Blowup and deformation groupoids in relation with index theory

D. & Skandalis - Blowup constructions for Lie groupoids and a Boutet de Monvel type calculus

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IMJ-PRG

Orléans 2019 : Jean's feast

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It defines : $0 \to C^*(\mathcal{G}_M^t|_{M\times]0,1]}) \to C^*(\mathcal{G}_M^t) \overset{e_0}{\to} C^*(\mathcal{G}_M^t|_{M\times\{0\}}) \to 0$

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Let $e_1 : C^*(\mathcal{G}_M^t) \to C^*(\mathcal{G}_M^t|_{M\times\{1\}}) = C^*(M \times M) \simeq \mathcal{K}$.

The index element

$$\mathrm{Ind}_{M\times M} := [e_0]^{-1} \otimes [e_1] \in KK(C^*(TM), \mathcal{K}) \simeq K^0(C^*(TM)) \; .$$

The algebra $\Psi^*(G) = \Psi^*(M \times M)$ identifies with the C^*-algebra of order 0 pseudodifferential operators on M and

$$0 \longrightarrow \underset{\simeq \mathcal{K}}{C^*(M \times M)} \longrightarrow \Psi^*(M \times M) \longrightarrow C(\mathbb{S}^*TM) \longrightarrow 0$$

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Foliation $\mathcal{F}$ on M : Replace in the picture the groupoid $M \times M$ by the holonomy groupoid $Hol(M, \mathcal{F})$ (i.e. the “smallest” Lie groupoid over M whose orbits are the leaves of the foliation) [Connes].

General Lie groupoid $G \rightrightarrows M$ [Monthubert-Pierrot, Nistor-Weinstein-Xu]
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- 0-calculus, (pseudodifferential) operators vanishing on V : replace $M \times M$ by $G_0 \rightrightarrows M$ equal to the pair groupoid on $M \setminus V$ outside V and isomorphic to $\mathcal{G}^t_V \rtimes \mathbb{R}^*_+$ around V.

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- b-calculus, (pseudodifferential) operators vanishing on the normal direction of V : replace $M \times M$ by $G_b \rightrightarrows M$ equal to $M \setminus V \times M \setminus V$ outside V and isomorphic to $V \times V \times \mathbb{R} \rtimes \mathbb{R}_+^*$ around V.

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Today, in this talk :

- Present general groupoid constructions involved in such situations.
- Compute, compare the corresponding index elements and connecting maps arising.

The Deformation to the Normal Cone construction

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It is endowed with a smooth structure thanks to the choice of an exponential map $\theta : U' \subset N_V^M \to U \subset M$ by asking the map

$$\Theta : (x,X,t) \mapsto \begin{cases} (\theta(x,tX),t) \ for \ t \neq 0 \\ (x,X,0) \ for \ t = 0 \end{cases}$$

to be a diffeomorphism from the open neighborhood $W' = \{(x,X,t) \in N_V^M \times \mathbb{R} \mid (x,tX) \in U'\}$ of $N_V^M \times \{0\}$ in $N_V^M \times \mathbb{R}$ on its image.

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We define similarly

$$DNC_+(M,V) = M \times \mathbb{R}_+^* \cup N_V^M \times \{0\}$$

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For any $x \in M$, $\lambda \in \mathbb{R}^*$, $y \in V$, $\xi \in T_yM/T_yV$:

- $p : DNC(M,V) \to M \times \mathbb{R} : \quad p(x,\lambda) = (x,\lambda)\,, \;\; p(y,\xi,0) = (y,0);$

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- $p : DNC(M,V) \to M \times \mathbb{R} : \quad p(x,\lambda) = (x,\lambda)\,,\;\; p(y,\xi,0) = (y,0);$
- given $f : M \to \mathbb{R}$, smooth with $f_{|V} = 0$,
  $\tilde{f} : DNC(M,V) \to \mathbb{R}, \quad \tilde{f}(x,\lambda) = \dfrac{f(x)}{\lambda}$, $\tilde{f}(y,\xi,0) = (df)_y(\xi)$

Functoriality of DNC

Consider a commutative diagram of smooth maps

$$\begin{array}{ccc} V & \hookrightarrow & M \\ {\scriptstyle f_V}\downarrow & & \downarrow{\scriptstyle f_M} \\ V' & \hookrightarrow & M' \end{array}$$

Where the horizontal arrows are inclusions of submanifolds. Let

$$\begin{cases} DNC(f)(x,\lambda) = (f_M(x),\lambda) & \text{for } x \in M,\ \lambda \in \mathbb{R}_* \\ DNC(f)(x,\bar{\xi},0) = (f_V(x),\overline{(df_M)_x(\xi)},0) & \text{for } x \in V,\ \bar{\xi} \in T_xM/T_xV \end{cases}$$

We get a smooth map $DNC(f) : DNC(M,V) \to DNC(M',V')$.

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$$DNC(G,\Gamma) = G \times \mathbb{R}^* \cup \mathcal{N}_\Gamma^G \times \{0\} \rightrightarrows G^{(0)} \times \mathbb{R}^* \cup N_{\Gamma^{(0)}}^{G^{(0)}} \times \{0\}$$

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$$\mathcal{T} = DNC\big(DNC(E \times E, E \underset{M}{\times} E), \Delta E \times \{0\}\big) \rightrightarrows E \times \mathbb{R} \times \mathbb{R}$$

Let $\mathcal{T}^{\Box} = \mathcal{T}|_{E \times [0,1] \times [0,1]}$ and $\mathcal{T}hom = \mathcal{T}|_{E \times \{0\} \times [0,1]}$.

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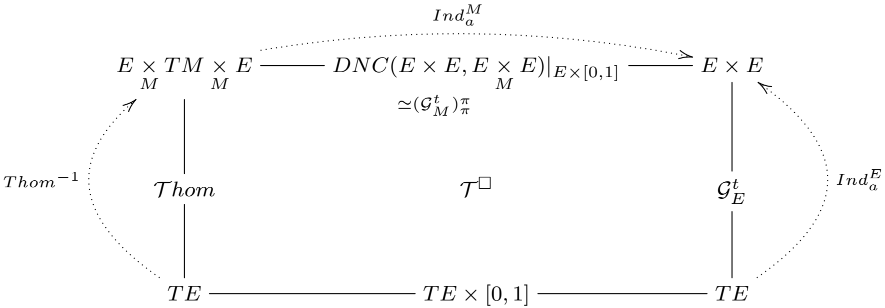

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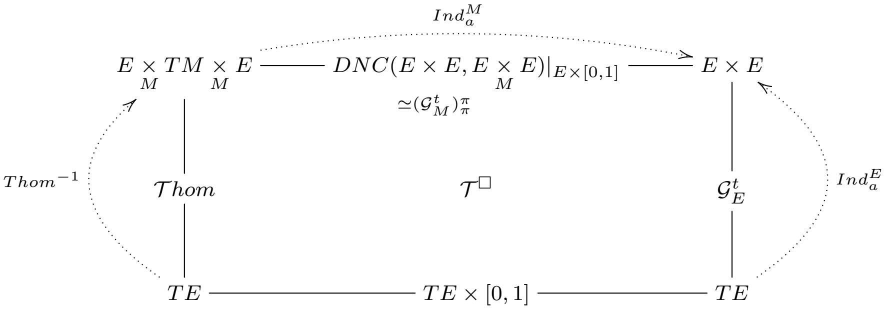

Gives $Ind_a^M = Ind_t^M$ [D.-Lescure-Nistor].

The Blowup construction

The scaling action of $\mathbb{R}^*$ on $M \times \mathbb{R}^*$ extends to the gauge action on $DNC(M,V) = M \times \mathbb{R}^* \cup N_V^M \times \{0\}$:

$$\begin{array}{ccc} DNC(M,V) \times \mathbb{R}^* & \longrightarrow & DNC(M,V) \\ (z,t,\lambda) & \mapsto & (z,\lambda t) \; for \; t \neq 0 \\ (x,X,0,\lambda) & \mapsto & (x,\frac{1}{\lambda}X,0) \; for \; t = 0 \end{array}$$

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$$Blup(M,V) = \big(DNC(M,V) \setminus V \times \mathbb{R}\big)/\mathbb{R}^* = M \setminus V \cup \mathbb{P}(N_V^M)$$

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The gauge action is free and proper on the open subset $DNC(M,V) \setminus V \times \mathbb{R}$ of $DNC(M,V)$. We let :

$$Blup(M,V) = \big(DNC(M,V) \setminus V \times \mathbb{R}\big)/\mathbb{R}^* = M \setminus V \cup \mathbb{P}(N_V^M) \ \text{ and}$$

$$SBlup(M,V) = \big(DNC_+(M,V) \setminus V \times \mathbb{R}_+\big)/\mathbb{R}_+^* = M \setminus V \cup \mathbb{S}(N_V^M) \ .$$

Functoriality of *Blup*

$$\begin{array}{ccc} V & \hookrightarrow & M \\ \downarrow{\scriptstyle f_V} & & \downarrow{\scriptstyle f_M} \\ V' & \hookrightarrow & M' \end{array} \quad \text{gives } DNC(f) : DNC(M,V) \to DNC(M',V')$$

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Then $DNC(f)$ passes to the quotient :

$$Blup(f) : Blup_f(M, V) \to Blup(M', V')$$

Analogous constructions hold for $SBlup$.

Blowup groupoid

Let Γ be a closed Lie subgroupoid of a Lie groupoid $G \overset{t,s}{\rightrightarrows} G^{(0)}$. Define

$$\widetilde{DNC(G},\Gamma) = U_t(G,\Gamma) \cap U_s(G,\Gamma)$$

elements whose image by $DNC(s)$ and $DNC(t)$ are not in $\Gamma^{(0)} \times \mathbb{R}$.

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$$Blup_{t,s}(G,\Gamma) = \widetilde{DNC(G,\Gamma)}/\mathbb{R}^* \rightrightarrows Blup(G^{(0)},\Gamma^{(0)})$$

is naturally a Lie groupoid; its source and range maps are $Blup(s)$ and $Blup(t)$ and its product is $Blup(m)$.

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Remark

Let $\mathring{\mathcal{N}}_\Gamma^G$ be the restriction of $\mathcal{N}_\Gamma^G \rightrightarrows N_{\Gamma^{(0)}}^{G^{(0)}}$ to $N_{\Gamma^{(0)}}^{G^{(0)}} \setminus \Gamma^{(0)}$ and $\mathring{G}$ the restriction of G to $G^{(0)} \setminus \Gamma^{(0)}$.
$\mathring{\mathcal{N}}_\Gamma^G/\mathbb{R}^*$ inherits a structure of Lie groupoid : $\mathcal{P}\mathcal{N}_\Gamma^G \rightrightarrows \mathbb{P}N_{\Gamma^{(0)}}^{G^{(0)}}$.

$$Blup_{r,s}(G,\Gamma) = \mathring{G} \cup \mathcal{P}\mathcal{N}_\Gamma^G \rightrightarrows G^{(0)} \setminus \Gamma^{(0)} \cup \mathbb{P}N_{\Gamma^{(0)}}^{G^{(0)}} .$$

Examples of blowup groupoids

1. Take $G \rightrightarrows G^{(0)}$ a Lie groupoid ans $\mathbb{G} = G \times \mathbb{R} \times \mathbb{R} \rightrightarrows G^{(0)} \times \mathbb{R}$.

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$$Blup_{r,s}(\mathbb{G}, \mathbb{G}^{(0)} \times \{(0,0)\}) = DNC(G, G^{(0)}) \rtimes \mathbb{R}^* \rightrightarrows G^{(0)} \times \mathbb{R}$$

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2. Let $V \subset M$ be a hypersurface.

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Iterate these constructions to go to the study of manifolds with corners. Or consider a foliation with no holonomy on V. Define the holonomy groupoid of a manifold with iterated fibred corners.

Exact sequences coming from deformations and blowups

Let $\Gamma \rightrightarrows V$ be a closed Lie subgroupoid of a Lie groupoid $G \overset{t,s}{\rightrightarrows} M$, suppose that Γ is amenable and let $\mathring{M} = M \setminus V$. Let $\mathring{\mathcal{N}}_\Gamma^G$ be the restriction of the groupoid $\mathcal{N}_\Gamma^G \rightrightarrows \mathcal{N}_V^M$ to $\mathcal{N}_V^M \setminus V$.

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$$DNC_{+}(G,\Gamma) = G \times \mathbb{R}_{+}^{*} \cup \mathcal{N}_{\Gamma}^{G} \times \{0\} \rightrightarrows M \times \mathbb{R}_{+}^{*} \cup \mathcal{N}_{V}^{M}$$

$$\widetilde{DNC_{+}(G},\Gamma) = G_{\mathring{M}}^{\mathring{M}} \times \mathbb{R}_{+}^{*} \cup \mathring{\mathcal{N}}_{\Gamma}^{G} \times \{0\} \rightrightarrows \mathring{M} \times \mathbb{R}_{+}^{*} \cup \mathring{\mathcal{N}}_{V}^{M}$$

$$SBlup_{t,s}(G,\Gamma) = \widetilde{DNC_{+}(G},\Gamma)/\mathbb{R}_{+}^{*} = G_{\mathring{M}}^{\mathring{M}} \cup \mathcal{S}\mathcal{N}_{\Gamma}^{G} \rightrightarrows \mathring{M} \cup \mathbb{S}(\mathcal{N}_{V}^{M})$$

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$$0 \longrightarrow C^*(G \times \mathbb{R}_+^*) \longrightarrow C^*(DNC_+(G,\Gamma)) \longrightarrow C^*(\mathcal{N}_\Gamma^G) \longrightarrow 0$$

$$0 \longrightarrow C^*(G_{\mathring{M}}^{\mathring{M}} \times \mathbb{R}_+^*) \longrightarrow C^*(\widetilde{DNC_+(G,\Gamma)}) \longrightarrow C^*(\mathring{\mathcal{N}}_\Gamma^G) \longrightarrow 0$$

$$0 \longrightarrow C^*(G_{\mathring{M}}^{\mathring{M}}) \longrightarrow C^*(SBlup_{t,s}(G,\Gamma)) \longrightarrow C^*(\mathcal{S}\mathcal{N}_\Gamma^G) \longrightarrow 0$$

Connecting elements

$$0 \longrightarrow C^*(G \times \mathbb{R}_+^*) \longrightarrow C^*(DNC_+(G,\Gamma)) \longrightarrow C^*(\mathcal{N}_\Gamma^G) \longrightarrow 0 \quad \partial_{DNC_+}$$

$$0 \longrightarrow C^*(G_{\mathring{M}}^{\mathring{M}} \times \mathbb{R}_+^*) \longrightarrow C^*(DN\widetilde{C_+(G},\Gamma)) \longrightarrow C^*(\mathring{\mathcal{N}}_\Gamma^G) \longrightarrow 0 \quad \partial_{\widetilde{DNC_+}}$$

$$0 \longrightarrow C^*(G_{\mathring{M}}^{\mathring{M}}) \longrightarrow C^*(SBlup_{t,s}(G,\Gamma)) \longrightarrow C^*(\mathcal{S}\mathcal{N}_\Gamma^G) \longrightarrow 0 \quad \partial_{SBlup}$$

Connecting elements : $\partial_{DNC_+} \in KK^1(C^*(\mathcal{N}_\Gamma^G), C^*(G \times \mathbb{R}_+^*))$,
$\partial_{\widetilde{DNC_+}} \in KK^1(C^*(\mathring{\mathcal{N}}_\Gamma^G), C^*(G_{\mathring{M}}^{\mathring{M}} \times \mathbb{R}_+^*))$ and
$\partial_{SBlup} \in KK^1(C^*(\mathcal{S}\mathcal{N}_\Gamma^G), C^*(G_{\mathring{M}}^{\mathring{M}}))$.

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$$0 \longrightarrow C^*(G \times \mathbb{R}_+^*) \longrightarrow C^*(DNC_+(G,\Gamma)) \longrightarrow C^*(\mathcal{N}_\Gamma^G) \longrightarrow 0 \quad \partial_{DNC_+}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^*(G_{\mathring{M}}^{\mathring{M}} \times \mathbb{R}_+^*) & \longrightarrow & C^*(\widetilde{DNC_+}(G,\Gamma)) & \longrightarrow & C^*(\mathring{\mathcal{N}}_\Gamma^G) & \longrightarrow & 0 \quad \partial_{\widetilde{DNC_+}} \\
 & & \Big| \, \mathring{\beta} & & \Big| \, \beta & & \Big| \, \beta^\partial & & \\
0 & \longrightarrow & C^*(G_{\mathring{M}}^{\mathring{M}}) & \longrightarrow & C^*(SBlup_{t,s}(G,\Gamma)) & \longrightarrow & C^*(\mathcal{S}\mathcal{N}_\Gamma^G) & \longrightarrow & 0 \quad \partial_{SBlup}
\end{array}$$

The β's being KK-equivalences given by Connes-Thom elements.

Connecting elements

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C^*(G \times \mathbb{R}_+^*) & \longrightarrow & C^*(DNC_+(G,\Gamma)) & \longrightarrow & C^*(\mathcal{N}_\Gamma^G) & \longrightarrow & 0 \quad \partial_{DNC_+} \\
 & & \uparrow {\scriptstyle \mathring{j}} & & \uparrow {\scriptstyle j} & & \uparrow {\scriptstyle j^\partial} & & \\
0 & \longrightarrow & C^*(G_{\mathring{M}}^{\mathring{M}} \times \mathbb{R}_+^*) & \longrightarrow & C^*(\widetilde{DNC_+}(G,\Gamma)) & \longrightarrow & C^*(\mathring{\mathcal{N}}_\Gamma^G) & \longrightarrow & 0 \quad \partial_{\widetilde{DNC_+}} \\
 & & | {\scriptstyle \mathring{\beta}} & & | {\scriptstyle \beta} & & | {\scriptstyle \beta^\partial} & & \\
0 & \longrightarrow & C^*(G_{\mathring{M}}^{\mathring{M}}) & \longrightarrow & C^*(SBlup_{t,s}(G,\Gamma)) & \longrightarrow & C^*(\mathcal{S}\mathcal{N}_\Gamma^G) & \longrightarrow & 0 \quad \partial_{SBlup}
\end{array}
$$

The j's coming from inclusion.

Connecting elements

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C^*(G\times\mathbb{R}_+^*) & \longrightarrow & C^*(DNC_+(G,\Gamma)) & \longrightarrow & C^*(\mathcal{N}_\Gamma^G) & \longrightarrow & 0 \quad \partial_{DNC_+} \\
 & & \uparrow \mathring{j} & & \uparrow j & & \uparrow j^\partial & & \\
0 & \longrightarrow & C^*(G_{\mathring{M}}^{\mathring{M}}\times\mathbb{R}_+^*) & \longrightarrow & C^*(\widetilde{DNC_+}(G,\Gamma)) & \longrightarrow & C^*(\mathring{\mathcal{N}}_\Gamma^G) & \longrightarrow & 0 \quad \partial_{\widetilde{DNC_+}} \\
 & & \Big| \mathring{\beta} & & \Big| \beta & & \Big| \beta^\partial & & \\
0 & \longrightarrow & C^*(G_{\mathring{M}}^{\mathring{M}}) & \longrightarrow & C^*(SBlup_{t,s}(G,\Gamma)) & \longrightarrow & C^*(\mathcal{S}\mathcal{N}_\Gamma^G) & \longrightarrow & 0 \quad \partial_{SBlup}
\end{array}
$$

Proposition

$$\partial_{SBlup}\otimes\mathring{\beta}\otimes[\mathring{j}] = \beta^\partial\otimes[j^\partial]\otimes\partial_{DNC_+}\in KK^1(C^*(\mathcal{S}\mathcal{N}_\Gamma^G),C^*(G)).$$

Index type connecting elements

$$\begin{array}{cccccccccc}
0 & \longrightarrow & C^*(G\times\mathbb{R}_+^*) & \longrightarrow & \Psi^*(DNC_+(G,\Gamma)) & \longrightarrow & \Sigma_{DNC_+} & \longrightarrow & 0 & \widetilde{Ind}_{DNC_+} \\
 & & \mathring{j}\ \big\uparrow & & j\ \big\uparrow & & j^{\partial}\ \big\uparrow & & & \\
0 & \longrightarrow & C^*(G_{\mathring{M}}^{\mathring{M}}\times\mathbb{R}_+^*) & \longrightarrow & \Psi^*(\widetilde{DNC_+(G,\Gamma)}) & \longrightarrow & \Sigma_{\widetilde{DNC_+}} & \longrightarrow & 0 & \widetilde{Ind}_{\widetilde{DNC_+}} \\
 & & \mathring{\beta}\ \big| & & \beta\ \big| & & \beta^{\partial}\ \big| & & & \\
0 & \longrightarrow & C^*(G_{\mathring{M}}^{\mathring{M}}) & \longrightarrow & \Psi^*(SBlup_{r,s}(G,\Gamma)) & \longrightarrow & \Sigma_{SBlup} & \longrightarrow & 0 & \widetilde{Ind}_{SBlup}
\end{array}$$

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\begin{array}{ccccccccccc}
0 & \longrightarrow & C^*(G \times \mathbb{R}_+^*) & \longrightarrow & \Psi^*(DNC_+(G,\Gamma)) & \longrightarrow & \Sigma_{DNC_+} & \longrightarrow & 0 & & \widetilde{Ind}_{DNC_+} \\
 & & \uparrow \mathring{j} & & \uparrow j & & \uparrow j^{\partial} & & & & \\
0 & \longrightarrow & C^*(G_{\mathring{M}}^{\mathring{M}} \times \mathbb{R}_+^*) & \longrightarrow & \Psi^*(\widetilde{DNC_+(G,\Gamma)}) & \longrightarrow & \Sigma_{\widetilde{DNC_+}} & \longrightarrow & 0 & & \widetilde{Ind}_{\widetilde{DNC_+}} \\
 & & \Big| \mathring{\beta} & & \Big| \beta & & \Big| \beta^{\partial} & & & & \\
0 & \longrightarrow & C^*(G_{\mathring{M}}^{\mathring{M}}) & \longrightarrow & \Psi^*(SBlup_{r,s}(G,\Gamma)) & \longrightarrow & \Sigma_{SBlup} & \longrightarrow & 0 & & \widetilde{Ind}_{SBlup}
\end{array}
$$

Proposition

$$\widetilde{Ind}_{SBlup} \otimes \mathring{\beta} \otimes [\mathring{j}] = \beta^{\partial} \otimes [j^{\partial}] \otimes \widetilde{Ind}_{DNC_+} \in KK^1(C^*(\Sigma_{SBlup}), C^*(G)).$$

Thank you for your attention !