COHOMOLOGY FOR FREE MINIMAL $\mathbb{Z}^d$-ACTIONS ON THE CANTOR SET

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NOTATIONS FOR TODAY :

- X : THE CANTOR SET
- (X, φ) FREE MINIMAL ACTION OF $\mathbb{Z}^d$ ON X.

RECALL : VERY RICH CLASS

$$\begin{array}{c} (X, \varphi) \\ \Big\downarrow \text{FACTOR MAP} \\ (Y, \psi) \end{array}$$

FREE, MIN G-ACTION
Y COMPACT, METRIC

- $C(X, \mathbb{Z})$ ABELIAN GRP. OF CONT. INTEGER VALUED FUNCT. ON X. $\mathbb{Z}^d$-MODULE, VIA $n . f(x) = f(\varphi^n(x))$.

THEN $\quad H^*(X,\varphi) = H^*(\mathbb{Z}^d, C(X,\mathbb{Z}))$

GROUP COHOMOLOGY OF $\mathbb{Z}^d$ WITH COEF. IN $C(X,\mathbb{Z})$.

REM.: FIRST CONSIDERED BY FORREST-HUNTON AND KELLENDONK.

FACTS:
- 1) $H^*(X,\varphi) = 0 \quad \forall \ * > d$
- 2) $H^0(X,\varphi) = C(X,\mathbb{Z})^{\varphi\text{-INV}} \simeq \mathbb{Z}$
- 3) $H^d(X,\varphi) = C(X,\mathbb{Z}) \big/ \langle f - f\circ\varphi \rangle$ $\quad$ CO-INVARIANTS

4) $Z^1(X,\varphi) = \{ \theta : X \times \mathbb{Z}^d \to \mathbb{Z} ;$ CONT.

$\theta(x, n+m) = \theta(x,m) + \theta(\varphi^m(x), n) \}$

A COCYCLE θ IS A COBOUNDARY IF $\exists\, h \in C(X, \mathbb{Z})$

st. $\theta(x,n) = h(\varphi^n(x)) - h(x)$.

5) (X,φ) FREE MIN. $\mathbb{Z}^d$- CANTOR SYST.

THEN $H^1(X,\varphi)$ TORSION FREE.

[PF: SUPPOSE $\theta \in Z^1(X,\varphi)$ ST. $n\theta \in B^1(X,\varphi)$, WITH $n \geq 1$.

HENCE, $\exists\, h \in C(X,\mathbb{Z})$ ST $n\theta(x,k) = h(\varphi^k(x)) - h(x)$

FOR $0 \leq i \leq n-1$, SET $X_i = \{x \in X;\ h(x) \equiv i \pmod n\}$

AS $n \mid h(\varphi^{k}(x)) - h(x)$, $\varphi^{k}(X_i) = X_i$, $\forall k \in \mathbb{Z}^d$

φ MINIMAL $\Longrightarrow$ ALL X_i, BUT ONE ARE EMPTY

LET'S SAY X_j

THEN $\frac{1}{n}(h-j) \in C(X, \mathbb{Z})$ AND

$$\frac{1}{n}(h-j)(\varphi^{k}(x)) - \frac{1}{n}(h-j)(x) = \theta(x,k)$$

$\blacksquare$

REM: THE COHOMOLOGY $H^*(X,\varphi)$ IS AN INVARIANT OF CONT. ORBIT EQUIVALENCE (COE).

$\blacksquare$ $H^*(X,\varphi)$ IS THE GROUPOID COHOM. OF ÉTALE GROUPOID $X \times \mathbb{Z}^d$. $\blacksquare$

6) $\mu \in M_1(X,\varphi)$ INV. PROB. MEAS

$$\tilde{\iota}^1_\mu : Z^1(X,\varphi) \longrightarrow \mathrm{HOM}(\mathbb{Z}^d, \mathbb{R})$$
$$\theta \longmapsto \tilde{\iota}^1_\mu(\theta) : \begin{array}{c} n \\ \downarrow \\ \int \theta(x,n)\, d\mu(x) \end{array}$$

$\tilde{\iota}^1_\mu(B^1(X,\varphi)) = 0$, AS $\tilde{\iota}^1_\mu(\theta)(n) = \int h(\varphi^n(x))\, d\mu(x) - \int h(x)\, d\mu(x) = 0$

HENCE $\tilde{\iota}^1_\mu : H^1(X,\varphi) \longrightarrow \mathrm{HOM}(\mathbb{Z}^d, \mathbb{R})$

IDENTIFYING $\mathrm{HOM}(\mathbb{Z}^d, \mathbb{R}) \cong \mathbb{R}^d$, WE HAVE

$$\tilde{\iota}^1_\mu : H^1(X,\varphi) \longrightarrow \mathbb{R}^d \quad \text{GRP. HOMOM.}$$
$$[\theta] \longmapsto (\tilde{\iota}^1_\mu(\theta)(e_1), \dots, \tilde{\iota}^1_\mu(\theta)(e_d))$$

FOR $1 \leq j \leq d$, SET $\theta_j(x, e_i) = \delta_{ij}$. THEN $\tilde{\iota}^1_\mu([\theta_j]) = e_j$

HENCE $$\mathbb{Z}^d = \mathrm{Span}\{e_j\} \subset \hat{\iota}^1_\mu(H^1(X,\varphi)) \subset \mathbb{R}^d$$

Q? : CLASS OF $\mathbb{Z}^d \subset H \subset \mathbb{R}^d$, COUNTABLE, WHICH CAN BE REALIZED DYNAMICALLY, i.e.

$\exists\ (X,\varphi)$ FREE, MIN. $\mathbb{Z}^d$- ACTIONS UNIQUELY ERG. ST

$$\hat{\iota}^1_\mu(H^1(X,\varphi)) = H.$$

KNOWN: (CLARKE-SADUN)

$\exists\ (X,\varphi)$ FREE, MIN., UNIQ. ERG $\mathbb{Z}^2$-ACTION ST

$$\hat{\iota}^1_\mu(H^1(X,\varphi)) = \mathbb{Z}^2.$$

REM: $(d=1)$ $H^1(X,\varphi) = $ CO-INVARIANTS $= C(X,\mathbb{Z}) / \langle f - f\circ\varphi \rangle$

$H^1(X,\varphi) = D(X,\varphi) \; (= K_0(C^*(X,\varphi)))$ SIMPLE DIM. GRP.

$\mathbb{Z} \subset \tau_\mu^1(H^1(X,\varphi)) \subset \mathbb{R}$, DENSE.

RECALL: ① [HPS] ANY SIMPLE DIM. GRP CAN ARISE

REM: [HPS] GIVES MUCH MORE !!

② FOR ANY $\mathbb{Z} \subset H \subset \mathbb{R}$, DENSE, $\exists (X,\varphi)$ CMS

UNIQ. ERG. ST $\tau_\mu^1(H^1(X,\varphi)) = H$.

a) IF $\mathbb{Z} \subset H \subset \mathbb{Q}$, DENSE, THEN (X,φ) ODOMETER

b) IF $H \not\subset \mathbb{Q}$, THEN (X,φ) CAN BE CHOSEN AS

A DENJOY OR STURMIAN SYST.

REM: [GPS]. ANY UNIQ. ERG. CMS IS OE TO EITHER AN ODOMETER OR A DENJOY SYST.

CASE 1. $\mathbb{Z}^d \subset H \subset \mathbb{Q}^d$, H DENSE.

PROP: $\exists$ $\mathbb{Z}^d$-ODOMETER (X, φ) ST $\tau'_\mu(H'(X,\varphi)) = H$.

$\mathbb{Z}^d$-ODOMETER: M-I CORTEZ, C + S. PETITE, C + K. MEDYNETS

$\mathbb{Z}^d$

G-ODOM.

COE

G RES. FINITE

X. LI RIGIDITY.

RECALL: $d=1$. $p_n \geq 2$, $q_n = p_1 p_2 \cdots p_n$

$$X: \quad \mathbb{Z}/q_1 \leftarrow \mathbb{Z}/q_2 \leftarrow \cdots \qquad \varphi = +(1,1,1,\dots)$$

$H^1(X,\varphi)$ = CO-INVARIANTS $\cong \mathbb{Z}[\frac{1}{M}]$. WHERE

M GENERALIZED INTEGER ASS. TO $(p_n)_{n\geq 1}$.

REM: $\mathbb{Z} \supset q_1\mathbb{Z} \supset q_2\mathbb{Z} \cdots$, $\bigcap q_n \mathbb{Z} = \{0\}$.

FOR $d \geq 1$, $\mathfrak{g} = \{\mathbb{Z}^d \supset Z_1 \supset Z_2 \cdots$, FINITE INDEX, $\bigcap Z_n = \{0\}\}$.

$$X_\mathfrak{g}: \quad \mathbb{Z}^d/Z_1 \leftarrow \mathbb{Z}^d/Z_2 \leftarrow \cdots \qquad \varphi_\mathfrak{g}(n) = +(n,n,\dots,n,\dots)$$
$$n \in \mathbb{Z}^d.$$

FACTS: ① $(X_\mathfrak{g},\varphi_\mathfrak{g})$ FREE MIN. $\mathbb{Z}^d$-ACTION ON THE CANTOR SET

② $(X_\mathfrak{g},\varphi_\mathfrak{g})$ EQUICONT., UNIQ. ERG.

RECALL: $Z \subset \mathbb{Z}^d$ FINITE INDEX SUBGRP. $\rightsquigarrow$ LATTICE

$Z^* \supset \mathbb{Z}^d$ DUAL LATTICE $Z^* = \{ v \in \mathbb{R}^d ; (v|x) \in \mathbb{Z}, \forall x \in Z \} \subset \mathbb{Q}^d$

FACTS: 1) $[\mathbb{Z}^d ; Z] = [Z^*, \mathbb{Z}^d]$.

2) $\cdots \; Z_2 \subset Z_1 \subset \mathbb{Z}^d \subset Z_1^* \subset Z_2^* \subset \cdots$

3) $H = \bigcup_{n \geq 1} Z_n^* \subset \mathbb{Q}^d$, dense IFF $\bigcap Z_n = \{0\}$

ANOTHER CONST. OF $\mathbb{Z}^d$ ODOMETER:

1) $\mathbb{Z}^d \subset H \subset \mathbb{Q}^d$ ($\Rightarrow H/\mathbb{Z}^d$ TORSION GRP)

IF $H/\mathbb{Z}^d$ ENDOWED DISCRETE TOP, THEN $Y_H = \left(H/\mathbb{Z}^d \right)^\wedge$ COMPACT TOT. DISC.

2) IF $\varrho : H/\mathbb{Z}^d \longrightarrow \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ INCLUSION, THEN

$h + \mathbb{Z}^d \longmapsto (e^{2i\pi h_1}, \ldots, e^{2i\pi h_d})$

WITH $\hat{\varrho} : \mathbb{Z}^d \left(= (\mathbb{T}^d)^\wedge\right) \longrightarrow \left(H/\mathbb{Z}^d\right)^\wedge = Y_H$, WE DEFINE

AN ACTION Ψ_H OF $\mathbb{Z}^d$ ON Y_H, BY $\Psi_H(n) = + \hat{\varrho}(n)$

THM: $H_1 \subset H_2 \subset H_3 \subset \cdots$ $H = \bigcup_{n \geq 1} H_n$ FINITE INDEX INCL.

THEN (Y_H, Ψ_H) CONJUGATE (X_g, φ_g) WITH $g = \{\mathbb{Z}^d \supset H_1^* \supset H_2^* \supset \cdots\}$.

CONVERSELY, IF $g = \{\mathbb{Z}^d \supset Z_1 \supset Z_2 \supset \cdots \quad \bigcap Z_n = \{0\}\}$, THEN

(X_g, φ_g) CONJUG TO (Y_H, Ψ_H), WITH $H = \bigcup_{n \geq 1} Z_n^*$.

THM: $\mathbb{Z}^d \subset H \subset \mathbb{Q}^d$, DENSE, $d = 1$ OR 2. THEN

$\tau_\mu^1 : H^1(Y_H, \Psi_H) \longrightarrow H$ ISOMORPHISM.

THM : $\mathbb{Z}^d \subset H, H' \subset \mathbb{Q}^d$, DENSE, $d = 1, 2$.

$\mathbb{Z}^d$- ODOM. (H) AND $\mathbb{Z}^d$- ODOM. (H') ARE

CONJUGATE IFF $H = H'$

ISOMORPHIC IFF $\exists \alpha \in GL_d(\mathbb{Z}),\ \alpha(H) = H'$

COE IFF $\exists \alpha \in GL_d(\mathbb{Q}),\ |\det(\alpha)| = 1$.

ST $\alpha(H) = H'$.

Q? OE OF $\mathbb{Z}^d$- ODOMETERS.

DEF $\mathbb{Z}^d \subset H \subset \mathbb{Q}^d$. SUPERINDEX OF $\mathbb{Z}^d$ IN H

$$[[H : \mathbb{Z}^d]] = \{ [H' : \mathbb{Z}^d] ;\ \mathbb{Z}^d \subset H' \subset H,\ [H' : \mathbb{Z}^d] < \infty \}.$$

THM: $\mathbb{Z}^d \subset H \subset \mathbb{Q}^d$, DENSE. THEN

$$\iota^d_\mu : H^d(\mathbb{Z}^d\text{-ODOM}(H)) \longrightarrow \bigcup_{m \in [\![H : \mathbb{Z}^d]\!]} \frac{1}{m}\mathbb{Z}$$

$$\mathbb{Z}^d\text{-ODOM}(H) = D(\varphi_H, \varphi_H)$$

(pointed, - ORDER) - ISOM.

COR: $\mathbb{Z}^d \subset H \subset \mathbb{Q}^d$, $\mathbb{Z}^{d'} \subset H' \subset \mathbb{Q}^{d'}$, DENSE

$\mathbb{Z}^d$-ODOM(H) OE $\mathbb{Z}^{d'}$-ODOM(H') IFF SUPERINDEX(H) = SUPERINDEX(H')

CONSEQUENCE: CONJ $\Longrightarrow$ ISOM $\Longrightarrow$ COE $\Longrightarrow$ OE

$d=1$: $\Longleftarrow$ $\Longleftarrow$ $\Longleftarrow$

$d=2$: $\not\Longleftarrow$ $\not\Longleftarrow$ $\not\Longleftarrow$

CASE 2 : $\mathbb{Z}^2 \subset H \subset \mathbb{R}^2$, H DENSE.

RECALL: ① (X,φ) FREE, MIN., UNIQ. ERG $\mathbb{Z}^2$-CMS.

a) $H^1(X,\varphi)$ TORSION FREE.

b) $\tau^1_\mu : H^1(X,\varphi) \longrightarrow \mathbb{R}^2$ GRP. HOM. $\exists\ y_1, y_2 \in H^1(X,\varphi)$ st.

$$\tau^1_\mu(y_1) = (1,0) \text{ and } \tau^1_\mu(y_2) = (0,1).$$

② IF H COUNTABLE, TORSION FREE, ABEL. GRP, $u_1, u_2 \in H$ AND $\sigma : H \longrightarrow \mathbb{R}^2$ GRP HOM. ST. $\sigma(H) \subset \mathbb{R}^2$, DENSE. AND $\sigma(u_1) = (1,0)$, $\sigma(u_2) = (0,1)$

THEN H, WITH STRICT ORDERING FROM σ, (ie. $H^+ = \{ h \in H, \sigma(h) \in (\mathbb{R}^2)^{++} \}$ IS A SIMPLE DIM. GRP WITH TWO EXTR. STATES. θ_1, θ_2 WITH $\theta_i(u_j) = \delta_{ij}$.

THM H SIMPLE DIMENSION GROUP WITH TWO EXTREMAL STATES θ_1, θ_2, $u_1, u_2 \in H$, $\theta_i(u_j) = \delta_{ij}$.

THEN $\exists$ (X,φ) UNIQ ERG., FREE, MIN $\mathbb{Z}^2$-ACTION

ST

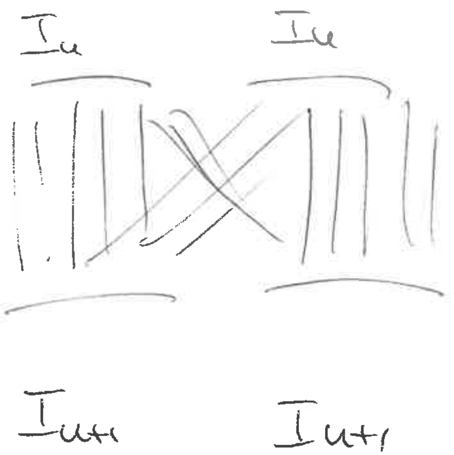
$$\begin{array}{ccc} H^1(X,\varphi) & \xrightarrow{\tau^1_\mu} & \mathbb{R}^2 \\ \wr\!\!| & & |\!\wr \\ H & \xrightarrow{(\theta_1,\theta_2)} & \mathbb{R}^2 \end{array}$$

STEPS OF THE PROOF:

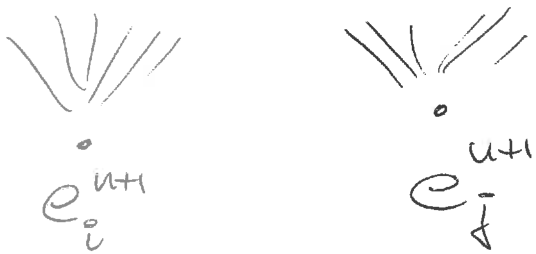
STEP 1: WRITE H AS A "NICE" INDUCTIVE LIMIT.

$$\cdots \longrightarrow \mathbb{Z}^{I_n} \oplus \mathbb{Z}^{I_n} \xrightarrow{E_n} \mathbb{Z}^{I_{n+1}} \oplus \mathbb{Z}^{I_{n+1}} \longrightarrow \cdots$$

WITH
$$E_n = \begin{pmatrix} E^n_{11} & E^n_{12} \\ E^n_{21} & E^n_{22} \end{pmatrix} = \begin{pmatrix} \text{large} & \text{small} \\ \text{small} & \text{large} \end{pmatrix}.$$



CHOOSE A "GOOD" LINEAR ORDER ON

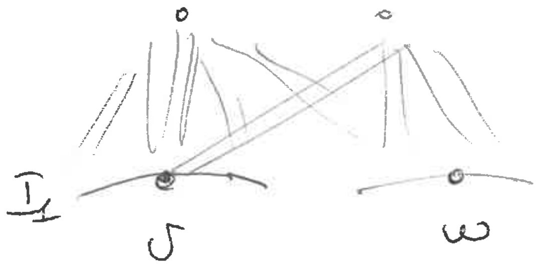


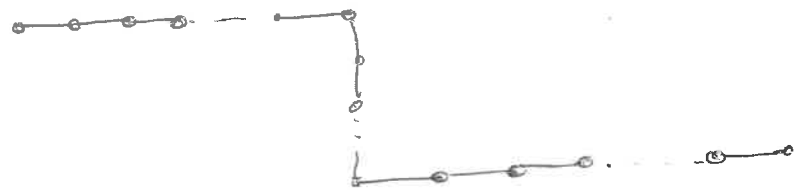
USE THIS ORDERING, TO ASSOCIATE TO ANY

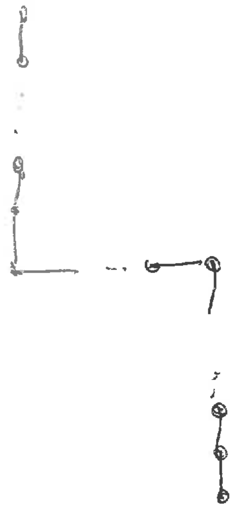
$v = (e_i^n, 0)$ A "HORIZONTAL" POLYGONAL PATH $b_n(v)$.

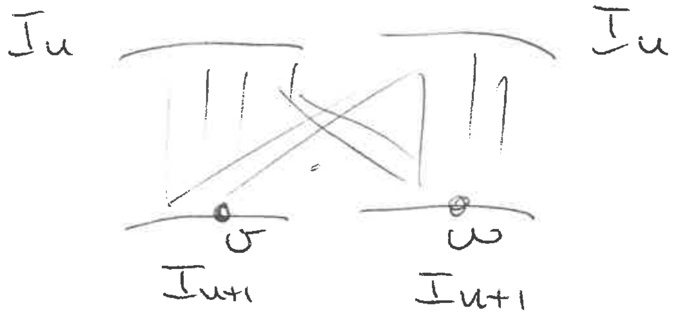
$w = (0, e_j^n)$ A "VERTICAL" ——— $b_n(w)$

ST $b_0(1,0)$  $b_0(0,1) =$ 

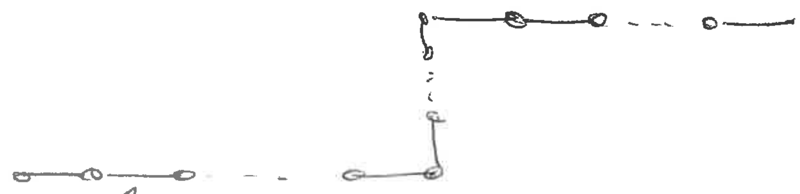


$b_1(v) =$ 

$b_1(w) =$ 



$b_{n+1}((v,0)) =$



path of level n

$$b_{n+1}((0,w)) = \cdots$$

<u>STEP 2</u>: USE THE HORIZONTAL AND VERTICAL PATHS TO CONSTRUCT A SIMPLE BRATTELI DIAG.

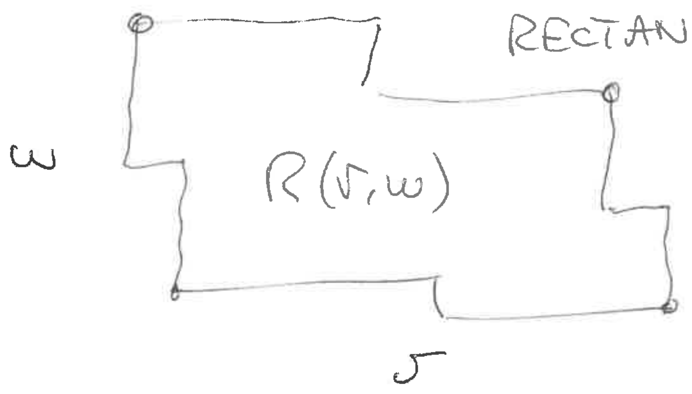
$$\mathcal{B} = (W, F), \quad W = \bigcup_{n \geq 0} W_n.$$

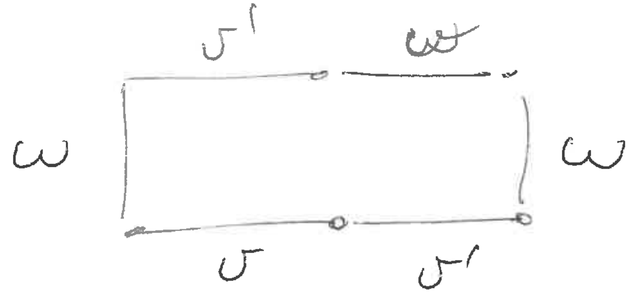
$$W_0 = \{ \square \} \; ; \quad W_1 = \{ R(v,w), \; H(v,v';w), \; H(v,w,w') \}.$$

RECTANGLE

HEXAGON.

vertical hexagon.

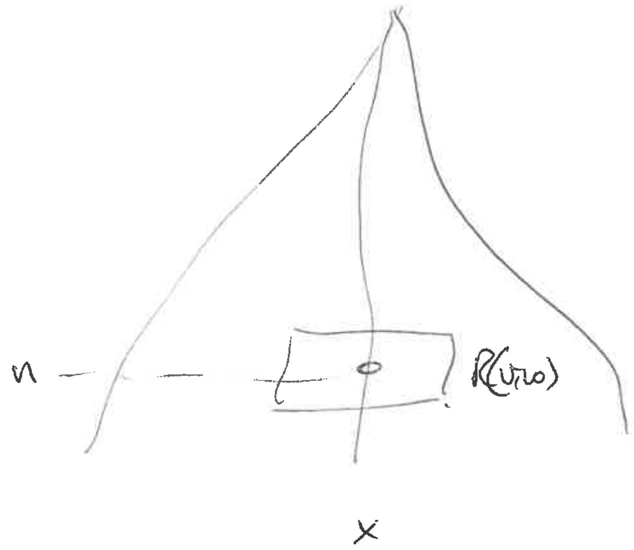




TO CONSTRUCT THE EDGES OF $B = (W, F)$, NEED TO DECOMPOSE TH POLYGONAL TILES OF W_n AS UNION OF TILES OF W_{n-1}.

(TRICKY !! USE PROPERTIES OF THE INDUCTIVE LIMIT DESCRIPTION OF H AND OF THE ORDER ON H.)

<u>STEP 3</u> : ON THE PATH SPACE OF THE SIMPLE BRATTELI DIAGRAM (W, F), DEFINE A FREE. MIN. ACTION $\mathbb{Z}^2$-ACTION



STEP 4:

ASSOCIATE TO ANY $J \in I_u \amalg I_u$. (OF H)

A COCYCLE $\theta_J \in Z^1(X_B, \varphi)$ AND SHOW

THAT

$$H^1(X, \varphi) \cong H.$$