Groupoid Methods and Singular Spaces in Free Analysis

Based on joint work with E. Griesenauer and B. Solel

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Overview

The Cast of Characters

- $M_n(\mathbb{C})^d$ – d-tuples of $n \times n$ complex matrices.
- For $\mathfrak{z} \in M_n(\mathbb{C})^d$, write $\mathfrak{z} = (Z_1, Z_2, \cdots Z_d)$, $Z_i \in M_n(\mathbb{C})$ and $\mathbf{Z}_i(\mathfrak{z}) := Z_i$.
- $\mathbb{G}(d,n) :=$ the algebra generated by$\{\mathbf{Z}_1, \mathbf{Z}_2, \cdots, \mathbf{Z}_d\}$ – an algebra of polynomial maps from $M_n(\mathbb{C})^d$ to $M_n(\mathbb{C})$ – the d-generic $n \times n$ complex matrices.
- $F(\mathfrak{z}) = \sum_{w \in \mathbb{F}_d^+} a_w \mathbf{Z}^w(\mathfrak{z})$. $\mathbf{Z}^w := \mathbf{Z}_{i_1}\mathbf{Z}_{i_2} \cdots \mathbf{Z}_{i_{|w|}}$, $w = i_1 i_2 \cdots i_{|w|}$.

The Problems

- Let $X \subseteq M_n(\mathbb{C})^d$ be compact and let $\mathfrak{A}(X)$ be the sup-norm closure of $\mathbb{G}(d,n)$ in $C(X, M_n(\mathbb{C}))$.

Problem 1 Describe $\mathfrak{A}(X)$.

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Problem 2 Determine the C^*-algebra generated by $\mathfrak{A}(X)$ - $C^*(\mathfrak{A}(X))$.

Problem 3 Identify the C^*-envelope (in the sense of Arveson) of $\mathfrak{A}(X)$, $C^*_e(\mathfrak{A}(X))$ - an intrinsic C^*-algebra containing $\mathfrak{A}(X)$ and is a quotient of any C^*-algebra generated by a completely isometric representation of $\mathfrak{A}(X)$.

Archetypal Example

- $d = n = 1$: $\mathbb{G}(d, n)$ is just the algebra of polynomial functions in one variable.
- Let X be the closed unit disc in $\mathbb{C}$.
- $\mathfrak{A}(X) = \{f \in C(X) \mid$ f is holomorphic on the interior of $X\}$
- $C^*(\mathfrak{A}(X)) = C(X)$.
- $C^*_e(\mathfrak{A}(X)) = C(|\mathfrak{z}| = 1)$.

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- Problems: What sort of functions are the $\widehat{r}$? What do they tell us about R? Where do these functions really live?
- Example: $\mathbb{G}(d, n) = \mathbb{C}\langle X_1, \cdots, X_d\rangle\widehat{}$ on its space of n-dim. representations.

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- Let $G \subseteq \mathrm{Aut}(S)$ be a subgroup, e.g. take $G = \mathrm{InnAut}(S)$. Then G acts on X as well as on S:

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- Further

$$\widehat{r}(\varphi \cdot g) = (\varphi \cdot g)(r) = \varphi(r) \cdot g = \widehat{r}(\varphi) \cdot g.$$

- Thus each $\widehat{r}$ is a "*G-concomitant*".

Concomitants

Definition

If a group G acts on two spaces X and Y, then a function $f : X \to Y$ is called a G-concomitant in case

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- Concomitants are also called covariants, fixed functions, invariant functions, equivariant functions, intertwiners. . . .

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Ineluctable Conclusion

- If one wants to study a ring R in terms of the *similarity classes* of its representations in a noncommutative ring S, one is naturally led to study $\widehat{R}$ as sections of the bundle whose base is $X/\mathrm{InnAut}(S)$ and whose total space is $(X \times S)/\mathrm{InnAut}(S)$.

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- The details are not straightforward - the quotient space $X/\mathrm{InnAut}(S)$ can be problematic and the $\mathrm{InnAut}(S)$ action on X need not be free.

Familiar facts

- If A is a C^*-algebra, if $X = \mathrm{Rep}_{irr}(A, B(H))$, and if G is the unitary group of $B(H)$, then whether or not X/G is countably separated as a Borel space is precisely the test for deciding if A is type I.

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- This fact and its consequences in group representation theory is Mackey's source of inspiration for his theory of virtual groups and the *entrée* of groupoids into operator algebra.
- The C^*-algebras that can be written as continuous sections of bundles of irreducible representations are the continuous trace C^*-algebras.

Generic Matrices as Concomitants

- $G := PGL(n, \mathbb{C})$, viewed as the automorphism group of M_n. Write $A \cdot g := g^{-1}Ag, \quad A \in M_n, \quad g \in G$.
- G acts on M_n^d (the diagonal action): $\mathfrak{z} \cdot g := (Z_1 \cdot g, Z_2 \cdot g, \cdots, Z_d \cdot g)$.
- Since $\mathbf{Z}_k(\mathfrak{z} \cdot g) = \mathbf{Z}_k(\mathfrak{z}) \cdot g$, each $\mathbf{Z}_k$ is a G-concomitant. Thus $\mathbb{G}(d, n)$ consists of G-concomitants.

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- What does the bundle perspective have to offer for these concomitants?

What is problematic about M_n^d/G?

- The set theoretic quotient M_n^d/G is of little use because the G-orbits in M_n^d need not be closed.

Example

$d = 1, n = 2$: The G-orbit of the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is

$$\left\{ U_1 \begin{bmatrix} 0 & \lambda \\ 0 & 0 \end{bmatrix} U_2 \mid \lambda \neq 0, \quad U_i \in U(2, \mathbb{C}) \right\}$$

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- The correct equivalence relation: $\mathfrak{z} \equiv \mathfrak{w}$ iff $\overline{G\cdot\mathfrak{z}} \cap \overline{G\cdot\mathfrak{w}} \neq \emptyset$. But how to view the quotient space?

Categorical Quotients

Definition

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Given a category $\mathcal{C}$ and an object X of $\mathcal{C}$ endowed with an action of a group G (by automorphisms of X in $\mathcal{C}$), a *categorical quotient* for the action is a pair (Y, π) where Y is an object of $\mathcal{C}$ and π is a morphism of $\mathcal{C}$ mapping X to Y such that

1. π is invariant; i.e., $\pi \circ \sigma = \pi \circ p_2$ where $\sigma : G \times X \to X$ is the given group action and p_2 is the projection of $G \times X$ onto X; and
2. (Y, π) has this universal property: any morphism $X \to Z$ satisfying 1) factors uniquely through π .

Categorical Quotients, continued

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Theorem (Mumford)

The categorical quotient $M_n^d//G$, called the Hilbert quotient, in the category of affine G-varieties is the (abstract) affine algebraic variety $Spec_{max}\{\mathbb{C}[M_n^d]^G\}$. It may be realized concretely as an embedded algebraic variety as follows: Choose a finite set of generators $p_1, p_2, \cdots, p_e$ for $\mathbb{C}[M_n^d]^G$ and define $\mathbf{p} : M_n^d \to \mathbb{C}^e$ by $\mathbf{p}(\mathfrak{z}) := (p_1(\mathfrak{z}), p_2(\mathfrak{z}), \cdots, p_e(\mathfrak{z}))$. The image of $\mathbf{p}$ is the embedded algebraic variety, $\mathbf{V}$, that is the common zeros of the polynomial relations among the p_i.

Some Properties of $M_n^d//G$

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- Orbit closures are separated by invariants, so $M_n//G$ may be thought of as the space of orbit closures.

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- Orbit closures are separated by invariants, so $M_n//G$ may be thought of as the space of orbit closures.
- Artin (1969): The orbit of $\mathfrak{z} = (Z_1, \cdots, Z_d)$ is closed if and only if $\{Z_1, Z_2, \cdots, Z_d\}$ generates a semi-simple subalgebra of M_n.

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- We identify $M_n^d//G$ with **V** and the quotient map π with **p**.
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- Artin (1969): The orbit of $\mathfrak{z} = (Z_1, \cdots, Z_d)$ is closed if and only if $\{Z_1, Z_2, \cdots, Z_d\}$ generates a semi-simple subalgebra of M_n.
- $\mathcal{V}(d,n) := \{\mathfrak{z} \in M_n^d \mid \{Z_1, Z_2, \cdots, Z_d\}$ generates $M_n\}$. $\mathcal{V}(d,n)$ is called the set of *irreducible points* in M_n^d.

M_n^d is Almost a Principal G-bundle.

Theorem (Procesi)

$\mathcal{V}(d,n)$ is the total space of a principal G bundle with bundle map $\mathbf{p}$ restricted to $\mathcal{V}(d,n)$ and base $Q_0(d,n) := \mathbf{p}(\mathcal{V}(d,n))$ – contained in the smooth points of $\mathbf{V}$.

- $\mathcal{V}(d,n)$ is Zariski dense in M_n^d. If n or d is greater than 2, then a function holomorphic on $\mathcal{V}(d,n)$ extends to be holomorphic on M_n^d. When $d = n = 2$, $\mathcal{V}(d,n)$ is a domain of holomorphy.

An Example

From J. J. Sylvester, 1883

- $\mathfrak{z} = (Z_1, Z_2) \in \mathcal{V}(2,2)$ if and only if $\det[Z_1, Z_2] \neq 0$.
- $\mathbb{C}[M_2^2]^G$ is generated by the following 5 functions: $\mathfrak{z} \to tr(Z_1)$, $\mathfrak{z} \to tr(Z_2)$, $\mathfrak{z} \to tr(Z_1^2)$, $\mathfrak{z} \to tr(Z_2^2)$, and $\mathfrak{z} \to tr(Z_1 Z_2)$.
- These functions are algebraically independent, so $M_2^2//G = \text{Spec}_{max}\{\mathbb{C}[M_2^2]^G\}$ is $\mathbb{C}^5$.

Other Categories

Theorem (Luna)

$(\mathbf{V}, \mathbf{p})$ serves also as the categorical quotient $(M_n^d//G, \pi)$ for the action of G on M_n^d in the following categories: T_1 spaces and continuous maps, T_2 spaces and continuous maps, and complex analytic varieties and holomorphic maps.

A More Concrete Model for $M_n^d//G$

- $K := PU_n$.
- $\mathfrak{N} = \mathfrak{N}(d, n) := \{\mathfrak{z} \in M_n^d \mid \sum_{k=1}^d [Z_k^*, Z_k] = 0\}$ - the normal points in M_n^d.

Theorem (Kempf-Ness)

$\mathfrak{N} = \{\mathfrak{z} \mid g \to \|\mathfrak{z} \cdot g\|_2$ *achieves its infimum at* $g = e\}$. ($\|\cdot\|_2 :=$ *Hilbert-Schmidt norm.*)

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$\mathfrak{N} = \{\mathfrak{z} \mid g \to \|\mathfrak{z} \cdot g\|_2$ *achieves its infimum at* $g = e\}$. ($\|\cdot\|_2 :=$ *Hilbert-Schmidt norm.)* $\mathbf{p}$ *is a* K*-equivariant map of* $\mathfrak{N}$ *onto* $M_n^d//G$, *inducing a homeomorphism* $\mathfrak{N}/K \simeq M_n^d//G$.

- The Kempf-Ness theorem is a substantial generalization of the fact that every diagonable matrix is similar to a normal matrix.

Important Contributions of Procesi

- *Trace Algebra* $\mathbb{S}(d,n) :=$ subalgebra of $M_n(\mathbb{C}[M_n^d])$ generated by $\mathbb{C}[M_n^d]^G$ and $\mathbb{G}(d,n)$.

Theorem (Procesi)

1. $\mathbb{C}[M_n^d]^G$ *is generated by* $\mathfrak{z} \to tr(Z^w)$, $|w| \leq 2^n - 1$.
2. $\mathbb{C}[M_n^d]^G = \mathfrak{Z}(\mathbb{S}(d,n))$ *and* $\mathbb{S}(d,n)$ *is generated as a module over* $\mathbb{C}[M_n^d]^G$ *by* $\mathbf{Z}^w$, $|w| \leq 2^{n-1}$.
3. $\mathbb{S}(d,n) = M_n(\mathbb{C}[M_n^d])^G$ – *all* G*-concomitants in* $M_n(\mathbb{C}[M_n^d])$.

The Bundle Perspective for Generic Matrices and Trace Algebras (continued)

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- First guess:

$$(M_n^d \times M_n)/G = \{[\mathfrak{z}, A] \mid \mathfrak{z} \in M_n^d, \ A \in M_n\},$$

which may be identified with $M_n^{(d+1)}/G$, with bundle projection: $\pi([\mathfrak{z}, A]) = [\mathfrak{z}]$.

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which may be identified with $M_n^{(d+1)}/G$, with bundle projection: $\pi([\mathfrak{z}, A]) = [\mathfrak{z}]$. Wrong!

- This has the same defect as M_n^d/G, plus there may be problems with isotropy.

A replacement for $(M_n^d \times M_n)/G$

- Let $\mathfrak{M} := \{[\mathfrak{z}, A] \in (\mathfrak{N} \times M_n)/K \mid A \in \{K_{\mathfrak{z}}\}'\}$, where $K_{\mathfrak{z}}$ is the isotropy group of $\mathfrak{z}$ in K. Let $\pi([\mathfrak{z}, A]) := [\mathfrak{z}]$ and let $\mathfrak{M}([\mathfrak{z}]) := \pi^{-1}([\mathfrak{z}])$.

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Proposition

$\{\mathfrak{M}([\mathfrak{z}])\}_{[\mathfrak{z}]\in\mathfrak{N}/K}$ has the structure of a continuous field of C^*-algebras over $\mathfrak{N}/K \simeq M_n^d//G$. A total family of continuous fields is given by $\{[\mathfrak{z}] \to [\mathfrak{z}, F(\mathfrak{z})]\}$ where F runs over the semigroup generated by $\{\mathbf{Z}_i, \mathbf{Z}_j^*\}_{i,j=1}^d$.

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- For every compact subset $Y \subseteq M_n^d//G$ (realized as $\mathfrak{N}/K$), the continuous sections of $\mathfrak{M}$ over Y, $\Gamma_c(Y,\mathfrak{M})$, is a C^*-algebra that is n-homogeneous when $Y \subseteq Q_0(d,n)$.

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- If $Y \subseteq M_n^d//G$, its $\mathbb{C}[M_n^d]^G$-*convex hull*, $\widehat{Y}$, is $\{\mathfrak{z} \in M_n^d//G \mid |f(\mathfrak{z})| \leq \sup\{|f(\mathfrak{x})| \mid \mathfrak{x} \in Y\},\ f \in \mathbb{C}[M_n^d]^G\}$.
- A pre-compact domain $\mathcal{D} \subseteq M_n^d//G$ is $\mathbb{C}[M_n^d]^G$-*convex*, if $\widehat{\overline{\mathcal{D}}} = \overline{\mathcal{D}}$.

The Tracial Function Algebra $\mathbb{S}(Y)$

Definition

The tracial function algebra *determined by a compact* $Y \subseteq \mathfrak{N}/K$ *is* $\mathbb{S}(Y) := \overline{\mathbb{S}(d,n)}^{cl}$ *in* $\Gamma_c(Y, \mathfrak{M})$.
$\mathbb{I}(Y) := \overline{\mathbb{C}[M_n^d]^G}^{cl}$ *in* $C(Y)$.

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Observation

$C^*(\mathbb{S}(Y)) = \Gamma_c(Y, \mathfrak{M})$

Theorem (Griesenauer, M, Solel)

Let $\overline{\mathcal{D}} \subseteq Q_0(d,n)$ *be a* $\mathbb{C}[M_n^d]^G$*-convex domain. Then*

1. $\overline{\mathcal{D}}$ *is the maximal ideal space of* $\mathbb{I}(\overline{\mathcal{D}})$.
2. $\mathbb{I}(\overline{\mathcal{D}})$ *is the center of* $\mathbb{S}(\overline{\mathcal{D}})$
3. $\mathbb{S}(\overline{\mathcal{D}})$ *is a rank* n^2*-Azumaya algebra over* $\mathbb{I}(\overline{\mathcal{D}})$.

The Tracial Function Algebra $\mathbb{S}(Y)$ (continued)

- If $\overline{\mathcal{D}} \subseteq \mathfrak{N}/K$ is compact and $\mathbb{C}[M_n^d]^G$-convex, then $\partial\mathcal{D} :=$ the Shilov boundary of $\overline{\mathcal{D}}$ viewed as the maximal ideal space of $\mathbb{I}(\overline{\mathcal{D}})$ and $\partial_e\mathcal{D}$ denotes its extreme (or Choquet) boundary.

Theorem (Griesenauer, M, Solel – extended)

If $\overline{\mathcal{D}}$ is $\mathbb{C}[M_n^d]^G$-convex, then:

1. *The boundary representations of $C^*(\mathbb{S}(\overline{\mathcal{D}})) \supseteq \partial_e\overline{\mathcal{D}}$.*
2. $C_e^*(\mathbb{S}(\overline{\mathcal{D}})) = \Gamma_c(\partial\mathcal{D}, \mathfrak{M})$.

Back to $\mathfrak{A}(X)$, $X \subseteq M_n(\mathbb{C})^d$

- Let X be a compact, K-invariant subset of $M_n(\mathbb{C})^d$. What can be said about $\mathfrak{A}(X) :=$ the closure of $\mathbb{G}(d, n)$ in $C(X, M_n(\mathbb{C}))$?

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Theorem (Neeman's Retraction Theorem)

There is a K-invariant deformation retraction $\{\varphi_t\}_{t\in[0,1]}$ of $M_n(\mathbb{C})^d$ onto $\mathfrak{N}$, with $\varphi_0 = id$, $\varphi_1 : M_n(\mathbb{C})^d \to \mathfrak{N}$, such that $\varphi_t(\mathfrak{z}) \in G\mathfrak{z}$ for $0 \leq t < 1$ and $\varphi_1(\mathfrak{z}) \in \overline{G\mathfrak{z}}$.

An Isomorphism

Theorem (M & Solel)

Suppose $X \subseteq \mathcal{V}(d,n)$. Then Neeman's retraction φ_1 induces a homomorphic conditional expectation Φ from $C(X, M_n(\mathbb{C}))^K$ onto $C(\varphi_1(X), M_n(\mathbb{C}))^K$, which in turn is $*$-isomorphic to $\Gamma_c(\varphi_1(X)/K, \mathfrak{M})$. Further, Φ restricted to $\mathfrak{A}(X)$ induces a completely isometric isomorphism between $\mathfrak{A}(X)$ and $\mathbb{S}(\varphi_1(X)/K)$.

Some (Vexing) Unfinished Business

- When $X \subseteq \mathcal{V}(d,n)$, $\mathfrak{A}(X) = \overline{\mathbb{S}(d,n)}^{cl}$ in $C(X, M_n(\mathbb{C}))$. But if X is not contained in $\mathcal{V}(d,n)$, they may be different.
- It is important to know when $\varphi_1(X) = X \cap \mathfrak{N}(d,n)$. In particular, if $\overline{\mathbb{D}(d,n)}^{cl} := \{\mathfrak{z} \in M_n(\mathbb{C})^d \mid I - \mathfrak{z}\mathfrak{z}^* \leq 1\}$, does $\varphi_1(\overline{\mathbb{D}(d,n)}^{cl}) = \overline{\mathbb{D}(d,n)}^{cl} \cap \mathfrak{N}(d,n)$?

Thank You for Listening!