Classification of regular subalgebras of the hyperfinite II_1 factor

(joint work with D. Shlyakhtenko and S. Popa)

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Regular subalgebras of the hyperfinite II_1 factor

Theorem (Connes, 1976)

Up to isomorphism, there is a unique amenable II_1 factor.

- But, **subalgebras** of R can encode very rich structures!
- **Example** (not today's topic): Jones' theory of subfactors $N \subset M$ and its connections to knot theory, low dimensional topology, etc.

Topic today. Subalgebras $B \subset R$ that

- are regular: the normalizer $\mathcal{N}_R(B) = \{u \in \mathcal{U}(R) \mid uBu^* = B\}$ generates R ;
- and satisfy: $B' \cap R = \mathcal{Z}(B)$.

$\rightsquigarrow$ Very natural as the two extreme cases B abelian and B factorial will show.

Cartan subalgebras: the case B abelian

Definition

A **Cartan subalgebra** B of a II_1 factor M is a regular subalgebra that is maximal abelian: $B' \cap M = B$.

- Feldman-Moore: $B \subset M$ is isomorphic with $L^\infty(X) \subset L(\mathcal{R}, u)$, where
- $\mathcal{R}$ is a countable, probability measure preserving equivalence relation on (X, μ), and u is a scalar 2-cocycle on $\mathcal{R}$.

$\rightsquigarrow$ This means: $\mathcal{R} \subset X \times X$ is a Borel set and an equivalence relation with countable equivalence classes; μ-preserving condition.

Theorem (Connes-Feldman-Weiss, 1981)

There is a unique **amenable**, ergodic, countable, probability measure preserving equivalence relation $\mathcal{R}$ with infinite orbits.

Its 2-cohomology is trivial.

Thus: the hyperfinite II_1 factor R has a unique Cartan subalgebra.

Cocycle crossed products: the case B factorial

If $B \subset M$ is regular and **irreducible**, $B' \cap M = \mathbb{C}1$, then it is of the form $B \subset B \rtimes_{\alpha,u} \Gamma$, where

- Γ is the countable group $\mathcal{N}_M(B)/\mathcal{U}(B)$,
- (α, u) is a free cocycle action of Γ on B : $\alpha_g \circ \alpha_h = \operatorname{Ad} u(g,h) \circ \alpha_{gh}$ and a 2-cocycle relation.

Theorem (Ocneanu, 1985)

Assume that Γ is amenable and that B is amenable (i.e. $B \cong R$).

- The 2-cocycle is always a coboundary.
- The group Γ has a unique free action on B up to cocycle conjugacy.

In other words: the irreducible, regular subalgebras B of the hyperfinite II_1 factor R are completely classified by the group $\Gamma = \mathcal{N}_R(B)/\mathcal{U}(B)$.

General regular inclusions

A regular subalgebra $B \subset M$ with $B' \cap M = \mathcal{Z}(B)$ is described by

- a discrete measured groupoid $\mathcal{G}$ with unit space $\mathcal{G}^{(0)} = X$ given by $\mathcal{Z}(B) = L^\infty(X)$,
- a direct integral decomposition $B = \int_X^\oplus B_x d\mu(x)$,
- a free cocycle action of $\mathcal{G}$ on B :
  - $\alpha_g : B_{s(g)} \to B_{t(g)}$,
  - $\alpha_g \circ \alpha_h = \mathrm{Ad}\, u(g,h) \circ \alpha_{gh}$ if $s(g) = t(h)$.

Theorem (Popa-Shlyakhtenko-V, 2018)

The regular subalgebras $B \subset R$ of the hyperfinite II_1 factor with $B' \cap R = \mathcal{Z}(B)$ are completely classified by the associated groupoid $\mathcal{G}$ and the type of B.

$\rightsquigarrow$ 2-cocycle vanishing (PSV 2018) and
uniqueness of free action (Sutherland-Takesaki 1984).

2-cohomology vanishing

Theorem (Popa-Shlyakhtenko-V, 2018)

Let $\mathcal{G}$ be an amenable discrete measured groupoid and (α, u) a free cocycle action on **any** field of II_1 factors.

Then, u is a coboundary.

Popa (2018): if Γ is an amenable group and (α, u) is a free cocycle action on **any** II_1 factor B, then u is a coboundary.

Popa's remarkable proof:

- There exist $v_g \in \mathcal{U}(B)$ and a copy of the hyperfinite II_1 factor $R \subset B$ such that $(\mathrm{Ad}\, v_g \circ \alpha_g)(R) = R$.
- Then apply Ocneanu's theorem.
- The existence of such a globally invariant copy of R characterizes amenability of Γ.

2-cohomology vanishing for amenable groupoids

Discrete measured groupoid $\mathcal{G}$:

- Space of units $\mathcal{G}^{(0)}$ with probability measure μ.
- Source and target maps $s, t : \mathcal{G} \to \mathcal{G}^{(0)}$.
- Composition gh is defined if $s(g) = t(h)$.

Isotropy groups Γ_x for $x \in \mathcal{G}^{(0)}$:

- $\Gamma_x = \{g \in \mathcal{G} \mid s(g) = x = t(g)\}$.

Structure of amenable discrete measured groupoids

An amenable discrete measured groupoid $\mathcal{G}$ is the semidirect product of

- a measurable field of amenable groups $(\Gamma_x)_{x \in X}$,
- a countable amenable equivalence relation $\mathcal{R}$ on (X, μ),
- isomorphisms $\delta_{(x,y)} : \Gamma_y \to \Gamma_x$ for all $(x, y) \in \mathcal{R}$.

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Let (α, u) be a free cocycle action of $\mathcal{G}$ on $(B_x)_{x \in X}$.

- By Popa's theorem, the 2-cocycle is a coboundary on a.e. Γ_x.

  This means that $u_x = \partial v_x$.
- **Problem:** to prove that u is a coboundary, we need to choose v_x equivariantly w.r.t. $\alpha_{(x,y)} : B_y \to B_x$.
- If v_x were unique, this would be automatic. But v_x is intrinsically non-unique: determined up to a 1-cocycle.

An equivariant choice lemma

Theorem (Popa-Shlyakhtenko-V, 2018)

Let $(G_x \curvearrowright P_x)_{x \in X}$ be a measurable field of continuous actions of **Polish groups** G_x on **Polish spaces** P_x.

Let $\mathcal{R}$ be a countable **amenable** equivalence relation on (X, μ) that acts on $(G_x \curvearrowright P_x)_{x \in X}$ by conjugacies.

IF $G_x \curvearrowright P_x$ has dense orbits, there exists a section $x \mapsto \pi(x) \in P_x$ that is equivariant w.r.t. a cocycle twist of the action.

- This is an abstract/general version of the cohomology lemmas of Jones-Takesaki and Sutherland (1980s).
- The proof of 2-cohomology vanishing can be finished **IF** we can prove that 1-cocycles are approximately coboundary.

Approximate vanishing of 1-cocycles

Let Γ be a countable group and $(\alpha_g)_{g\in\Gamma}$ an outer action on a II_1 factor B.

- **1-cocycle :** unitaries $v_g \in \mathcal{U}(B)$ satisfying $v_{gh} = v_g\,\alpha_g(v_h)$.
- **Coboundary :** $v_g = w\,\alpha_g(w^*)$ for some $w \in \mathcal{U}(B)$.
- **Approximate coboundary :** $v_g = \lim_n w_n\,\alpha_g(w_n^*)$ for $w_n \in \mathcal{U}(B)$.

Theorem (Popa-Shlyakhtenko-V, 2018)

Let Γ be a countable group. TFAE

- Γ is amenable.
- For every outer action of Γ on a II_1 factor, every 1-cocycle is approximately coboundary.
- No outer action of Γ on a II_1 factor B is strongly ergodic : there are many $w_n \in \mathcal{U}(B)$ with $w_n - \alpha_g(w_n) \to 0$ for every $g \in \Gamma$.

Approximate vanishing of 1-cocycles

Assume that $\Gamma \curvearrowright^{\alpha} B$ is an outer action.

- Then Γ acts naturally on the ultrapower B^{ω}.
- Strong ergodicity means: $(B^{\omega})^{\alpha} = \mathbb{C}1$.

If $(B^{\omega})^{\alpha}$ is always factor, then every 1-cocycle is approx coboundary

Connes' 2×2 matrix trick.

Let $\Gamma \curvearrowright^{\alpha} B$ be an outer action and $(v_g)_{g \in \Gamma}$ a 1-cocycle.

- Then $\beta_g \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha_g(a) & \alpha_g(b) v_g^* \\ v_g \alpha_g(c) & v_g \alpha_g(d) v_g^* \end{pmatrix}$ is an action on $M_2(B)$.
- The projections $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are equivalent in the factor $M_2(B^{\omega})^{\beta}$.

Approximate vanishing of 1-cocycles

Actions of amenable groups are never strongly ergodic

Let $\Gamma \curvearrowright^{\alpha} B$ be an outer action on the II_1 factor B.

- We have $B' \cap B \rtimes \Gamma = \mathbb{C}1$.
- Popa: there exists a unitary $u \in \mathcal{U}(B^\omega)$ that is **free** w.r.t. $B \rtimes \Gamma$.
- Then all $\alpha_g(uBu^*)$ are free inside B^ω.
- We thus find $\mathop{*}_{g \in \Gamma} B \hookrightarrow B^\omega$.
- For amenable Γ, we can concretely construct almost invariant elements in $\mathop{*}_{g \in \Gamma} B$.

Conclusion: end of proof of 2-cocycle vanishing for amenable groupoids.

When does 2-cohomology vanishing hold ?

Popa (2018)

Denote by $\mathcal{VC}$ the class of countable groups Γ such that for every free cocycle action (α, u) on a II_1 factor, u is coboundary.

- Amenable groups belong to $\mathcal{VC}$.
- $\mathcal{VC}$ is stable under amalgamated free products over finite groups.
- $\mathcal{VC}$ does not contain property (T) groups. $\mathcal{VC}$ does not contain nonamenable direct product groups.
- All groups in $\mathcal{VC}$ have the Haagerup approximation property.

Wild guess: $\mathcal{VC}$ consists of **treeable groups.**

Triple inclusions and subequivalence relations

Consider: II_1 factor M and $A \subset B \subset M$ such that

- $A \subset M$ is Cartan,
- $B \subset M$ is regular.

$\rightsquigarrow$ $L^\infty(X) \subset L(\mathcal{S}, u) \subset L(\mathcal{R}, u)$ where $\mathcal{S} \subset \mathcal{R}$ is **strongly normal** (in the sense of FSZ).

- $A \subset B$ is the direct integral of $(A_x \subset B_x)_{x \in X}$ with B_x a factor.
- There is a quotient groupoid $\mathcal{G}$ with unit space $\mathcal{G}^{(0)} = X$.
- There is a cocycle action of $\mathcal{G}$ on the field $A_x \subset B_x$, and thus on the equivalence relations $\mathcal{S}_x$.
- Identify M with the cocycle crossed product.

Triple inclusions and subequivalence relations

Let R be the hyperfinite II_1 factor.

Theorem (Popa-Shlyakhtenko-V, 2018)

Inclusions $A \subset B \subset R$ with $A \subset R$ Cartan and $B \subset R$ regular are precisely classified by the associated groupoid and the type of B.

Corollary

Let $B \subset R$ be a regular subalgebra satisfying $B' \cap R = \mathcal{Z}(B)$.

Then there exists $A \subset B$ such that $A \subset R$ is Cartan and A is unique, up to an automorphism that preserves B.

- Vanishing of 2-cohomology for cocycle actions of amenable groupoids on arbitrary equivalence relations.
- Amenable groupoids have a unique (up to cocycle conjugacy) free action on a field of amenable equivalence relations.

2-cohomology vanishing and treeability

Recall (Popa): class $\mathcal{VC}$ of groups with 2-cohomology vanishing for all free cocycle actions on II_1 factors.

Similarly (PSV): class $\mathcal{VC}_{\mathsf{Cartan}}$ of groups with 2-cohomology vanishing for all free cocycle actions on ergodic II_1 equivalence relations.

Using (variants of) Connes-Jones cocycles:

- Popa: if $\Gamma \in \mathcal{VC}$, then $L(\Gamma) \hookrightarrow L(\mathbb{F}_\infty)$.
- PSV: if $\Gamma \in \mathcal{VC}_{\mathsf{Cartan}}$, then Γ is treeable.

$\rightsquigarrow$ Similar definitions for groupoids instead of discrete groups.

$\rightsquigarrow$ Two notions of treeability for an equivalence relation.

$\rightsquigarrow$ Treeable $\Rightarrow$ strongly treeable?