Groupoids and groupoid C^*-algebras from minimal dynamical systems

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GROUPOIDS AND OPERATOR ALGEBRAS
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- Classification of C^*-algebras

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- Minimal dynamical systems

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- Minimal equivalence relations

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Let A, B be simple separable unital C^*-algebras with finite nuclear dimension and which satisfiy the UCT. Then $A \cong B$ if and only if $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$.

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In particular, $\dim(C(X)) = \dim X$ for any locally compact Hausdorff space X.

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3. (X, φ) finitely many ergodic measures $\implies$ mean dimension zero.

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Example: If M is a manifold with nonzero Euler characteristic then M does not admit a minimal homeomorphism [Fuller, 1953].

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Theorem (Deeley, Putnam, S.)

Let Y be a finite CW complex. Then there exists an infinite compact metric space X admitting a minimal homeomorphism and satisfying $K^*(Y) \cong K^*(X)$.

The first secret ingredient - minimal "point-like" systems

[DPS '15] For every $d \in \mathbb{N} \setminus \{0\}$ there exists a compact metric space Z admitting a minimal homeomorphism $\zeta : Z \to Z$ such that

$$\dim Z \geq d, \quad K^0(Z) \cong \mathbb{Z}, \quad K^1(Z) = 0.$$

Moreover, we can arrange that (Z, ζ) has any finite number of ergodic probability measures.

Another ingredient - Skew products

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Put $X := Z \times Y$. Let $\mathrm{Homeo}_s(X) \subset \mathrm{Homeo}(X)$ denote the subset of homeomorphisms $G : X \to X$ of the form

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Let

$$\mathcal{S}(\varphi) := \{G^{-1} \circ (\varphi \times \mathrm{id}) \circ G \mid G \in \mathrm{Homeo}_s(Y).\}$$

Theorem (Glasner–Weiss '79)

Let Γ be a subgroup of Homeo(Y) which is pathwise connected and such that (Y, Γ) is minimal. Then for a residual subset $\mathcal{R} \subset \overline{\mathcal{S}(\varphi)}$, the system (X, α) is minimal for every $\alpha \in \mathcal{R}$.

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In particular, if Y is a finite CW complex then $Y \times Q$ [West, '70].

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Thus, let $X := Z \times Q \times Y$ where (Z, ζ) is a minimal point-like system. Then $K^*(X) \cong K^*(Y)$ and there exists $\varphi : X \to X$ minimal. Furthermore

$$K_0(\mathrm{C}^*(\mathcal{R}_\varphi)) \cong K_1(\mathrm{C}^*(\mathcal{R}_\varphi)) \cong K^0(Y) \oplus K^1(Y).$$

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There are other interesting (simple separable unital nuclear) C^*-algebras we can associate to (X, φ), which we call *orbit-breaking algebras*.

They are groupoid C^*-algebras defined by certain minimal equivalence relations $\mathcal{R} \subset \mathcal{R}_\varphi$.

Orbit-breaking

The ingredients for our orbit-breaking algebras are an infinite compact metric space X, a minimal homeomorphism $\varphi : X \to X$ and a nonempty closed subset $W \subset X$ meeting every φ-orbit at most once.

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Then $\mathrm{C}^*(\mathcal{R}_W)$ is simple, separable, unital, nuclear.

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$$\mathrm{C}^*(\mathcal{R}_W) = \mathrm{C}^*(C(X), uC_0(X \setminus W)).$$

Theorem (DPS)

Let Y be a finite CW complex. Then there exists a minimal dynamical system (X, φ) and a closed nonempty subset $W \subset X$ meeting every φ-orbit at most once such that

$$K_0(\mathrm{C}^*(\mathcal{R}_W)) \cong K^0(Y), \qquad K_1(\mathrm{C}^*(\mathcal{R}_W)) \cong K^1(Y).$$

Furthermore, the C^*-algebras $\mathrm{C}^*(\mathcal{R}_W)$ has no nontrivial projections.

Theorem (DPS)

Let G_0 and G_1 be arbitrary countable abelian groups and W a finite-dimensional compact metric space such that

$$K^0(W) = \mathbb{Z} \oplus G_0, \quad K^1(W) = G_1,$$

and let Δ be a finite-dimensional Choquet simplex. Then there exists a minimal dynamical system (Z, ζ) with $W \subset Z$ meets every ζ-orbit at most once such that

$$K_0(\mathrm{C}^*(\mathcal{G}_W)) \cong \mathbb{Z} \oplus G_0, \qquad K_1(\mathrm{C}^*(\mathcal{G}_W)) \cong G_1, \qquad T(\mathrm{C}^*(\mathcal{G}_W)) \cong \Delta.$$

Again, $\mathrm{C}(\mathcal{G}_W)$ has no nontrivial projections.

Theorem (DPS)

Let G_0 be a simple countable dimension group, T a countable abelian torsion group, G_1 a countable abelian group. Then there exists a minimal dynamical system (X, φ) and a nonempty closed subset $Y \subset X$ meeting every φ-orbit at most once such that

$$K_0(\mathrm{C}^*(\mathcal{R}_Y)) \cong T \oplus G_0, \qquad K_1(\mathrm{C}^*(\mathcal{R}_Y)) \cong G_1.$$

Moreover, $\mathrm{C}^*(\mathcal{R}_Y)$ has real rank zero.

Orbit-breaking: projectionless, infinite-dimensional

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[Putnam] There is a six term exact sequence given by

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Let (Z, ζ) be a uniquely ergodic and minimal with $K_0(Z) = \mathbb{Z}$ and $K_1(Z) = 0$ and let $z_0 \in Z$. Then $\mathrm{C}^*(\mathcal{R}_{\{z_0\}}) \cong \mathcal{Z}$.

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More generally, let $X = Z \times Q \times Y$ as before. Let $W := \{z_0\} \times Q \times Y$ for some $z_0 \in Z$. Then

$$K_*(\mathrm{C}^*(\mathcal{R}_W)) \cong K^*(Y).$$

More orbit-breaking: projectionless and finite-dimensional

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Proposition

There exists an embedding $\iota : Y \hookrightarrow Z$ in such a way that $\iota(Y)$ meets every ζ-orbit at most once.

More orbit-breaking: projectionless and finite-dimensional

With (Z, ζ) and Y as above, we break the orbit at Y to get $\mathrm{C}^*(\mathcal{R}_Y) \subset \mathrm{C}^*(\mathcal{R}_\zeta)$ which satisfies

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$\dim(Z) < \infty$ so $\mathrm{C}^*(\mathcal{R}_Y)$ will always have finite nuclear dimension. Furthermore, $\mathrm{C}^*(\mathcal{R}_\zeta)$ is always projectionless, so $\mathrm{C}^*(\mathcal{R}_Y)$ is always projectionless.

More orbit-breaking: projectionless and finite-dimensional

With (Z, ζ) and Y as above, we break the orbit at Y to get $\mathrm{C}^*(\mathcal{R}_Y) \subset \mathrm{C}^*(\mathcal{R}_\zeta)$ which satisfies

$$K_0(\mathrm{C}^*(\mathcal{R}_Y)) \cong \mathbb{Z} \oplus G_0, \quad K_1(\mathrm{C}^*(\mathcal{R}_Y)) \cong G_1.$$

$\dim(Z) < \infty$ so $\mathrm{C}^*(\mathcal{R}_Y)$ will always have finite nuclear dimension. Furthermore, $\mathrm{C}^*(\mathcal{R}_\zeta)$ is always projectionless, so $\mathrm{C}^*(\mathcal{R}_Y)$ is always projectionless.

Since we can arrange that (Z, ζ) has any number of ergodic measures, we can arrange that $\mathrm{C}^*(\mathcal{R}_Y)$ have any number, up to a continuum, of extreme tracial states.

Orbit-breaking: many projections (real rank zero)

The ingredients in this construction are a Cantor minimal system (K, φ) and a compact connected finite-dimensional metric space Y. We will fatten up the Cantor system until we can embed Y.

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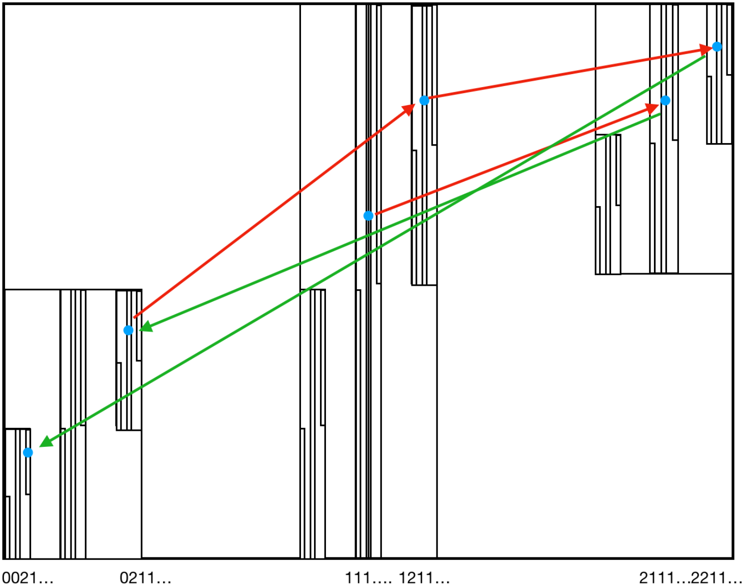
We will then embed Y into one of the fibres over the Cantor system; we will break the orbit at Y.

Factoring onto 3-odometer



Dynamical aside...

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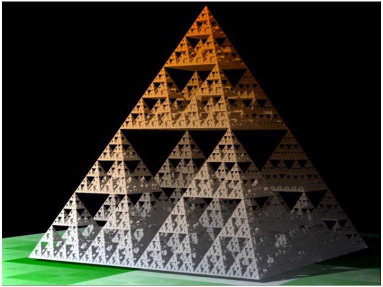
So, for example we can have a dynamical system on a space that looks like a Cantor set from one angle and Menger curve from another! (Other examples include the Sierpiński gasket, Sierpiński pyramid, another Cantor set, ...)



Orbit-breaking algebras with projections

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There is (K, φ) a Cantor minimal system with

$$K_0(\mathrm{C}^*(\mathcal{R}_\varphi)) = G_0,$$

and a compact connected finite-dimensional metric space Y such that

$$K^0(Y) = \mathbb{Z} \oplus T, \quad K^1(Y) = G_1.$$

Orbit-breaking algebras with projections

Let $n >> \dim(Y)$. Then we can embed Y into I^n, and there is some point $y_0 \in K$ whose fibre in X is I^n. So we consider Y as a subset of this fibre, hence as a subset of our space X.

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Theorem

$\mathrm{C}^*(\mathcal{R}_Y)$ *has real rank zero and*

$$K_0(\mathrm{C}^*(\mathcal{R}_Y)) \cong T \oplus G_0, \qquad K_1(\mathrm{C}^*(\mathcal{R}_Y)) \cong G_1.$$

Merci!