{\iota} \mathscr{G}$$

Holonomy groupoid

Equivalence relation: Different choices of $\alpha_1, \ldots, \alpha_n$ give maps

$$\psi : U \to U' \text{ such that } \varphi' \circ \psi = \varphi$$

Proposition-Definition (Holonomy groupoid)

This is an equivalence relation. Obtain diffeological groupoid

$$H^{\mathscr{G}}(\mathcal{B}) = \cup_i U_i / \sim$$

Depends on choice of $\mathscr{G}$ integrating A.

Example: Let $(M, \mathcal{F})$ singular foliation.

1. Holonomy groupoid $H(\mathcal{F}) = H^{M \times M}(\mathcal{F})$
2. Monodromy-ish groupoid $H^{\Pi(M)}(\mathcal{F})$

Properties

- $\Phi : H^{\mathscr{G}}(\mathcal{B}) \to \mathscr{G}$ continuous. Injective iff $(\mathscr{G}, \overrightarrow{\mathcal{B}})$ has trivial holonomy.
- $H(\mathscr{G}, \overrightarrow{\mathcal{B}}) = H^{\mathscr{G}}(\mathcal{B}) \times_{\mathbf{t},\mathbf{s}} \mathscr{G}$
- $H^{\mathscr{G}}(\mathcal{B})$ Lie groupoid $\Leftrightarrow$ $\mathcal{B}$ (locally) projective. Then $d\Phi$ gives isomorphism $\Gamma_c(\mathscr{A}(H^{\mathscr{G}}(\mathcal{B}))) \simeq \mathcal{B}$.
- Period bounding lemma $\Rightarrow$ $\mathbf{s}$-fiber $H^{\mathscr{G}}(\mathcal{B})_x$ always smooth
- Covering groupoid $\pi : \tilde{\mathscr{G}} \to \mathscr{G}$, then $H^{\tilde{\mathscr{G}}}(\mathcal{B}) \simeq (H^{\mathscr{G}}(\mathcal{B}) \times_{\Phi,\pi} \tilde{\mathscr{G}})_0$

DG for diffeological groupoids: Integration

H: "submersive" diffeological groupoid, $\Phi : H \to \mathscr{G}$ continuous morphism.

$\{\mathbf{b}_t\}_{t \in I}$: 1-parameter group of bisections of H, $\mathbf{b}_0 = \mathrm{id}$

- $\{\mathbf{b}_t\}_{t \in I}$ smooth iff $\{\Phi \circ \mathbf{b}_t\}_{t \in I}$ smooth
- **injectivity:** $\Phi \circ \mathbf{b}_t = \mathrm{id}$ for all $t \Rightarrow \mathbf{b}_t = \mathrm{id}$ for all t
- **surjectivity:** Every $\boldsymbol{\alpha} \in \mathcal{B}$ is $\boldsymbol{\alpha} = \frac{d}{dt}\big|_{t=0}(\Phi \circ \mathbf{b}_t)$.

Theorem (IA-Zambon)

$H^{\mathscr{G}}(\mathcal{B})$ integrates $\mathcal{B}$ as a topological groupoid.

NCG for "submersive" diffeological groupoids (with G. Skandalis)

General principle

- Work with the atlas (plots);
- Constructions must descend to the groupoid.

$(A\mathscr{G}, \mathcal{B})$, atlas $\{\phi_i : U_i \to \mathscr{G}\}_{i\in I}$. Put

$$C^{\infty}(\mathscr{G}, \mathcal{B}) = \oplus_{i\in I} C^{\infty}(U_i; \Omega^{1/2} U_i)/\sim$$

Get $C^*(\mathcal{B}), C^*_r(\mathcal{B})$

Desintegration 1

Let $(M, \mathcal{F})$ singular foliation, put $G = H(\mathcal{F})$.

Definition

- Let (U, t, s) bisubmersion and $\lambda^s \in C^\infty(U; \ker ds)$, $\lambda^t \in C^\infty(U; \ker dt)$ positive Borel measures
- Let μ measure on M. For any $f \in C^\infty(U)$, put

$$\mu \circ \lambda^t(f) = \int_M \left(\int_{t^{-1}(\{x\})} f(y) d\lambda^t_x(y) \right) d\mu(x)$$

- μ is quasi-invariant if for all (U_i, t_i, s_i) the measures $\mu \circ \lambda^{s_i}$ and $\mu \circ \lambda^{t_i}$ are equivalent.

Proposition Can choose measures $\lambda^{s_i}, \lambda^{t_i}$ so that μ defines a Radon-Nikodym derivative $D : G \to \mathbb{R}^+$.

Desintegration 2

Definition

A representation of G is $(\mu, \mathcal{H}, \pi)$ where:

- μ quasi-invariant and $\mathcal{H} = (H_x)_{x \in M}$ μ-measurable field of Hilbert spaces
- π^U is $(\mu \circ \lambda)$-measurable section of the field of unitaries $\pi^U_u : H_{s(u)} \to H_{t(u)}$

such that:

1. π is "defined on G": if $f : U \to V$ morphism then $\pi^U_u = \pi^V_{f(u)}$
2. π is a homomorphism: $\pi^{U \circ V}_{u,v} = \pi^U_u \pi^V_v$.

Proposition Any repn $(\mu, \mathcal{H}, \pi)$ induces a repn of $C^*(M, \mathcal{F})$ on the space of sections $\mathcal{H} = \int_M^{\oplus} H_x \, d\mu(x)$:

Desintegration 3

- $f : N \to M$ submersion. Define Hilbert $C_0(M)$-module $\mathcal{E}_f$ by completion of $C_c(N; \Omega^{1/2}\ker(df))$ with $C_0(M)$-valued inner product
$$\langle \xi, \eta \rangle(x) = \int_{z \in f^{-1}(x)} \overline{\xi(z)}\eta(z)$$
- For (U_i, t_i, s_i), get Hilbert modules $\mathcal{E}_{t_i}$ and $\mathcal{E}_{s_i}$ over $C_0(M)$.
- As $C_0(M) \subset \mathcal{M}(\mathbb{C}^*(G))$, every repn π_G of $C^*(G)$ on a Hilbert space $\mathcal{H}$ gives rise to a repn π_M of $C_0(M)$. Gives a measure (class) μ on M.
- π_G characterized by π_M and, for every $i \in I$, a unitary $V_i \in \mathcal{L}(\mathcal{E}_{s_i} \otimes_{C_0(M)} \mathcal{H}, \mathcal{E}_{t_i} \otimes_{C_0(M)} \mathcal{H})$ intertwining the representations of $C_0(U)$.
- Get measurable family of unitaries $U_u : H_{s_i(u)} \to H_{t_i(u)}$. We require that U_u only depends on class of $q_i(u)$ in G and determines a repn of G.

$H(\mathcal{F})$-C^*-algebras

Definition

An G-algebra is a $C_0(M)$-algebra A together with an isomorphism of $C_0(U_i)$-algebras $\alpha^i : s_i^* A \to t_i^* A$ for every $i \in I$.

1. The isomorphism α^i is a family $(\alpha^i_u)_{i \in I}$ of isomorphisms $\alpha^i_u : A_{s_i(u)} \to A_{t_i(u)}$. We require that if $\gamma \in G$ is represented by two elements $u_i \in U_i$ and $u_j \in U_j$, then $\alpha^i_{u_i} = \alpha^j_{u_j}$.
2. By (1), we get a well defined isomorphism $\alpha_\gamma : A_{s(\gamma)} \to A_{t(\gamma)}$. We require that for every composable $\gamma, \gamma' \in G$, we have $\alpha_{\gamma\gamma'} = \alpha_\gamma \circ \alpha_{\gamma'}$.

Example: $C_0(Y)$, where $Y \subseteq M$ saturated. For every $i \in I$, get $s_i^*(C_0(Y)) = t_i^*(C_0(Y))$. Action is the identity.

Covariant representations 2

- Let $G \circlearrowright A$ and take repn $\pi_A : A \to \mathcal{L}(\mathcal{H})$. Extend π_A to multipliers.
- Using $C_0(M) \to \mathcal{M}(A)$ obtain a repn of $C_0(M)$ to $\mathcal{L}(\mathcal{H})$. Image of $C_0(M)$ sits in $\mathcal{ZM}(A)$.
- For every $i \in I$, get repns

$$\pi_A^{s_i} : s_i^*(A) \to \mathcal{L}(\mathcal{E}_{s_i} \otimes_{C_0(M)} \mathcal{H}) \quad \pi_A^{t_i} : t_i^*(A) \to \mathcal{L}(\mathcal{E}_{t_i} \otimes_{C_0(M)} \mathcal{H})$$

Definition

A *covariant representation* of G and A is given by a representation of π_G of $C^*(G)$ and a representation π_A of A in the same Hilbert space $\mathcal{H}$ such that the two representations of $C_0(M)$ agree and, for every $i \in I$, the unitary V_i intertwines $\pi_A^{s_i} \circ \alpha^i$ with $\pi_A^{t_i}$.

Crossed product algebras

The closed linear span of $\pi_A(a)\pi_G(x)$ where a runs over A and x over $C^*(G)$ is a $*$-subalgebra of $\mathcal{L}(\mathcal{H})$.

Definition

The *full crossed product* $A \rtimes G$ is the completion of this linear span with respect to the supremum norm over all covariant representations.

- Using the “regular representations” on $L^2(G_x)$, one may also construct a natural reduced crossed product.

G-actions on Hilbert modules

Let (A, α) be a G-algebra and $\mathcal{E}$ a Hilbert module over A.

Definition

An action of G on $\mathcal{E}$ is given by an action of G on the C*-algebra $\mathcal{K}(\mathcal{E} \oplus A)$ so that natural morphism $A \to \mathcal{K}(\mathcal{E} \oplus A)$ is equivariant.

- This amounts to giving, for any $i \in I$, an isomorphism $\tilde{\alpha} : \mathcal{E} \otimes_A s_i^* A \to \mathcal{E} \otimes_A t_i^* A$ of Banach spaces: family of isomorphisms $\tilde{\alpha}_u : \mathcal{E}_{s_i(u)} \to \mathcal{E}_{t_i(u)}$.
- Need compatibility with α: For $x \in A_{s_i(u)}$ and $\xi, \zeta \in \mathcal{E}_{s_i(u)}$:

$$\tilde{\alpha}_u(\xi x) = \tilde{\alpha}_u(\xi)\alpha_u(x) \quad \alpha_u(\langle \xi | \zeta \rangle) = \langle \tilde{\alpha}_u(\xi) | \tilde{\alpha}_u(\zeta) \rangle$$

- Also $\tilde{\alpha}_u$ depends only on class γ of u in G and $\tilde{\alpha}_\gamma : \mathcal{E}_{s(\gamma)} \to \mathcal{E}_{t(\gamma)}$ is morphism of groupoids: $\tilde{\alpha}_{\gamma\gamma'} = \tilde{\alpha}_\gamma \tilde{\alpha}_{\gamma'}$.

Constructions

- Equivariant Kasparov cycles and $KK_G(A, B)$
- Descent morphism: $KK_G(A, B) \to KK(A \rtimes G, B \rtimes G)$.
- Kasparov product in KK_G.
- Calculation of $K(C^*(\mathcal{F}))$: Baum-Connes...

Best wishes Jean!