On non-standard limits of graphs and some orbit equivalence invariants

Damien Gaboriau
joint work with Alessandro Carderi and Mikael de la Salle

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Definition

A p.m.p. discrete groupoid over (X, μ) is a groupoid $\mathcal{G}$ over (X, μ) equipped with a σ-algebra $\mathscr{A}$ and a measure $\widetilde{\mu}$ such that

- $(\mathcal{G}, \mathscr{A}, \widetilde{\mu}, s)$ and $(\mathcal{G}, \mathscr{A}, \widetilde{\mu}, t)$ are both countable fibred spaces over (X, μ) (i.e. isomorphic with measurable subsets of $X \times \mathbb{N} \to X$ equipped with product σ-algebra and product measure $\mu\times$counting)
- the maps $g \in \mathcal{G} \mapsto g^{-1} \in \mathcal{G}$ and $(g, h) \in (\mathcal{G}, s) * (\mathcal{G}, t) \mapsto gh$ are measurable.

In particular for measurable $A \subset \mathcal{G}$

$$\widetilde{\mu}(A) = \int_X |s^{-1}(x) \cap A| \; d\mu(x) = \int_X |t^{-1}(x) \cap A| \; d\mu(x).$$

We restrict ourselves to measured groupoids with countable fibers, a.k.a. countable, or r-discrete (Anantharaman-Renault 2000); but we do not assume that the base space X is standard.

The notion of cost of p.m.p. groupoids (on standard Borel spaces) has been considered by several authors (for instance Ueda 2006, Carderi (master thesis 2011), Abért-Nikolov 2012, Takimoto 2015).

Definition

Let $\mathcal{G}$ be a p.m.p. groupoid.
A graphing $\Phi = (\varphi_i)_{i\in I}$ is a collection of measurable bisections* indexed by some $\{1, 2, \cdots, N\}$ ("finite size") or by $I = \mathbb{N}$.
The (groupoid) cost of $\mathcal{G}$ is the infimum of the costs of its generating graphings:

$$\mathcal{C}(\mathcal{G}) = \inf \{\mathcal{C}(\Phi) \colon \Phi \text{ generating graphings of } \mathcal{G}\}.$$

* bisection=measurable subset $\varphi \subset \mathcal{G}$ such that the restrictions of s and t to φ are injective.

A generating graphing defines a $[0, \infty]$-valued distance on $\mathcal{G}$: $d(g, g') = \min$ length of a product w of elements of $\varphi_i^{\pm 1}$ s.t. to $w = g^{-1}g'$.

- + a "Cayley" graph structure $\Phi[x]$ on each fibre $s^{-1}(x)$.

L-Lipschitz cost

The L-Lipschitz cost of the graphed groupoid $(\mathcal{G}, \Phi)$ is the infimum of the groupoid costs of all graphings of $\mathcal{G}$ which are L-Lipschitz equivalent to Φ, $\mathcal{C}_L(\mathcal{G}, \Phi) := \inf \left\{ \mathcal{C}(\Psi) \colon \Psi \overset{\mathcal{L}ip}{\sim}_L \Phi \right\}$.

Lemma (G. 2000)

Let $(\mathcal{G}, \Phi)$ be a graphed groupoid with finite size graphing Φ. The groupoid cost of $\mathcal{G}$ can be computed inside the Lipschitz class of Φ:

$$\mathcal{C}(\mathcal{G}) = \inf_L \mathcal{C}_L(\mathcal{G}, \Phi).$$

Of course, the speed of convergence $\mathcal{C}_L(\mathcal{G}, \Phi) \underset{L \to \infty}{\longrightarrow} \mathcal{C}(\mathcal{G})$ depends on the graphed groupoid.

This dependance makes the spice of the notion of combinatorial cost for a sequence of graphed groupoids :

-1- work with a uniform Lipschitz scale L along the sequence (thus obtaining limits in n of $\mathcal{C}_L(\mathcal{G}_n, \Phi)$), and then

-2- let the Lipschitz constant L vary.

Combinatorial cost vs cost

For one finite connected graph $G = (V, E)$: $\mathcal{G}$ is the transitive equivalence relation on V_n and Φ_n is given by the edges (v, v')

$$\lim_L \mathcal{C}_L(\mathcal{G}, \Phi) = 1 - \frac{1}{|V|} = \mathcal{C}(\mathcal{G})$$

attained for instance as soon as $L \geq |V|$.

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Initially, Elek's combinatorial cost was introduced as an invariant for sequences of finite graphs $G_n = (V_n, E_n)$ with uniformly bounded degree $\leq D$:

$$c\mathcal{C}((G_n)_n) := \inf_L \liminf_{n\to\infty} \mathcal{C}_L(\mathcal{G}_n, \Phi_n)$$

where $\mathcal{G}_n$ is the equivalence relation on V_n "belonging to the same connected components" and Φ_n is given by the edges.

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QUESTION Interpret $c\mathcal{C}((G_n)_n)$ in terms of genuine cost?

Combinatorial cost vs cost

Given a non-principal ultrafilter $\mathfrak{u}$, a sequence $(\mathcal{G}_n, \Phi_n)_n$ of p.m.p. graphed groupoids on (X_n, μ_n) gives an ultraproduct p.m.p. groupoid $\mathcal{G}_\mathfrak{u}$ on $(X_\mathfrak{u}, \mu_\mathfrak{u})$ and a graphing $\Phi_\mathfrak{u}$ generating $\mathcal{G}_\mathfrak{u}$.

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Corollary (Carderi-G. -de la Salle)

If $(G_n)_n$ is a sequence of finite graphs with uniformly bounded degree, the combinatorial cost is the infimum over all non principal ultrafilters of the costs of the ultraproduct groupoids $\mathcal{G}_\mathfrak{u}$ of the associated groupoids $\mathcal{G}_n$:

$$c\mathcal{C}((G_n)_n) = \inf\{\mathcal{C}(\mathcal{G}_\mathfrak{u}) : \mathfrak{u} \textit{ non principal ultrafilter}\}.$$

It is not clear to us whether the infimum is realized by some $\mathfrak{u}$.

Cost and $\mathfrak{u}$-combinatorial cost

Definition (inspired by Elek 2007)

Let $(\mathcal{G}_n, \Phi_n)_n$ be a sequence of graphed p.m.p. groupoids of bounded size. The $\mathfrak{u}$-combinatorial cost of the sequence is

$$c\mathcal{C}_{\mathfrak{u}}((\mathcal{G}_n, \Phi_n)_n) := \inf_L \lim_{n \in \mathfrak{u}} \mathcal{C}_L(\mathcal{G}_n, \Phi_n).$$

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Theorem (Carderi-G.-de la Salle)

For every sequence $(a_n)_n$ of p.m.p. actions of a finitely generated group Γ we have (for groupoid cost):

$$c\mathcal{C}_{\mathfrak{u}}((a_n)_n) = \mathcal{C}(a_{\mathfrak{u}}) \geq \lim_{\mathfrak{u}} \mathcal{C}(a_n) \geq \liminf_{n\to\infty} \mathcal{C}(a_n) \geq \mathscr{C}_*(\Gamma).$$

If the ultraproduct action $a_{\mathfrak{u}}$ is free and Γ has fixed price, then

$$\mathcal{C}(a_{\mathfrak{u}}) = c\mathcal{C}_{\mathfrak{u}}((a_n)_n) = \lim_{\mathfrak{u}} \mathcal{C}(a_n) = \mathscr{C}_*(\Gamma).$$

Cost and $\mathfrak{u}$-combinatorial cost

The $\mathfrak{u}$-combinatorial cost of the sequence $(\mathcal{G}_n, \Phi_n)_n$

$$c\mathcal{C}_{\mathfrak{u}}((\mathcal{G}_n, \Phi_n)_n) := \inf_L \lim_{n\in\mathfrak{u}} \mathcal{C}_L(\mathcal{G}_n, \Phi_n).$$

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Fails without finite generation!
<u>EXAMPLE</u> : $\Gamma = \mathbf{F}_\infty \times \mathbb{Z}$ with a decreasing sequence $(\Gamma_n)_n$ of finite index normal subgroups with trivial intersection.
Sequence of actions $a_n : \Gamma \curvearrowright \Gamma/\Gamma_n$ (groupoid costs!)

$$\underbrace{c\mathcal{C}_{\mathfrak{u}}((a_n)_n)}_{=+\infty} \neq \underbrace{\mathcal{C}(a_{\mathfrak{u}})}_{=1} \not\geq \underbrace{\lim_{\mathfrak{u}} \mathcal{C}(a_n)}_{=+\infty} \geq \underbrace{\liminf_{n\to\infty} \mathcal{C}(a_n)}_{=+\infty} \geq \underbrace{\mathscr{C}_*(\Gamma)}_{=1}.$$

Cost and $\mathfrak{u}$-combinatorial cost

This **THM** for actions of a f.g. Γ

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Without any boundedness assumption we still have:

$$c\mathcal{C}_{\mathfrak{u}}((\mathcal{G}_n, \Phi_n)_n) \quad \geq \quad \begin{cases} \lim_{\mathfrak{u}} \mathcal{C}(\mathcal{G}_n) \\ \mathcal{C}(\mathcal{G}_{\mathfrak{u}}) \end{cases}$$

Contrarily to the other inequalities (previous example):

$$c\mathcal{C}_{\mathfrak{u}}((\mathcal{G}_n, \Phi_n)_n) \leq \quad \mathcal{C}(\mathcal{G}_{\mathfrak{u}}) \quad \geq \lim_{\mathfrak{u}} \mathcal{C}(\mathcal{G}_n)$$

Cost and $\mathfrak{u}$-combinatorial cost – reversing inequality ?

THM

$$c\mathcal{C}_{\mathfrak{u}}((\mathcal{G}_n, \Phi_n)_n) = \mathcal{C}(\mathcal{G}_{\mathfrak{u}}) \geq \lim_{\mathfrak{u}} \mathcal{C}(\mathcal{G}_n).$$

• The reverse inequality $\mathcal{C}(\mathcal{G}_{\mathfrak{u}}) \leq \lim_{\mathfrak{u}} \mathcal{C}(\mathcal{G}_n)$ does not hold even under finiteness assumptions.

EXAMPLE $\Gamma = \mathbf{F}_2$ finite generating set S + a decreasing sequence $(\Gamma_n)_n$ of finite index normal subgroups with trivial intersection.

– Sequence of finite Schreier graphs $(\Gamma/\Gamma_n, S)_n$
– sequence $\Phi_n = (\varphi_s)_{s\in S}$ of graphings on (finite sets) $X_n = \Gamma/\Gamma_n$
– transitive equivalence relation $\mathcal{G}_n$ thus $\mathcal{C}(\mathcal{G}_n) = 1 - \frac{1}{[\Gamma:\Gamma_n]}$.
– Ultraproduct groupoid $\mathcal{G}_{\mathfrak{u}}$ =free action of $\Gamma \curvearrowright X_{\mathfrak{u}} \rightsquigarrow \mathcal{C}(\mathcal{G}_{\mathfrak{u}}) = 2$.

$$c\mathcal{C}_{\mathfrak{u}}((\mathcal{G}_n, \Phi_n)_n) = \underbrace{\mathcal{C}(\mathcal{G}_{\mathfrak{u}})}_{=2} \not\leq \underbrace{\lim_{\mathfrak{u}} \mathcal{C}(\mathcal{G}_n)}_{=1}$$

This type of example justifies the introduction of Elek's combinatorial cost (2007) or of the groupoid cost (Abért-Nikolov 2012) (isotropy subgroups are taken into account).

Cost and $\mathfrak{u}$-combinatorial cost – boundedness assumptions?

THM

$c\mathcal{C}_{\mathfrak{u}}((\mathcal{G}_n, \Phi_n)_n) = \mathcal{C}(\mathcal{G}_{\mathfrak{u}}) \geq \lim_{\mathfrak{u}} \mathcal{C}(\mathcal{G}_n)$.

- The inequality $\mathcal{C}(\mathcal{G}_{\mathfrak{u}}) \geq \lim_{\mathfrak{u}} \mathcal{C}(\mathcal{G}_n)$ does not hold without $\lim_{\mathfrak{u}} \mathcal{C}(\Phi_n) = \mathcal{C}(\Phi_{\mathfrak{u}}) < \infty$.

EXAMPLE Again, $\Gamma = \mathbf{F}_2$ + finite Schreier graphs $(\Gamma/\Gamma_n, S)_n$.
Instead of $\Phi_n = (\varphi_s)_{s \in S}$ subdivide the bisections φ_s into singletons, $\Phi'_n = (\varphi'_{v,s} := (v, sv))_{v \in \Gamma/\Gamma_n, s \in S}$ each $\tilde{\mu}(\varphi'_{v,s}) = \frac{1}{[\Gamma:\Gamma_n]}$.
Thus $\mathcal{C}(\Phi'_n) = |S|$ and $\mathcal{C}(\mathcal{G}_n) = 1 - \frac{1}{[\Gamma:\Gamma_n]}$.
But $\mathcal{G}_{\mathfrak{u}}$ is the trivial groupoid $X_{\mathfrak{u}} \rightsquigarrow$ cost 0,

$$\underbrace{\mathcal{C}(\mathcal{G}_{\mathfrak{u}})}_{=0} \not\geq \underbrace{\lim_{\mathfrak{u}} \mathcal{C}(\mathcal{G}_n)}_{=1}.$$

Loss of mass at infinity
$\rightsquigarrow$ Uniform summability is needed even even if $\sup_n \mathcal{C}(\Phi_n) < \infty$

Farber sequences

Farber sequences= sequ. of finite index subgroups $(\Gamma_n)_n$ of Γ s. t. the ultraproduct action $a_{\mathfrak{u}}$ of the actions $\Gamma \curvearrowright^{a_n} \Gamma/\Gamma_n$ is free.

Theorem

Let Γ be a group finitely generated by S. For every non necessarily nested Farber sequence $(\Gamma_n)_n$ of finite index subgroups, we have:
$\mathscr{C}^*(\Gamma) \geq c\mathcal{C}((\mathrm{Sch}(\Gamma/\Gamma_n, S))_n) \geq \liminf_{n\to\infty} \frac{\mathrm{rank}(\Gamma_n)-1}{[\Gamma:\Gamma_n]} + 1 \geq \mathscr{C}_*(\Gamma) \geq \beta_1^{(2)}(\Gamma) + 1.$

The convergence of the sequence $\left(\frac{\mathrm{rank}(\Gamma_n)-1}{[\Gamma:\Gamma_n]}\right)_n$ is not at all clear in general.

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Corollary

If Γ is finitely generated by S and $(\Gamma_n)_n$ is a nested Farber sequence, then:

$$\mathcal{C}(a_{\mathfrak{u}}) = c\mathcal{C}_{\mathfrak{u}}(a_n) = c\mathcal{C}(\mathrm{Sch}(\Gamma/\Gamma_n, S)_n) = \lim_{n\to\infty} \mathcal{C}(a_n).$$

Farber sequences

Corollary

If Γ *is finitely generated by* S, *has fixed price* $\mathscr{C}_*(\Gamma)$, *and* $(\Gamma_n)_n$ *is any non necessarily nested Farber sequence, then:*

$$\lim_{n\to\infty} \frac{\mathrm{rank}(\Gamma_n) - 1}{[\Gamma : \Gamma_n]} + 1 = c\mathcal{C}(\mathrm{Sch}(\Gamma/\Gamma_n, S)) = \mathscr{C}_*(\Gamma).$$

-An extension of a theorem of Abért and Nikolov (2012) (nested).
- Gives a converse to Abért-Nikolov-Gelander (2017) where an inequality $\lim_{n\to\infty} \frac{\mathrm{rank}(\Gamma_n)-1}{[\Gamma:\Gamma_n]} + 1 \leq c\mathcal{C}(\mathrm{Sch}(\Gamma/\Gamma_n, S))$ was obtained.
- Similar (independent) results of Abért-Tóth for sequences of graphs.

Stuck-Zimmer groups

A countable group Γ is called a Stuck-Zimmer group if any p.m.p. Γ-action on a standard Borel space with no finite orbits has a.s. finite stabilizers.

The main source for Stuck-Zimmer property examples come from algebraic groups with higher rank conditions such as irreducible lattices in semisimple real Lie groups.

Theorem

Let Γ be a torsion free f.g. Stuck-Zimmer group and let $(\Gamma_n)_n$ be a sequence of finite index subgroups of Γ such that $[\Gamma : \Gamma_n] \to \infty$.

1. *The action $\Gamma \curvearrowright \Gamma/\Gamma_n$ defines a sofic approximation of Γ.*
2. *If Γ has fixed price $\mathscr{C}_*(\Gamma)$, then $\lim_n \frac{(\mathrm{rank}(\Gamma_n)-1)}{[\Gamma:\Gamma_n]} = \mathscr{C}_*(\Gamma) - 1$.*

Relatively Stuck-Zimmer groups

Theorem (Carderi-G. -de la Salle)

Let $\Lambda < \mathrm{SL}(d,\mathbb{Z})$ a subgroup whose action on $\mathbb{R}^d$ is $\mathbb{Z}$-strongly irreducible. Let $(\Gamma_n)_n$ be any sequence of finite index subgroups of $\Gamma = \Lambda \ltimes \mathbb{Z}^d$ such that $[\mathbb{Z}^d : \Gamma_n \cap \mathbb{Z}^d] \underset{n\to\infty}{\longrightarrow} \infty$. Then it defines a sofic approximation of Γ and*

$$\lim_n \frac{(\mathrm{rank}(\Gamma_n) - 1)}{[\Gamma : \Gamma_n]} = 0.$$

* No finite union of infinite index subgroups of $\mathbb{Z}^d$ is Λ-invariant.

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$$\lim_n \frac{(\mathrm{rank}(\Gamma_n) - 1)}{[\Gamma : \Gamma_n]} = 0.$$

* No finite union of infinite index subgroups of $\mathbb{Z}^d$ is Λ-invariant.

Corollary (Carderi-G. -de la Salle)

Let Λ *be any subgroup of* $\mathrm{SL}(2,\mathbb{Z})$ *containing a hyperbolic element. Any sequence* $(\Gamma_n)_n$ *of finite index subgroups of* $\Gamma := \Lambda \ltimes \mathbb{Z}^2$ *such that* $[\mathbb{Z}^2 : \Gamma_n \cap \mathbb{Z}^2] \underset{n\to\infty}{\longrightarrow} \infty$

$$\lim_n \frac{(\mathrm{rank}(\Gamma_n) - 1)}{[\Gamma : \Gamma_n]} = 0.$$

Factors, actions, fixed price

Factors, actions

Let $\mathcal{G}, \mathcal{G}'$ be p.m.p. groupoids over prob. spaces $(X, \mu), (X', \mu')$.

Definition

A homomorphism $T : \mathcal{G}' \to \mathcal{G}$ is a measurable map commuting with source, target, products and inversion. It sends units to units ($X = \{g \in \mathcal{G} : s(g) = t(g), g^2 = g\}$).
T is p.m.p. if the induced map $X' \to X$ sends μ' to μ.

Definition (Factor map)

A homorphism $T : \mathcal{G}' \to \mathcal{G}$ is a factor map if T maps $\widetilde{\mu}'$ to $\widetilde{\mu}$.

A factor map is thus essentially onto and "class-bijective", i.e. $T(g')$ is a unit iff g' itself is a unit, $\widetilde{\mu}'$-a.e. $g' \in \mathcal{G}'$.
EXAMPLE A p.m.p. action of Γ on X is just a factor $\mathcal{G}_{\Gamma \curvearrowright X} \to \Gamma$ sending (γ, x) to γ.

Cost, fixed price

A factor $\mathcal{G}' \to \mathcal{G}$ is interpreted as a p.m.p. action $\mathcal{G} \curvearrowright X'$.
The action is free when $\mathcal{G}'$ is a principal groupoid (i.e. equivalence relation).

The infimum cost $\mathscr{C}_*(\mathcal{G})$ of a p.m.p. groupoid $\mathcal{G}$ is the infimum of the costs of all p.m.p. groupoids $\mathcal{H}$ which factor onto it:

$$\mathscr{C}_*(\mathcal{G}) := \inf\{\mathcal{C}(\mathcal{H}) : \mathcal{H} \text{ p.m.p. groupoid which factors onto } \mathcal{G}\}.$$

The p.m.p. groupoid $\mathcal{G}$ has fixed price if $\mathcal{C}(\mathcal{H}) = \mathscr{C}_*(\mathcal{G})$ for every principal p.m.p. groupoid $\mathcal{H}$ which factors onto $\mathcal{G}$.

Question (Fixed price problem for p.m.p. goupoids)

Does there exist a p.m.p. groupoid that does not have fixed price?

A similar question has been raised by Popa-Shlyakhtenko-Vaes.

L^2-Betti numbers

L^2-Betti numbers

We obtain extension of the approximation theorems of Lück and Farber.

Theorem (Carderi-G. -de la Salle)

Let Γ be a count. group acting freely cocompactly on a k-connected simplicial complex. Let $(\Gamma_n)_n$ be a (non nec. nested) Farber sequence of finite index subgroups. Then for every $i \leq k$

$$\lim_{n\to\infty} \frac{b_i(\Gamma_n)}{[\Gamma : \Gamma_n]} = \beta_i^{(2)}(\Gamma).$$

L^2-Betti numbers

More generally, we obtain a generalization of Bergeron-Gab. for non nested, non Farber sequences:

Theorem (Carderi-G. -de la Salle)

Let Γ be a count. group acting freely cocompactly on a k-connected simplicial complex Σ. Let $(\Gamma_n)_n$ be any sequence of finite index subgroups Let $a_\mathfrak{u} : \Gamma \curvearrowright X_\mathfrak{u}$ be the ultraproduct of the actions $a_n : \Gamma \curvearrowright \Gamma/\Gamma_n$ and $\mathcal{N}_\mathfrak{u} := \{(x, \gamma) \in X_\mathfrak{u} \times \Gamma : \gamma x = x\}$ its totally isotropic subgroupoid. Then for every $i \leq k$

$$\lim_{n \in \mathfrak{u}} \frac{b_i(\Gamma_n \backslash \Sigma)}{[\Gamma : \Gamma_n]} = \beta_i^{(2)}(\mathcal{N}_\mathfrak{u} \backslash (X_\mathfrak{u} \times \Sigma), \mathcal{R}_\mathfrak{u}),$$

where $\mathcal{R}_\mathfrak{u} = \mathcal{G}_\mathfrak{u}/\mathcal{N}_\mathfrak{u}$ is the p.m.p. equivalence relation of the action $a_\mathfrak{u}$.

Coarse structures and ultraproduct groupoids

Inspired by by Roe's topological analogues

Definition

A coarse structure on a p.m.p. groupoid $\mathcal{G}$ is a collection $\mathcal{E}$ of measurable subsets of $\mathcal{G}$ (= controlled sets), which contains the units and is closed under measurable subsets, inverses, products and finite unions.

A coarse structure $\mathcal{E}$ on $\mathcal{G}$
– is generating if $\mathcal{G}$ is the union of all controlled sets.
– has bounded geometry if for every $E \in \mathcal{E}$, $\exists M \in \mathbb{N}$ s.t. $|E \cap s^{-1}(x)| < M$, $\forall^* x \in X$.

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Definition

A coarse structure on a sequence $(\mathcal{G}_n)_n$ of p.m.p. groupoids over (X_n, μ_n) is a collection $\mathcal{E}$ of sequences $(E_n)_n$ of measurable subsets $E_n \subset \mathcal{G}_n$ (= controlled sequences) which is closed under the same operations.

A coarse structure $\mathcal{E}$ on $(\mathcal{G}_n)_n$ has bounded geometry if for every controlled sequence $(E_n)_n \in \mathcal{E}$, $\exists M \in \mathbb{N}$ s.t. $|E_n \cap s^{-1}(x)| < M$ $\forall n, \forall^* x \in X_n$.

Example (Coarse structure given by a graphing)

If $(\mathcal{G}, \Phi)$ is a graphed p.m.p. groupoid, then the collection $\mathcal{E}_\Phi$ of measurable subsets $E \subset \mathcal{G}$ on which the length function is bounded

$$\mathcal{G} \ni g \overset{\ell_\Phi}{\longmapsto} \min \left\{ k \in \mathbb{N} : g \in \overline{\{\varphi_1^{\pm 1}, \dots, \varphi_k^{\pm 1}\}^k} \right\}$$

(i.e. $\min k$ s.t. g is a product of $\leq k$ of the first k letters) gives a countably generated coarse structure with bounded geometry.

countably generated: there is a sequence $(E_k)_k \in \mathcal{E}$ such that every controlled set E is contained in one of the E_k (for some k).
bounded geometry: if for every $E \in \mathcal{E}$, $\exists M \in \mathbb{N}$ s.t. $|E \cap s^{-1}(x)| < M$, $\forall^* x \in X$.

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(i.e. $\min k$ s.t. g is a product of $\leq k$ of the first k letters) gives a countably generated coarse structure with bounded geometry.

When a group G is generated by S, take $\Phi = S$. If $|S| < \infty$ then ℓ_Φ is the usual word length. If $|S| = \infty$, then ℓ_Φ is not bounded on S and thus S it is not controlled for the coarse structure.

Example (Coarse structure given by a graphing)

If $(\mathcal{G}, \Phi)$ is a graphed p.m.p. groupoid, then the collection $\mathcal{E}_\Phi$ of measurable subsets $E \subset \mathcal{G}$ on which the length function is bounded

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Example (Coarse structure given by a sequence of graphings)

Similarly, if $(\mathcal{G}_n, \Phi_n)_n$ is a sequence of graphed groupoids, then a natural countably generated coarse structure with bounded geometry on $(\mathcal{G}_n)_n$ is given by the collection of all the sequences $(E_n \subset \mathcal{G}_n)_n$ satisfying $\sup_{n\in\mathbb{N}, g_n \in E_n} \ell_{\Phi_n}(g_n) < \infty$.

countably generated: there is a sequence $(E_{k,n})_k \in \mathcal{E}$ such that every controlled sequence $(E_n)_n$ is contained in one of the $(E_{k,n})_n$ (for some k).

Proposition (Ultraproducts of coarse p.m.p. groupoids)

Let $(\mathcal{G}_n)_n$ be a sequence of p.m.p. groupoids on (X_n, μ_n), $\mathcal{E}$ a countably generated coarse structure with bounded geometry on $(\mathcal{G}_n)_n$.

- *The ultraproduct $X_\mathfrak{u} = \left(\prod_{n\in\mathbb{N}} X_n\right) / \sim_\mathfrak{u}$ of the sequence $(X_n, \mu_n)_n$ has a probability measure $\mu_\mathfrak{u}$ s.t. $\mu_\mathfrak{u}([A_n]_\mathfrak{u}) = \lim_{n\in\mathfrak{u}} \mu_n(A_n)$.*
- *p.m.p. groupoid over $(X_\mathfrak{u}, \mu_\mathfrak{u})$ for the natural groupoid operations.*

$$\mathcal{G}_\mathfrak{u} := \{(g_n)_n : \exists (E_n)_n \in \mathcal{E}, g_n \in E_n\} / \sim_\mathfrak{u}$$

- *$\mathcal{E}_\mathfrak{u} := \{[E_n]_\mathfrak{u} : (E_n)_n \in \mathcal{E}\}$ is a countably generated and generating coarse structure with bounded geometry on $\mathcal{G}_\mathfrak{u}$.*
- *If $\mathcal{E}$ comes from a sequence of graphings $\Phi_n = (\varphi_n^i)_i$, then $\mathcal{G}_\mathfrak{u} = \{(g_n)_n : \sup_n \ell_{\Phi_n}(g_n) < \infty\} / \sim_\mathfrak{u}$ and $\mathcal{E}_\mathfrak{u}$ is the coarse structure associated to the generating graphing $\Phi_\mathfrak{u} = ([\varphi_n^i]_\mathfrak{u})_i$.*

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