Amenability of actions and Crossed Products

joint work with

Alcides Buss and Rufus Willett

Groupoids and Operator Algebras.

(The Jean Fest)

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Siegfried Echterhoff

Westfälische Wilhelms-Universität Münster
Motivation

Amenability of actions of groupoids on spaces or C*-algebras has been a central theme in the (partially joint) work of Jean Renault and Claire Anantharaman-Delaroche.
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$$G \text{ is amenable} \iff C^*_{\max}(G) = C^*_r(G).$$
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**Question:** Does a similar characterization of amenability hold for locally compact groupoids or crossed products by actions of groups or groupoids?

$$A \rtimes_{\text{max}} G = A \rtimes_r G \iff \alpha : G \to \text{Aut}(A) \text{ amenable}.$$
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$$A \rtimes_{\text{max}} G = A \rtimes_r G \iff \alpha : G \to \text{Aut}(A) \text{ amenable.}$$

In this talk we only consider actions of discrete groups $G$ (which includes the case of the groupoids $X \rtimes G$)!
Outline of this lecture

- Definition of amenability (after Anantharaman-Delaroche)
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- $G$-injectivity and the $G$-WEP
- On a conjecture of Ozawa.
Let $\alpha : G \to \text{Aut}(A)$ be an action. A function $\theta : G \to A$ is called of positive type, if for all $g_1, \ldots, g_l \in G$ we have

$$(\alpha_{g_i}(\theta(g_i^{-1}g_j)))_{i,j} \in M_l(A)^+. $$
Amenable actions (after Anantharaman-Delaroche)

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Definition (Anantharaman-Delaroche ’87) A \( G \)-algebra \( A \) is called \textit{amenable} if there exists a net \( \{\theta_i : G \to \mathbb{Z}(A^{**})\}_{i \in I} \) of \textit{finitely supported positive type} functions such that

1. \( \theta_i(e) \leq 1_{A^{**}} \) \( \forall i \in I \)
2. \( \theta_i(g) \to 1_{A^{**}} \) ultraweakly \( \forall g \in G \),
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Definition (Anantharaman-Delaroche ’87) A $G$-algebra $A$ is called amenable if there exists a net $\{\theta_i : G \to \mathbb{Z}(A^{**})\}_{i \in I}$ of finitely supported positive type functions such that

1. $\theta_i(e) \leq 1_{A^{**}} \quad \forall i \in I$
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We call $A$ strongly amenable if there exists a net of finitely supported positive type functions $\theta_i : G \to ZM(A)$ such that

1. $\theta_i(e) \leq 1_{M(A)} \quad \forall i \in I$
2. $\theta_i(g) \to 1_{M(A)}$ strictly $\forall g \in G.$
Amenable actions

The following results are all due to Anantharaman-Delaroche:

- If $A = C_0(X)$ is commutative, then
  $A$ strongly amenable $\iff A$ amenable $\iff X \rtimes G$ amenable.
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- If $A = C_0(X)$ is commutative, then
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- $A$ is amenable
  $\iff \exists$ norm one $G$-projection $P : \ell^\infty(G, A^{**}) \to A^{**}$
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• If $A$ is amenable, then $A \rtimes_{\max} G = A \rtimes_r G$. 
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- If $A$ is amenable, then $A \rtimes_{\text{max}} G = A \rtimes_{\text{r}} G$.

- If $A$ is nuclear, then
  $A$ is amenable $\iff A \rtimes_{\text{max}} G$ is nuclear $\iff A \rtimes_{\text{r}} G$ is nuclear.
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- If $A$ is amenable, then $A \rtimes_{\text{max}} G = A \rtimes_{r} G$.

- If $A$ is nuclear, then $A$ is amenable $\iff A \rtimes_{\text{max}} G$ is nuclear $\iff A \rtimes_{r} G$ is nuclear.

Example (Suzuki, 2018) For every non-amenable, exact, discrete group $G$ there exists a unital, simple, nuclear $G$-algebra $A$ such that $A \rtimes_{\text{max}} G = A \rtimes_{r} G$ is nuclear.

Thus: $A$ is amenable but not strongly amenable!
Matsumura’s theorem

Question: Does \( A \rtimes_{\text{max}} G = A \rtimes_r G \) imply amenability of \( A \)?
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Theorem (Matsumura 2014) Suppose \( G \) exact and \( A \) is a unital \( G \)-algebra.

- If \( A = C(X) \), then \( A \rtimes_{\text{max}} G = A \rtimes_{r} G \iff A \) amenable.
- If \( A \) is nuclear, then
  \[
  (A \otimes A^{op}) \rtimes_{\text{max}} G = (A \otimes A^{op}) \rtimes_{r} G \iff A \) amenable.
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**Idea of proof:** It suffices to show nuclearity of

$$A \rtimes_r G \xrightarrow{\pi} (A \rtimes_r G)^{**}$$

This implies that $A \rtimes_r G$ is nuclear and $A$ amenable!
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**Idea of proof:** Matsumura shows the existence of the diagram

\[
\begin{array}{ccc}
A \rtimes_r G & \xrightarrow{\pi} & A^{**} \rtimes_r G \\
& & \downarrow \psi \\
& & (A \rtimes_r G)^{**}
\end{array}
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Idea of proof: He then extends it to a diagram

\[
\begin{array}{ccc}
A \rtimes_{r} G & \xrightarrow{\pi} & (\ell^\infty(G) \otimes A) \rtimes_{r} G \\
\downarrow & & \downarrow \phi & \xrightarrow{A'^{*}} A'^{*} \rtimes_{r} G \\
(\ell^\infty(G) \otimes A) \rtimes_{r} G & \xrightarrow{\psi} & (A \rtimes_{r} G)^{*}
\end{array}
\]

This implies that $\pi : A \rtimes_{r} G \to (A \rtimes_{r} G)^{*}$ is nuclear!
The injective crossed product

**Definition** For a $G$-algebra $A$ the *injective crossed product* is defined as

$$A \rtimes_{\text{inj}} G := \overline{C_c(G, A)}_{\| \cdot \|_{\text{inj}}}$$

$$\| f \|_{\text{inj}} = \inf \{ \| \phi \circ f \|_{B \rtimes_{\text{max}} G} \mid \phi : A \hookrightarrow B \text{ a } G\text{-embedding.} \}.$$
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**Theorem (Buss-E-Willett, 2018)** $(A, \alpha) \mapsto A \rtimes_{\text{inj}} G$ is the largest (exotic) crossed-product functor which is injective in the sense

$$\phi : A \hookrightarrow B \text{ (G-embedding)} \quad \Rightarrow \quad \phi \rtimes G : A \rtimes_{\text{inj}} G \hookrightarrow B \rtimes_{\text{inj}} G.$$  

Moreover, if $G$ is exact, then $A \rtimes_{\text{inj}} G = A \rtimes r G.$
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**Proof:**

$$A \rtimes_{\text{inj}} G \hookrightarrow (\ell^\infty(G) \otimes A) \rtimes_{\text{inj}} G = (\ell^\infty(G) \otimes A) \rtimes_r G$$
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Proof: $A \rtimes_{\text{inj}} G \hookrightarrow (\ell^\infty(G) \otimes A) \rtimes_{\text{inj}} G = (\ell^\infty(G) \otimes A) \rtimes_r G$.

Notice:

$$A \rtimes_{\text{max}} G = A \rtimes_{\text{inj}} G \iff [A \hookrightarrow B \Rightarrow A \rtimes_{\text{max}} G \hookrightarrow B \rtimes_{\text{max}} G].$$
Injective covariant representations

**Definition** A covariant rep. \((\pi, u) : (A, G) \to \mathcal{B}(H)\) is injective if:

\[
\forall \phi : A \hookrightarrow B \text{ (}G\text{-hom.}) \exists \begin{cases} \text{ccp map } \sigma : B \to \mathcal{B}(H) \text{ s.t. } \sigma \circ \phi = \pi \\ \text{and } (\sigma, u) \text{ is covariant for } (B, G). \end{cases}
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\end{array} \right\}$$

Theorem (BEW) Let \(A\) be a \(G\)-algebra. TFAE:

(a) \(A \rtimes_{\text{max}} G = A \rtimes_{\text{inj}} G\).

(b) Every covariant rep \((\pi, u)\) of \((A, G)\) is injective.

(c) \(\exists\) an injective covariant rep. \((\pi, u)\) of \((A, G)\) such that \(\pi \rtimes u : A \rtimes_{\text{max}} G \to \mathcal{B}(H)\) is faithful.
**Injective covariant representations**

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\((c) \Rightarrow (a)\)

\[
\begin{array}{ccc}
A \rtimes_{\text{max}} G & \xrightarrow{\pi \rtimes u} & \mathcal{B}(H)
\end{array}
\]

\[
\begin{array}{ccc}
A \rtimes_{\text{max}} G & \xrightarrow{\phi \rtimes G} & B \rtimes_{\text{max}} G \\
& \xrightarrow{\sigma \rtimes u} & \\
& \xrightarrow{\sigma \rtimes G} &
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**Theorem (BEW)** Let \(A\) be a \(G\)-algebra. TFAE:

(a) \(A \rtimes_{\text{max}} G = A \rtimes_{\text{inj}} G\).

(b) Every covariant rep \((\pi, u)\) of \((A, G)\) is injective.

(c) \(\exists\) an injective covariant rep. \((\pi, u)\) of \((A, G)\) such that \(\pi \rtimes u : A \rtimes_{\text{max}} G \to \mathcal{B}(H)\) is faithful.

\[(c) \Rightarrow (a)\hspace{1cm} (a) \Rightarrow (b)\hspace{1cm} \sigma := \pi \rtimes u|_B\]

\[
\begin{array}{ccc}
A \rtimes_{\text{max}} G & \xrightarrow{\pi \ltimes u} & \mathcal{B}(H) \\
\phi \rtimes G & \downarrow & \sigma \rtimes u \\
A \rtimes_{\text{max}} G & \xrightarrow{\pi \ltimes u} & \mathcal{B}(H)
\end{array}
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\end{array}
\]
Lemma. Let \((\pi, u) : (A, G) \rightarrow \mathcal{B}(H)\) be injective with \(\pi\) nondeg.
Let \(C\) be any unital \(G\)-algebra.

Then there exists a ucp map

\[ \phi : C \rightarrow \pi(A)' \subseteq \mathcal{B}(H) \quad \text{s.t.} \quad (\phi, u) \text{ is covariant for } (C, G). \]

This applies in particular to \(C = \ell^\infty(G)\).
Injective covariant representations

**Lemma.** Let \((\pi, u) : (A, G) \to \mathcal{B}(H)\) be injective with \(\pi\) nondeg.. Let \(C\) be any unital \(G\)-algebra. Then there exists a ucp map

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This applies in particular to \(C = \ell^\infty(G)\).

**Proof:** Consider \(\iota_A, \iota_C : A, C \hookrightarrow M(C \otimes A)\). Injectivity of \((\pi, u)\) implies:

\[ \exists \ \text{ucp map } \sigma : M(C \otimes A) \to \mathcal{B}(H) \quad \text{s.t.} \quad \sigma \circ \iota_A = \pi \quad \text{and} \quad (\sigma, u) \text{ covariant.} \]

Put \(\phi = \sigma \circ \iota_C\). Then \((\phi, u)\) is covariant and \(\phi(C') \subseteq \pi(A)'

(notice that \(\iota_A(A)\) lies in the multiplicative domain of \(\sigma\)!)
Commutant amenability

**Definition** \((\pi, u) : (A, G) \to \mathcal{B}(H)\) is commutant amenable if there exists a net of finitely supported positive type functions \(\theta_i : G \to \pi(A)'\) (with resp. to \(\beta = \text{Ad} u\)) such that

(i) \(\theta_i(e) \leq 1\), and

(ii) \(\forall g \in G : \theta_i(g) \to 1\) ultraweakly as \(i \to \infty\)

\[\theta_i : G \to \pi(A)'\]
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We say \(A\) is commutant amenable if this holds for all \((\pi, u)\).
Commutant amenability

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Remark (a) \(A\) amenable \(\Rightarrow\) \(A\) commutant amenable.
Commutant amenability

**Definition** $(\pi, u) : (A, G) \to B(H)$ is **commutant amenable** if there exists a net of finitely supported positive type functions

$$\theta_i : G \to \pi(A)'$$ (with resp. to $\beta = \text{Ad } u$)

such that

(i) $\theta_i(e) \leq 1$, and

(ii) $\forall g \in G : \theta_i(g) \to 1$ ultraweakly as $i \to \infty$

We say $A$ is commutant amenable if this holds for all $(\pi, u)$.

**Remark** (a) $A$ amenable $\Rightarrow A$ commutant amenable.

(b) wlog $\exists$ finitely supported functions $\xi_i \in \ell^2(G, \pi(A)')$ s.t,

$$\theta_i(g) = \langle \xi_i, \tilde{\beta}_g(\xi_i) \rangle_{\pi(A)'}, \quad \forall g \in G.$$
Commutant amenability

**Theorem (BEW)** If the $G$-algebra $A$ is commutant amenable, then

$$A times_{\text{max}} G = A times_{\text{inj}} G = A times_r G.$$  

If $G$ is **exact**, the converse holds as well!

**Proof of converse:**
Assume that $G$ is exact and $A times_{\text{max}} G = A times_{\text{inj}} G$. Then every cov. rep. $(\pi, u)$ is injective. Thus there exists a $G$-ucp map

$$\phi : \ell^\infty(G) \to \pi(A)^{'}$$

$G$ being exact, $\ell^\infty(G)$ is amenable. Hence, there exist fin. sup. positive type functions

$$\eta_i : G \to \ell^\infty(G) \quad \text{satisfying (i) and (ii)}.$$

Put

$$\theta_i = \phi \circ \eta_i : G \to \pi(A)^{'}. $$
Haagerup’s standard form

**Theorem (Haagerup)** Let $A$ be a $G$-algebra. Then there exist normal, unital, and faithful reps.

$$\pi : A^{**} \to \mathcal{B}(H), \quad \pi^{op} : (A^{op})^{**} \to \mathcal{B}(H)$$

and a unitary rep $u : G \to U(H)$ such that

(i) $(\pi, u)$ and $(\pi^{op}, u)$ are covariant;

(ii) $\pi(A)' = \pi^{op}((A^{op})^{**}) \cong (A^{op})^{**}$, \quad $\pi^{op}(A^{op})' = \pi(A^{**}) \cong A^{**}$;

(iii) if $A$ is commutative, then $\pi(A)' \cong A^{**} = Z(A^{**})$. 

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Haagerup’s standard form

**Theorem (Haagerup)** Let $A$ be a $G$-algebra. Then there exist normal, unital, and faithful reps.

$$
\pi : A^{**} \to \mathcal{B}(H), \quad \pi^{op} : (A^{op})^{**} \to \mathcal{B}(H)
$$

and a unitary rep $u : G \to U(H)$ such that

(i) $(\pi, u)$ and $(\pi^{op}, u)$ are covariant;

(ii) $\pi(A)' = \pi^{op}((A^{op})^{**}) \cong (A^{op})^{**}$,

(iii) if $A$ is commutative, then $\pi(A)' \cong A^{**} = Z(A^{**})$.

**Corollary (a)** $A$ amenable $\iff A \otimes_{\text{max}} A^{op}$ commutant amenable.

(since (ii) implies $Z(A^{**}) = \pi \otimes \pi^{op}(A \otimes_{\text{max}} A^{op})'$).

(b) If $A$ is commutative, then (since $\pi(A)' \cong A^{**} = Z(A^{**})$)

$A$ amenable $\iff A$ commutant amenable.
Matsumura’s theorem revisited

**Theorem (BEW)** Suppose that $G$ is *exact* and $A$ is a $G$-algebra. Then the following are equivalent:

(i) $A$ is amenable.

(ii) $A \otimes_{\max} A^{op}$ is amenable.

(iii) $A \otimes_{\max} A^{op}$ is commutant amenable.

(iv) $(A \otimes_{\max} A^{op}) \rtimes_{\max} G = (A \otimes_{\max} A^{op}) \rtimes_{r} G.$
Matsumura’s theorem revisited

Theorem (BEW) Suppose that $G$ is exact and $A$ is a $G$-algebra. Then the following are equivalent:

(i) $A$ is amenable.
(ii) $A \otimes_{\max} A^{\text{op}}$ is amenable.
(iii) $A \otimes_{\max} A^{\text{op}}$ is commutant amenable.
(iv) $(A \otimes_{\max} A^{\text{op}}) \rtimes_{\max} G = (A \otimes_{\max} A^{\text{op}}) \rtimes_r G$.

Assume, in addition, that $A$ is commutative. Then the following are equivalent:

(a) $A$ is (strongly) amenable.
(b) $A$ is commutant amenable.
(c) $A \rtimes_{\max} G = A \rtimes_r G$.

In particular: $C^*_\max(X \rtimes G) \cong C^*_r(X \rtimes G) \iff X \rtimes G$ is amenable.
Theorem (BEW) Let $A$ be unital and commutant amenable. Then:

(i) $A$ nuclear $\Rightarrow A \rtimes_r G \hookrightarrow (A \otimes A^{op}) \rtimes_r G$ is nuclear.

(ii) $A$ exact $\Rightarrow A \rtimes_r G$ exact.

(iii) $G$ is exact.
The Kaminker-Ozawa theorem

Theorem (BEW) Let $A$ be unital and commutant amenable. Then:

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(ii) $A$ exact $\Rightarrow$ $A \rtimes_r G$ exact.
(iii) $G$ is exact.

Proof: Use $\pi(A)' = \pi^{op}((A^{op})^{**})$ in Haagerup’s rep. to show:

$\exists \xi_i : G \rightarrow A^{op}$ with finite supp. s.t. $\theta_i(g) = \langle \xi_i, \tilde{\alpha}^{op}_g(\xi_i) \rangle \rightarrow 1_{A^{op}}$
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We then get

$$A \rtimes_r G \xrightarrow{(\iota_A \otimes 1) \times G} (A \otimes_{\text{max}} A^{op}) \rtimes_r G$$

$$\psi_F \downarrow \quad \phi_F$$

$$A \otimes M_F$$

for $F \subseteq G$ finite, such that $\phi_f \circ \psi_F$ approximates $(\iota_A \otimes 1) \times G$. 
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(ii) $A$ exact $\Rightarrow A \rtimes_r G$ exact.
(iii) $G$ is exact.

Proof: If $A$ is exact, we consider the extended diagram:

$$
\begin{array}{c}
A \rtimes_r G^C \\
\downarrow \psi_F \\
A \otimes M_F \\
\downarrow \phi_F \\
(A \otimes_{max} A^{op}) \rtimes_r G^C \\
\downarrow \\
B(H \otimes \ell^2(G'))
\end{array}
$$

Hence $A \rtimes_r G$ is exact!
The Kaminker-Ozawa theorem

Theorem (BEW) Let $A$ be unital and commutant amenable. Then:

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(ii) $A$ exact $\Rightarrow A \rtimes_r G$ exact.

(iii) $G$ is exact.

Proof: If $G$ is exact, restrict the diagram to $C^*_r(G) \subseteq A \rtimes_r G$:

$$
\begin{array}{c}
C^*_r(G) \subseteq A \rtimes_r G \xrightarrow{(\iota A \otimes 1) \times G} (A \otimes_{\text{max}} A^{op}) \rtimes_r G \xrightarrow{\phi_F} \mathcal{B}(H \otimes \ell^2(G))
\end{array}
$$

Hence $C^*_r(G)$ (and therefore $G$) is exact!
The Kaminker-Ozawa theorem

Corollary (cf Kaminker-Ozawa theorem) TFAE

(a) $G$ is exact.
(b) There exists a compact amenable $G$-space $X$.
(c) There exists a (strongly) amenable, unital $G$-algebra $A$.
(d) There exists a commutant amenable, unital $G$-algebra $A$. 
Injective $G$-algebras and the $G$-WEP

Definition Let $A$ be a $G$-algebra, $\iota : A \hookrightarrow A^{**}$ the canonical map.

(a) $A$ is called $G$-injective, if

$$\forall \psi : A \hookrightarrow B \text{ (}G\text{-emb.)} \exists \text{ cond. exp. } P : B \to A \text{ s.t. } P \circ \psi = \text{id}_A.$$ 

(b) $A$ has the $G$-WEP, if

$$\forall \psi : A \hookrightarrow B \text{ (}G\text{-emb.)} \exists \text{ ccp map } P : B \to A^{**} \text{ s.t. } P \circ \psi = \iota.$$
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**Examples**

(1) $\ell^\infty(G, B)$ is $G$-injective for any injective $B$ (e.g., $B = B(H)$).
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**Examples**

(1) $\ell^\infty(G, B)$ is $G$-injective for any injective $B$ (e.g., $B = \mathcal{B}(H)$).

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**Examples**

1. $\ell^\infty(G, B)$ is $G$-injective for any injective $B$ (e.g., $B = B(H)$).

2. $A$ has the $G$-WEP if $\exists$ $G$-injective $C$ such that $A \hookrightarrow C \hookrightarrow A^{**}$

3. If $A$ has the $G$-WEP (or $A$ is $G$-injective), then

$$A \rtimes_{\text{max}} G = A \rtimes_{\text{inj}} G.$$
Injective $G$-algebras and the $G$-WEP

Theorem (BEW) Let $A$ be a $G$-algebra.

(a) If $G$ is exact TFAE:

(i) $A$ is amenable and has the WEP.

(ii) $A$ has the $G$-WEP.
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(a) If $G$ is exact TFAE:

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(b) For a $G$-algebra $A$ TFAE

(i) $A$ is amenable and $A^{**}$ is injective.

(ii) $A^{**}$ is $G$-injective.
Injective $G$-algebras and the $G$-WEP

Theorem (BEW) Let $A$ be a $G$-algebra.

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(b) For a $G$-algebra $A$ TFAE
   (i) $A$ is amenable and $A^{**}$ is injective.
   (ii) $A^{**}$ is $G$-injective.

(ii) $\Rightarrow$ (i) Consider the inclusion $\iota : A^{**} \hookrightarrow \ell^{\infty}(G, A^{**})$. By $G$-injectivity of $A^{**}$ we obtain a $G$-conditional exp.

$$P : \ell^{\infty}(G, A^{**}) \rightarrow A^{**}$$

hence one of the equivalent characterizations of amenability.
The $G$-injective envelope

For a $G$-algebra $A$ let $I_G(A)$ denote Hamana’s $G$-injective envelope of $A$, i.e., If $A \hookrightarrow B$ and $B$ is $G$-injective, then

$$A \hookrightarrow I_G(A) \hookrightarrow B.$$ 

Note that $I_G(A)$ is always unital!
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Theorem (BEW) For a discrete group $G$ TFAE:

(i) $G$ is exact.

(ii) $I_G(A)$ is (strongly) amenable for every $G$-algebra $A$.

(iii) $I_G(A)$ is (strongly) amenable for some $G$-algebra $A$. 
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Idea for (i) $\Rightarrow$ (ii): Use $G$-injectivity of $I_G(A)$ to the inclusion

$$A \hookrightarrow \ell^\infty(G) \otimes A$$

to obtain a ucp map $\ell^\infty(G) \rightarrow Z(I_G(A))$. 

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Ozawa’s conjecture

Conjecture (Ozawa) For every exact C*-algebra $B$ there exists a nuclear C*-algebra $N(B)$ such that

$$B \subseteq N(B) \subseteq I(B) \quad (I(B) \text{ inj. env. of } B)$$

Kalantar & Kennedy 2017 The conjecture holds for $C^*_r(G)$.
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**Corollary (BEW-Suzuki)** Suppose $G$ is exact and $A$ is nuclear.
Then Ozawa’s conjecture holds for $A \rtimes_r G$. 
Ozawa’s conjecture

Conjecture (Ozawa) For every exact $C^*$-algebra $B$ there exists a nuclear $C^*$-algebra $N(B)$ such that

$$B \subseteq N(B) \subseteq I(B) \quad (I(B) \text{ inj. env. of } B)$$

Kalantar & Kennedy 2017 The conjecture holds for $C^*_r(G)$.

Corllary (BEW-Suzuki) Suppose $G$ is exact and $A$ is nuclear. Then Ozawa’s conjecture holds for $A \rtimes_r G$.

Proof: $I_G(A)$ strongly amenable $\Rightarrow Z(I_G(A))$ amenable. Then

$$D := C^*(A \cup Z(I_G(A))) \subseteq I_G(A)$$

is nuclear and amenable. Hence $N(B) := D \rtimes_r G$ is nuclear, and

$$A \rtimes_r G \subseteq N(B) = D \rtimes_r G \subseteq I_G(A) \rtimes_r G^{Hamana} \subseteq I(A \rtimes_r G).$$
References


My best wishes to Jean!
Thank you for your friendship!