Psuedodifferential operators from groupoids

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(joint work with Erik van Erp)

Université Clermont Auvergne

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The work of Connes & Debord-Skandalis

\[ TM = (M \times M) \times \mathbb{R}^\times \sqcup TM \times \{0\} \]
Connes ’90s: $\Psi$DOs & their symbols live on this picture.
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\[ TM = (M \times M) \times \mathbb{R}^x \quad \mathcal{D} \quad TM \times \{0\} \]

**Connes ’90s:** \( \Psi \text{DOs} \) & their symbols live on this picture.

**Debord-Skandalis ’14:** \( \Psi \text{DOs} \) are characterized by this picture.
Ellipticity and Fredholmness

$M$ — closed manifold,
$P$ — differential operator on $M$,
$H^s(M)$ — Sobolev space of order $s$. 

Theorem $P$: $H^s_pM$ $\rightarrow$ $H_{s'}^{-p}M$ Fredholm ($P$) elliptic.

BUT... the definition of $H^s_pM$ is biased towards ellipticity.
Ellipticity and Fredholmness

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**Theorem**

\[ P : H^s(M) \rightarrow H^{s-m}(M) \text{ Fredholm} \iff P \text{ elliptic.} \]
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Fredholm Operators

Definition

\( P \) — unbounded self-adjoint operator on a Hilbert space \( H \).

\( P \) is Fredholm if the bounded operator \( P/(1 + P^2)^{1/2} : H \to H \) is Fredholm.
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Theorem (Dave-Haller ’17)

Let $P$ be a Rockland operator on a compact filtered manifold $M$. Then $P$ is Fredholm.
Filtered manifolds

Definition

A Lie filtration on a manifold $M$ is a filtration of $TM$ by sub-bundles

$$0 = T^0M \leq T^1M \leq \cdots \leq T^N M = TM$$

such that

$$[\Gamma^{\infty}(T^i M), \Gamma^{\infty}(T^j M)] \subseteq \Gamma^{\infty}(T^{i+j} M) \quad \forall i, j.$$
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Fundamental calculation: $X \in \Gamma^\infty(T^i M)$, $Y \in \Gamma^\infty(T^j M)$, $f, g \in C^\infty(M)$,

$$r(fX + gY) = r(fX) + g(r(fX) - f(r(X))) \mod \Gamma^\infty(T^i M).$$
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$\leadsto$ The associated graded bundle $tM := \text{gr}(TM)$ inherits a Lie bracket which is $C^\infty(M)$-bilinear $\Rightarrow$ pointwise.
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$\Rightarrow$ The associated graded bundle $\mathfrak{t}M := \text{gr}(TM)$ inherits a Lie bracket which is $C^\infty(M)$-bilinear $\Rightarrow$ pointwise.

$\Rightarrow$ $\mathfrak{t}M$ is a bundle of nilpotent Lie algebras (Lie algebroid with trivial anchor).
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$\implies$ $tM$ is a bundle of nilpotent Lie algebras (Lie algebroid with trivial anchor).

$\implies$ $TM$ the associated bundle of simply connected nilpotent Lie groups: “osculating groupoid”.

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Example 1: *CR*-manifolds.

\[ X \subset \mathbb{C}^n \text{ — strongly pseudoconvex domain (eg, the unit ball).} \]
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CR-manifolds admit natural subelliptic analogues of \(\bar{\partial}\) and \(\Delta\).
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\[ T^1M_x \simeq H^{2n-1} \] — Heisenberg group.

CR-manifolds admit natural subelliptic analogues of \( \bar{\partial} \) and \( \Delta \).

They are of Rockland type, so Fredholm [Kohn, Folland-Stein, ...].
Example II: $M^5$ with generic distribution of rank 2

$M$ — closed manifold of dimension 5.

$H \subset TM$ — rank 2 distribution.
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Locally, $H = \text{Vect}\{X_1, X_2\}$ for some $X_1, X_2 \in \Gamma^\infty(TM)$. 

$T_M^x$ — nilpotent radical of a parabolic subgroup of $G$. 

The sublaplacian $\Delta \mid \mathcal{X}_2 \mid \mathcal{X}_2$ is Rockland, so Fredholm.
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**Definition**

The distribution $H$ is **generic** if $X_1, X_2, Y, Z_1, Z_2$ span $TM_x$ at all $x \in M$. 
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$TM_x \cong$ nilpotent radical of a parabolic subgroup of $G_2$. 

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The sublaplacian $\Delta = -X_1^2 - X_2^2$ is Rockland, so Fredholm.
**Example III : Flag varieties**

\[ G = \text{semisimple Lie group} \ (\text{eg, } G = SL(n, \mathbb{C})) \]

\[ B_- = \text{Borel subgroup} \ (\text{eg, } B_- = \{ \text{lower } \Delta^r \text{ matrices} \}) \]

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\[ \implies \quad M \text{ is a filtered manifold with } TM_x \cong N \text{ for all } x. \]
\( TM: \)
Diff. Ops: The groupoid view

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<thead>
<tr>
<th>Lie groupoid</th>
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Diff. op.: filtered DNC

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The Rockland Condition

$M$ — filtered manifold,

$P$ — diff. op. on $M$,

$P_x \in \mathcal{U}(tM_x)$ — frozen coefficient operator at $x \in M$. 

Definition (Rockland Condition)

$P$ is called a Rockland operator if for all $x \in M$ we have:

$\partial \pi P z T M x z t 0 u$ nontrivial unitary representation of $T M x$,

$\pi P x$ is invertible on $H 8 \pi$.

Theorem (Helffer-Nourrigat, ... Melin, ... Dave-Haller)

For $M$ closed, Rockland $\subset$ Fredholm.
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**Theorem (Helffer-Nourrigat, ... Melin, ... Dave-Haller)**

For \( M \) closed, Rockland \( \Rightarrow \) Fredholm.
**Rockland Theorem**

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Sketch of proof:
Theorem (Helffer-Nourrigat,… Melin,… Dave-Haller)
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Sketch of proof:

1. $P \in \mathcal{U}(\mathfrak{g})$ left-inv. diff. op. on a graded nilpotent Lie group $N$:
   
   $P$ Rockland $\Rightarrow \exists$ parametrix $Q$ s.t. $PQ - I, \; QP - I$ smoothing.  
   [Helffer-Nourrigat ’70s]
Rockland Theorem

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3. $P$ Rockland $\Rightarrow P$ admits a parametrix $Q \in \Psi^{-m}(M)$ s.t.
   
   $PQ - I$, $QP - I \in \Psi^{-1}(M)$.
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   [Helffer-Nourrigat ’70s]

2. Construct a pseudodiff. calculus $\Psi^\bullet(M) \supset \mathcal{DO}^\bullet(M)$ adapted to the filtration.  

   [Beals-Greiner,... Melin, Van Erp-Y.]

3. $P$ Rockland $\Rightarrow P$ admits a parametrix $Q \in \Psi^{-m}(M)$ s.t.

   $PQ - I$, $QP - I \in \Psi^{-1}(M)$.

4. Show that $\Psi^{-1}(M) \subset \mathcal{K}(L^2(M))$.  

   [Folland-Stein,Ponge, Dave-Haller]
Classical pseudodifferential operators
Recall: Definition via symbols...
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**Definition (Hörmander)**

A pseudodifferential operator on $\mathbb{R}^n$ is

$$P : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$$

$$Pu(x) = \int_{\mathbb{R}^n} e^{i\langle \xi, x \rangle} a(x, \xi) \hat{u}(\xi) d\xi,$$

where $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ belongs to a good ”symbol class”...
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NB. $P$ is a diff. op.

$\iff a(x, \xi)$ is polynomial in $\xi$
Classical pseudodifferential operators

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NB. $P$ is a diff. op.

$\iff$ $a(x, \xi)$ is polynomial in $\xi$

$\iff$ $a(x, \xi) = \sum_{i=0}^m a_m(x, \xi)$ with $a(x, \xi)$ homogeneous in $\xi$:

$$a(x, s\xi) = s^m a(x, \xi) \text{ for all } s \in \mathbb{R}_+^\times.$$
Symbol class

Denote by $S_{m,p_1,0}^{q}$ the set of functions $a$ in $\mathcal{S}_p \mathbb{R}^n \hat{\otimes} \mathbb{R}^n$ verifying:

$$\partial_{\alpha,\beta} a, b \in N \text{ multi-index, } D_{\mathcal{C}} a, b \approx 0 \text{ s.t. } |B_{\alpha,\beta} \xi B_{\alpha,\beta} x a^p x, \xi q| \lesssim C_{\alpha,\beta} p_1 \|\xi\|^q m \|a\|.$$ 

Polyhomogeneous symbols

Denote by $S_{m,phg}^{p}$ the set of functions $a$ in $\mathcal{S}_m^{p_1,0} \mathbb{R}^n \hat{\otimes} \mathbb{R}^n$ which admit an asymptotic expansion $a_{\mathbb{R}^n \hat{\otimes} \mathbb{R}^n}$, i.e.,

$$\partial_{k} a_{\mathbb{R}^n \hat{\otimes} \mathbb{R}^n} \approx \sum_{j=0}^{\infty} a_j$$

with $a_j$ homogeneous of order $m_j$ in $\xi$ outside a compact set:

$$\partial_{s} a_{\mathbb{R}^n \hat{\otimes} \mathbb{R}^n} \approx s_{m_j} a_{\mathbb{R}^n \hat{\otimes} \mathbb{R}^n} \text{ has compact support in } \xi.$$
Symbol class

Symbols of type $(1, 0)$:

Denote by $S_{(1,0)}^m(\mathbb{R}^n \times \mathbb{R}^n)$ the set of $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ verifying:

$\forall \alpha, \beta \in \mathbb{N}^n$ multi-index, $\exists C_{\alpha, \beta} > 0$ s.t.

$$|\partial^\alpha_x \partial^\beta_\xi a(x, \xi)| \leq C_{\alpha, \beta} (1 + |\xi|)^{m-|\alpha|}.$$
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$$a(x, \xi) - \sum_{j=0}^{k-1} a_j(x, \xi) \in S^{m-k}(\mathbb{R}^n \times \mathbb{R}^n)$$

with $a_j$ homogeneous of order $m-j$ in $\xi$ outside a compact set: $\forall s \in \mathbb{R}_+^\times$. 
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Lemma (van Erp-Y.)

The following are equivalent:

\[ a \in \mathcal{P}^c_{\mathcal{S}^p} \mathbb{R}^n \]

\[ a = a_p \pi^\dagger_{1, q} \]

for some function \( a \in \mathcal{P}^c_{\mathcal{S}^p} \mathbb{R}^n \)

homogeneous of degree \( m \) modulo Schwartz.

Homogeneous modulo Schwartz means:

\[ a_p(s, \xi) \pi_{1, q} \]

\[ a_p(\xi, t) \pi_{\mathbb{R}^n} \]

\[ a_p(\xi, t) \pi_\mathbb{R}^\wedge \]

\[ a_p(\xi) \pi_{1, q} \]

homogeneous mod \( \mathcal{S}^p \mathbb{R}^n \).
Lemma (van Erp-Y.)

The following are equivalent:

- $a \in C^\infty(\mathbb{R}^n)$ is polyhomogeneous of order $m$,
- $a = a(\cdot, 1)$ for some function $a \in C^\infty(\mathbb{R}^{n+1})$ homogeneous of degree $m$ modulo Schwartz.
Polyhomogeneous functions

Lemma (van Erp-Y.)

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- \( a \in C^\infty(\mathbb{R}^n) \) is polyhomogeneous of order \( m \),
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Homogeneous modulo Schwartz means:

- \( a(s\xi, st) - s^ma(\xi, t) \in \mathcal{S}(\mathbb{R}^{n+1}), \quad \forall s \in \mathbb{R}_+^\times \).
Polyhomogeneous functions

**Lemma (van Erp-Y.)**

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Classical pseudodifferential operators

To begin with, consider $M = \mathbb{R}^n$. 
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- $a(x, \xi)$ — symbol of $P$, 

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Kernels ($F^{-1}_1 \xi$):
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- $a(x, \xi) \mid$ symbol of $P$,
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Classical pseudodifferential operators

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**Symbols:**

**Kernels ($\mathcal{F}_\xi^{-1}$):**
Pseudodifferential operators from the tangent groupoid

For $G \Rightarrow M$ a Lie groupoid, we write

$$\mathcal{E}^r(G) = \{ r\text{-fibred distributions with proper support} \} = \{ u : C^\infty(G) \to C^\infty(M) \mid C^\infty(M)\text{-linear} \}.$$
For $G \rightrightarrows M$ a Lie groupoid, we write
\[ E'_r(G) = \{ r\text{-fibred distributions with proper support} \} = \{ u : C^\infty(G) \to C^\infty(M) \mid C^\infty(M)\text{-linear} \}. \]

**Definition**

1. $\Psi^m(M) = \{ p \in E'_r(TM) \mid \alpha_s^* p - s^m p \in C_c^\infty(TM) \quad \forall s \in \mathbb{R}_+^\times \}. $
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**Definition**

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**Theorem (van Erp-Y.)**

\[ \Psi^m(M)|_{t=1} = \Psi^m_{cl}(M) \]
For $G \rightrightarrows M$ a Lie groupoid, we write

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**Definition**

1. $\Psi^m(M) = \{ p \in \mathcal{E}_r'(TM) \mid \alpha_s^* p - s^m p \in C_c^\infty(TM) \quad \forall s \in \mathbb{R}_+^\times \}$.

2. $\Psi^m(M) = \Psi^m(M)|_{t=1}$.

**Theorem (van Erp-Y.)**

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Pseudodifferential operators from the tangent groupoid

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**Definition**

1. $\Psi^m(M) = \{ p \in \mathcal{E}_r'(TM) \mid \alpha_{s \ast} p - s^m p \in C^\infty_c(TM) \ \forall s \in \mathbb{R}_+^\times \}$.
2. $\Psi^m(M) = \Psi^m(M)|_{t=1}$.
3. $\Sigma^m(M) = \Psi^m(M)|_{t=0} \cup C^\infty_p(TM)$

**Theorem (van Erp-Y.)**

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**Definition**

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3. $\Sigma^m(M) = \Psi^m(M)|_{t=0} / C^\infty_p(TM)$

**Theorem (van Erp-Y.)**

$$\Psi^m(M)|_{t=1} = \Psi^m_{cl}(M)$$

**Philosophy:** In order to construct a pseudodifferential calculus, it suffices to construct an appropriate tangent groupoid.
Pseudodifferential operators from the tangent groupoid

The exact sequence

\[ 0 \rightarrow \mathcal{E}_r'(TM) \xrightarrow{\times t} \mathcal{E}_r'(TM) \xrightarrow{\text{rest}=0} \mathcal{E}_r'(TM) \rightarrow 0 \]
Pseudodifferential operators from the tangent groupoid

The exact sequence

$$0 \longrightarrow \mathcal{E}'_r(TM) \xrightarrow{\times t} \mathcal{E}'_r(TM) \xrightarrow{\text{rest} = 0} \mathcal{E}'_r(TM) \longrightarrow 0$$

gives

$$0 \longrightarrow \Psi^{m-1}(M) \longrightarrow \Psi^m(M) \xrightarrow{\sigma^m} \Sigma^m(M) \longrightarrow 0.$$
The exact sequence

\[ 0 \longrightarrow \mathcal{E}'_r(\mathbb{T}M) \xrightarrow{\times t} \mathcal{E}'_r(\mathbb{T}M) \xrightarrow{\text{rest}=0} \mathcal{E}'_r(TM) \longrightarrow 0 \]

gives

\[ 0 \longrightarrow \Psi^{m-1}(M) \longrightarrow \Psi^m(M) \xrightarrow{\sigma^m} \Sigma^m(M) \longrightarrow 0. \]

**Corollary**

\[ \sigma^m(P) \in \Sigma^m(M) \text{ invertible} \Rightarrow P \text{ has a parametrix mod } \Psi^{-1}(M) \]
Variations: Other pseudodifferential calculi
Example I: Pseudodifferential calculus on a filtered manifold

$M$ — closed filtered manifold,
$tM = \text{gr}(TM)$ — osculating Lie algebroid,
$TM$ — osculating groupoid.
Example I: Pseudodifferential calculus on a filtered manifold

\[ M \] — closed filtered manifold,
\[ tM = \text{gr}(TM) \] — osculating Lie algebroid,
\[ \mathcal{T}M \] — osculating groupoid.
\[ \delta_s \] — graded dilations on \( tN = \mathcal{T}N \):
\[ \delta_s(\xi) = s^k \xi \quad \text{if } \deg \xi = k. \]
Example I: Pseudodifferential calculus on a filtered manifold

\[ M \] — closed filtered manifold,
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\[ TM \] — osculating groupoid.
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Theorem (Choi-Ponge, van Erp-Y, Higson-Sadegh, Mohsen)

\[ \exists! \text{ smooth structure on the groupoid } \mathbb{T}M = (M \times M) \times \mathbb{R}^\times \sqcup TM \times \{0\} \text{ s.t. } \forall X \in \Gamma^\infty(T^kM) \text{ the sections} \]

\[ X_t = \begin{cases} t^k X & \in \Gamma^\infty(TM), \quad t \neq 0 \\ \text{gr}_k X & \in \Gamma^\infty(tM), \quad t = 0 \end{cases} \]

define \( \mathcal{C}^\infty \) sections of the Lie algebroid.
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\end{cases}
\]

define \( C^\infty \) sections of the Lie algebroid.

We again have an action of \( \mathbb{R}^\times_+ \):
\( \alpha_s(x, y, t) = (x, y, s^{-1}t), \quad t \neq 0 \)
\( \alpha_s(x, \xi, 0) = (x, \delta_s\xi, 0), \quad t = 0 \).
Example I: Pseudodiff. calculus on a filtered manifold

Def’n.

$$\Sigma^m(M) \xrightarrow{\sigma^m \text{ rest}=0} \Psi^m(M) \xrightarrow{\text{ rest}=1} \Psi^m(M) \subseteq \mathcal{E}'(TM) - \text{ hmg. mod } C^\infty_c(M)$$
Example I: Pseudodiff. calculus on a filtered manifold

Def’n. \( \Sigma^m(M) \xrightarrow{\sigma^m} \Psi^m(M) \xrightarrow{\text{rest}=0} \Psi^m(M) \xrightarrow{\text{rest}=1} \Sigma^m(M) \subseteq \mathcal{E}'(\mathbb{T}M) \rightarrow \text{hmg. mod } C_c^\infty(M) \)

\( \Rightarrow \) \( \Psi^D \) calculus for filtered manifolds (cf. [Beals-Greiner, . . . , Melin]).
Example I: Pseudodiff. calculus on a filtered manifold

Def’n. $\Psi^m(M) \subseteq \mathcal{E}'(TM) \rightleftharpoons \text{hmg. mod } C_c^\infty(M)$

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$\Rightarrow \Psi D$ calculus for filtered manifolds (cf. [Beals-Greiner, . . . , Melin]).

Theorem (van Erp-Y ’17)

$\sigma^m(P) \in \Sigma^m(M) \text{ invertible } \Rightarrow P \text{ admits a parametrix } Q \in \Psi^{-m}(M).$
Variations: Other pseudodifferential calculi

Example I: Pseudodiff. calculus on a filtered manifold

Def’n. \[ \Sigma^m(M) \xleftarrow{\sigma^m} \Psi^m(M) \xrightarrow{\text{rest}=1} \subseteq \mathcal{E}'_r(TM) \text{ — hmg. mod } C^\infty_c(M) \]

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Theorem (van Erp-Y ’17)

\[ \sigma^m(P) \in \Sigma^m(M) \text{ invertible } \Rightarrow \) \( P \) admits a parametrix \( Q \in \Psi^{-m}(M) \).

Theorem (Dave-Haller ’17)

Let \( M \) be a filtered manifold and let \( P \in DO^m \subseteq \Psi^m(M) \). Then \( \sigma^m(P) \) is invertible in \( \Sigma^m(M) \iff P_x \) satisfies the Rockland condition \( \forall x \in M \).
Example I : Pseudodiff. calculus on a filtered manifold

Def’n.

\[ \Sigma^m(M) \xleftarrow{\sigma^m} \Psi^m(M) \xrightarrow{\text{rest}=1} \Psi^m(M) \]

\[ \subseteq \mathcal{E}'(\mathbb{T}M) \quad \text{hmg. mod } C_c^\infty(M) \]

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Corollary

Pointwise Rockland condition \( \Rightarrow \) Fredholm.
Example II: Melrose’s $b$-calculus

$M$ — manifold with boundary $\partial M \neq \emptyset$. 
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**Melrose:** $\Gamma^\infty(\mathcal{b}TM) = \{ X \in \Gamma^\infty(M) \mid X|_{\partial M} \text{ is tangent to } \partial M \}$. 
Example II: Melrose’s $b$-calculus

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**Melrose**: $\Gamma^\infty(bTM) = \{ X \in \Gamma^\infty(M) \mid X|_{\partial M} \text{ is tangent to } \partial M \}$.

It is the space of sections of a Lie algebroid $bTM$. 
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**Monthubert:** $\mathcal{b}G(M) = \text{associated Lie groupoid.}$
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Adiabatic groupoid:

$$\mathbb{A}^bG(M) = bG(M) \times \mathbb{R} \bigtimes \setminus bTM \times \{0\}.$$
Example II: Melrose’s $b$-calculus

$M$ — manifold with boundary $\partial M \neq \emptyset$.

**Melrose**: $\Gamma^\infty(^bTM) = \{ X \in \Gamma^\infty(M) \mid X|_{\partial M} \text{ is tangent to } \partial M \}$. It is the space of sections of a Lie algebroid $^bTM$.

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Adiabatic groupoid:

$$\mathbb{A}^bG(M) = ^bG(M) \times \mathbb{R}^\times \sqcup ^bTM \times \{0\}.$$ 

Define:

- $\Psi^m(^bG(M)) = \{ p \in \mathcal{E}'_r(\mathbb{A}^bG(M)) \mid \alpha_{s*}p - s^m p \in C^\infty_c(TM) \}$. 
Example II: Melrose’s $b$-calculus

$M$ — manifold with boundary $\partial M \neq \emptyset$.

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$$\mathbb{A}^{\mathring{b}}G(M) = \mathring{b}G(M) \times \mathbb{R}^\times \sqcup \mathring{b}TM \times \{0\}.$$ 

Define:

- $\Psi^m(\mathring{b}G(M)) = \{ p \in \mathcal{E}'(\mathbb{A}^{\mathring{b}}G(M)) \mid \alpha_s*p - s^m p \in C^\infty_c(\mathbb{T}M) \}$.
- $\Psi^m(\mathring{b}G(M)) = \Psi^m(\mathring{b}G(M))|_{t=1}$. 

Robert Yuncken (UCA)
Example II: Melrose’s $b$-calculus

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Adiabatic groupoid:

$$A^bG(M) = bG(M) \times \mathbb{R}^\times \sqcup bTM \times \{0\}.$$  

Define:

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- $\Psi^m(bG(M)) = \Psi^m(bG(M))|_{t=1}$.

**Theorem**

$\Psi^m(bG(M)) = b$-calculus.
Example III: Rodino’s bisingular calculus

Consider $M = M_1 \times M_2$. 
Example III: Rodino’s bisingular calculus

Consider $M = M_1 \times M_2$.

$\Rightarrow C^\infty(M) = C^\infty(M_1) \otimes C^\infty(M_2)$. 
Example III: Rodino’s bisingular calculus

Consider $M = M_1 \times M_2$.

$\Rightarrow C^\infty(M) = C^\infty(M_1) \otimes C^\infty(M_2)$.

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Theorem

$\mathbf{\Psi}^{m_1,m_2}(M) = \text{Rodino’s bisingular calculus.}$