Stability of Lie groupoid C*-algebras and of C*-algebras of singular foliations

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joint work with Claire Debord

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For Jean

Summary

1. A question

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2. Lie groupoids

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2. Lie groupoids
3. Two questions

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Question

Does the groupoid C^*-algebra depend on the Haar system?

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Existence of this Morita equivalence for C^*-algebras does not depend on Haar systems.
In particular, two different Haar systems on G give rise to Morita equivalent C^*-algebras.

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Remark

If the orbits are "discrete" (countable), more precisely if $\{x \in G;\ r(x) = s(x)\}$ is open in G, then there is no choice of Haar system (up to equivalence).

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At the other extreme, we can say that the orbits are "continuous" if for every $x \in G^{(0)}$, G_x^x has empty interior in G_x.

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Theorem (Debord-S)

The C^*-algebra of a singular foliation with no leaves reduced to a point is stable.

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- $\chi_{V_j} \in \mathcal{M}(C^*(G))$ because the boundary of V_j is of zero measure in all the orbits.
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- $\sum \chi_{V_j}$ converges strongly in $\mathcal{L}(C^*(G))$.
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3. Two questions

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   Does this imply $C^*(G, \lambda) \simeq C^*(G)$?

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P.S. J'espère que tu as pris de bonnes notes.