# *Amenability of actions and Crossed Products*

*joint work with*
*Alcides Buss and Rufus Willett*

Groupoids and Operator Algebras.

(The Jean Fest)

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Amenability of actions of groupoids on spaces or C*-algebras has been a central theme in the (partially joint) work of Jean Renault and Claire Anantharaman-Delaroche.

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Question: Does a similar characterization of amenability hold for locally compact groupoids or crossed products by actions of groups or groupoids?

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In this talk we only consider actions of discrete groups $G$ (which includes the case of the groupoids $X \rtimes G$)!

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- On a conjecture of Ozawa.

## Amenable actions (after Anantharaman-Delaroche)

Let $\alpha : G \to \mathrm{Aut}(A)$ be an action. A function $\theta : G \to A$ is called *of positive type*, if for all $g_1, \ldots, g_l \in G$ we have

$$\big(\alpha_{g_i}(\theta(g_i^{-1}g_j))\big)_{i,j} \in M_l(A)^+.$$

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Defininitition (Anantharaman-Delaroche '87) A $G$-algebra $A$ is called *amenable* if there exists a net $\{\theta_i : G \to Z(A^{**})\}_{i \in I}$ of *finitely supported positive type* functions such that

1. $\theta_i(e) \leq 1_{A^{**}} \quad \forall i \in I$
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We call $A$ *strongly amenable* if there exists a net of *finitely supported* positive type functions $\theta_i : G \to ZM(A)$ such that

1. $\theta_i(e) \le 1_{M(A)} \quad \forall i \in I$
2. $\theta_i(g) \to 1_{M(A)}$ strictly $\forall g \in G$.

# Amenable actions

The following results are all due to Anantharaman-Delaroche:

- If $A = C_0(X)$ is commutative, then
  $A$ strongly amenable $\Leftrightarrow$ $A$ amenable $\Leftrightarrow$ $X \rtimes G$ amenable.

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  $A$ is amenable $\Leftrightarrow$ $A \rtimes_{\max} G$ is nuclear $\Leftrightarrow$ $A \rtimes_r G$ is nuclear.

Example (Suzuki, 2018) For every non-amenable, exact, discrete group $G$ there exists a unital, simple, nuclear $G$-algebra $A$ such that $A \rtimes_{\max} G = A \rtimes_r G$ is nuclear.

Thus: $A$ is amenable but not strongly amenable!

# Matsumura's theorem

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Theorem (Matsumura 2014) Suppose $G$ exact and $A$ is a unital $G$-algebra.

- If $A = C(X)$, then $A \rtimes_{\max} G = A \rtimes_r G \Leftrightarrow A$ amenable.
- If $A$ is nuclear, then $(A \otimes A^{op}) \rtimes_{\max} G = (A \otimes A^{op}) \rtimes_r G \Leftrightarrow A$ amenable.

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Idea of proof: It suffices to show nuclearity of

$$A \rtimes_r G \xrightarrow{\ \pi\ } (A \rtimes_r G)^{**}$$

This implies that $A \rtimes_r G$ is nuclear and $A$ amenable!

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Idea of proof: Matsumura shows the existence of the diagram

$$\begin{array}{ccc} A \rtimes_r G & & \\ & \searrow \quad \overset{\pi}{\searrow} & \\ & A^{**} \rtimes_r G \xrightarrow[\psi]{} & (A \rtimes_r G)^{**} \end{array}$$

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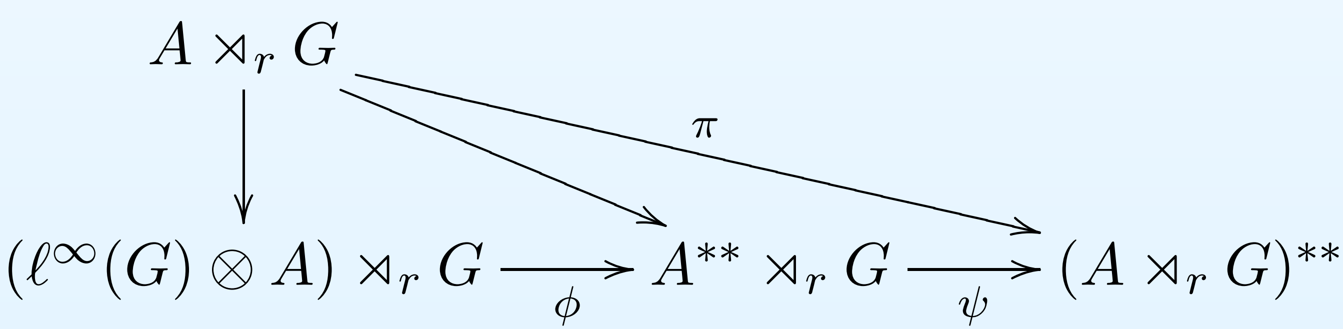
Idea of proof: He then extends it to a diagram




This implies that $\pi : A \rtimes_r G \to (A \rtimes_r G)^{**}$ is nuclear!

# The injective crossed product

Definition For a $G$-algebra $A$ the *injective crossed product* is defined as

$$A \rtimes_{\mathrm{inj}} G := \overline{C_c(G, A)}^{\|\cdot\|_{\mathrm{inj}}}$$

$$\|f\|_{\mathrm{inj}} = \inf\{\|\phi \circ f\|_{B \rtimes_{\max} G} \mid \phi : A \hookrightarrow B \text{ a } G\text{-embedding.}\}.$$

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Theorem (Buss-E-Willett, 2018) $(A, \alpha) \mapsto A \rtimes_{\mathrm{inj}} G$ is the largest (exotic) crossed-product functor which is injective in the sense

$$\phi : A \hookrightarrow B \quad (G\text{-embedding}) \quad \Rightarrow \quad \phi \rtimes G : A \rtimes_{\mathrm{inj}} G \hookrightarrow B \rtimes_{\mathrm{inj}} G.$$

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Moreover, if $G$ is exact, then $A \rtimes_{\text{inj}} G = A \rtimes_r G$.

Proof: $\quad A \rtimes_{\text{inj}} G \hookrightarrow (\ell^\infty(G) \otimes A) \rtimes_{\text{inj}} G = (\ell^\infty(G) \otimes A) \rtimes_r G$

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Proof: $A \rtimes_{\rm inj} G \hookrightarrow (\ell^\infty(G) \otimes A) \rtimes_{\rm inj} G = (\ell^\infty(G) \otimes A) \rtimes_r G$

Notice:

$$A \rtimes_{\max} G = A \rtimes_{\rm inj} G \iff \big[A \hookrightarrow B \Rightarrow A \rtimes_{\max} G \hookrightarrow B \rtimes_{\max} G\big].$$

# Injective covariant representations

Definition A covariant rep. $(\pi, u) : (A, G) \to \mathcal{B}(H)$ is injective if:

$$\forall \phi : A \hookrightarrow B \text{ ($G$-hom.) } \exists \left\{ \begin{array}{c} \text{ccp map } \sigma : B \to \mathcal{B}(H) \text{ s.t. } \sigma \circ \phi = \pi \\ \text{and } (\sigma, u) \text{ is covariant for } (B, G). \end{array} \right\}$$

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Theorem (BEW) Let $A$ be a $G$-algebra. TFAE:

(a) $A \rtimes_{\max} G = A \rtimes_{\text{inj}} G$.

(b) Every covariant rep $(\pi, u)$ of $(A, G)$ is injective.

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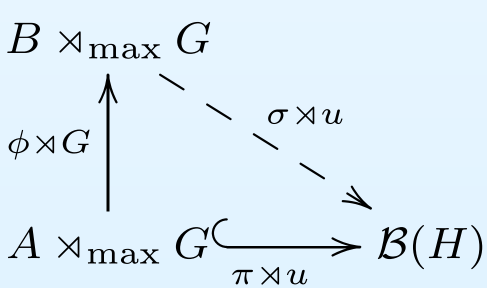

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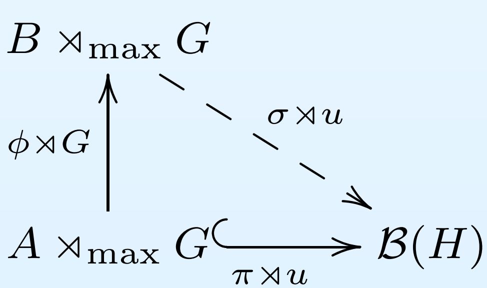


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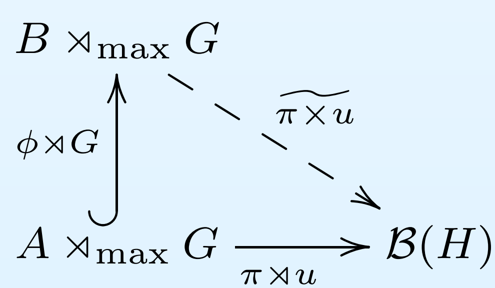


$\sigma := \widetilde{\pi \rtimes u}|_B$

# Injective covariant representations

Lemma. Let $(\pi, u) : (A, G) \to \mathcal{B}(H)$ be injective with $\pi$ nondeg..
Let $C$ be any unital $G$-algebra.

Then there exists a ucp map

$$\phi : C \to \pi(A)' \subseteq \mathcal{B}(H) \quad \text{s.t. } (\phi, u) \text{ is covariant for } (C, G).$$

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$Proof$ : Consider $\quad \iota_A, \iota_C : A, C \hookrightarrow M(C \otimes A)$.

Injectivity of $(\pi, u)$ implies:

$\exists$ ucp map $\sigma : M(C \otimes A) \to \mathcal{B}(H)$ s.t. $\sigma \circ \iota_A = \pi$ and $(\sigma, u)$ covariant.

Put $\phi = \sigma \circ \iota_C$. Then $(\phi, u)$ is covariant and $\phi(C) \subseteq \pi(A)'$

(notice that $\iota_A(A)$ lies in the multiplicative domain of $\sigma$!)

## Commutant amenability

Definition $(\pi, u) : (A, G) \to \mathcal{B}(H)$ is commutant amenable if there exists a net of finitely supported positive type functions

$$\theta_i : G \to \pi(A)' \quad \text{(with resp. to } \beta = \mathrm{Ad}\, u\text{)}$$

such that

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**Remark (a)** $A$ amenable $\Rightarrow$ $A$ commutant amenable.

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Remark (a) $A$ amenable $\Rightarrow$ $A$ commutant amenable.

(b) wlog $\exists$ finitely supported functions $\xi_i \in \ell^2(G, \pi(A)')$ s.t,

$$\theta_i(g) = \langle \xi_i, \tilde{\beta}_g(\xi_i) \rangle_{\pi(A)'}, \quad \forall g \in G.$$

# Commutant amenability

Theorem (BEW) If the $G$-algebra $A$ is commutant amenable, then

$$A \rtimes_{\max} G = A \rtimes_{\text{inj}} G = A \rtimes_r G.$$

If $G$ is exact, the converse holds as well!

Proof of converse:
Assume that $G$ is exact and $A \rtimes_{\max} G = A \rtimes_{\text{inj}} G$. Then every cov. rep. $(\pi, u)$ is injective. Thus there exists a $G$-ucp map

$$\phi : \ell^\infty(G) \to \pi(A)'$$

$G$ being exact, $\ell^\infty(G)$ is amenable. Hence, there exist fin. sup. positive type functions

$$\eta_i : G \to \ell^\infty(G) \quad \text{satisfying (i) and (ii).}$$

Put $\qquad \theta_i = \phi \circ \eta_i : G \to \pi(A)'.$

## Haagerup's standard form

Theorem (Haagerup) Let $A$ be a $G$-algebra. Then there exist normal, unital, and faithful reps.

$$\pi : A^{**} \to \mathcal{B}(H), \quad \pi^{op} : (A^{op})^{**} \to \mathcal{B}(H)$$

and a unitary rep $u : G \to U(H)$ such that

(i) $(\pi, u)$ and $(\pi^{op}, u)$ are covariant;

(ii) $\pi(A)' = \pi^{op}((A^{op})^{**}) \cong (A^{op})^{**}, \quad \pi^{op}(A^{op})' = \pi(A^{**}) \cong A^{**}$;

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Corollary (a) $A$ amenable $\Leftrightarrow$ $A \otimes_{\max} A^{op}$ commutant amenable.

(since (ii) implies $Z(A^{**}) = \pi \otimes \pi^{op}(A \otimes_{\max} A^{op})'$).

(b) If $A$ is commutative, then (since $\pi(A)' \cong A^{**} = Z(A^{**})$)

$A$ amenable $\Leftrightarrow$ $A$ commutant amenable.

# Matsumura's theorem revisited

**Theorem (BEW)** Suppose that $G$ is exact and $A$ is a $G$-algebra. Then the following are equivalent:

(i) $A$ is amenable.

(ii) $A \otimes_{\max} A^{op}$ is amenable.

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(iv) $(A \otimes_{\max} A^{op}) \rtimes_{\max} G = (A \otimes_{\max} A^{op}) \rtimes_r G$.

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Assume, in addition, that $A$ is commutative. Then the following are equivalent:

(a) $A$ is (strongly) amenable.

(b) $A$ is commutant amenable.

(c) $A \rtimes_{\max} G = A \rtimes_r G$.

In particular: $C^*_{\max}(X \rtimes G) \cong C^*_r(X \rtimes G) \Leftrightarrow X \rtimes G$ is amenable

## The Kaminker-Ozawa theorem

Theorem (BEW) Let $A$ be unital and commutant amenable. Then:

(i) $A$ nuclear $\Rightarrow A \rtimes_r G \hookrightarrow (A \otimes A^{op}) \rtimes_r G$ is nuclear.

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Proof: Use $\pi(A)' = \pi^{op}((A^{op})^{**})$ in Haagerup's rep. to show:

$\exists\, \xi_i : G \to A^{op}$ with finite supp. s.t. $\theta_i(g) = \langle \xi_i, \tilde{\alpha}_g^{op}(\xi_i)\rangle \xrightarrow{\|\cdot\|} 1_{A^{op}}$

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We then get

$$\begin{array}{ccc} A \rtimes_r G & \xrightarrow{(\iota_A \otimes 1) \rtimes G} & (A \otimes_{\max} A^{op}) \rtimes_r G \\ & {\scriptstyle \psi_F} \searrow \quad \nearrow {\scriptstyle \phi_F} & \\ & A \otimes M_F & \end{array}$$

for $F \subseteq G$ finite, such that $\phi_f \circ \psi_F$ approximates $(\iota_A \otimes 1) \rtimes G$.

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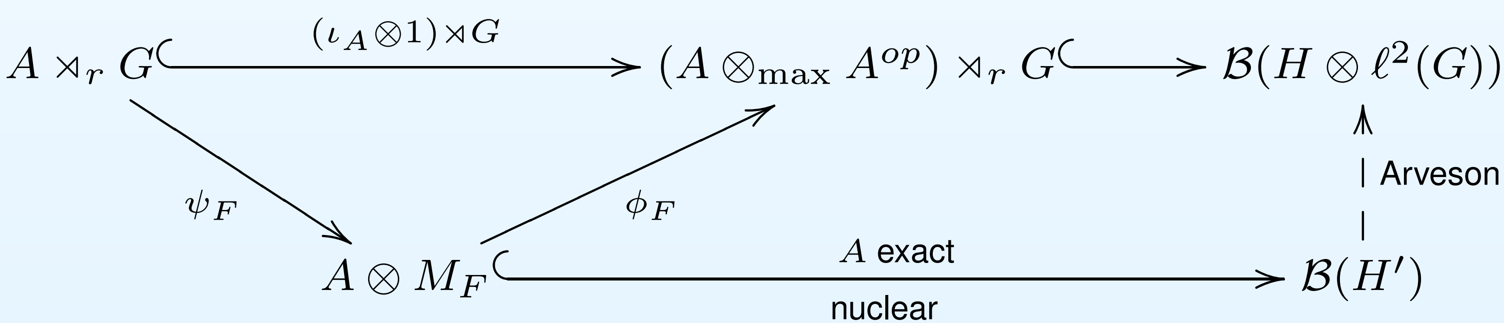
Proof: If $A$ is exact, we consider the extended diagram:




Hence $A \rtimes_r G$ is exact!

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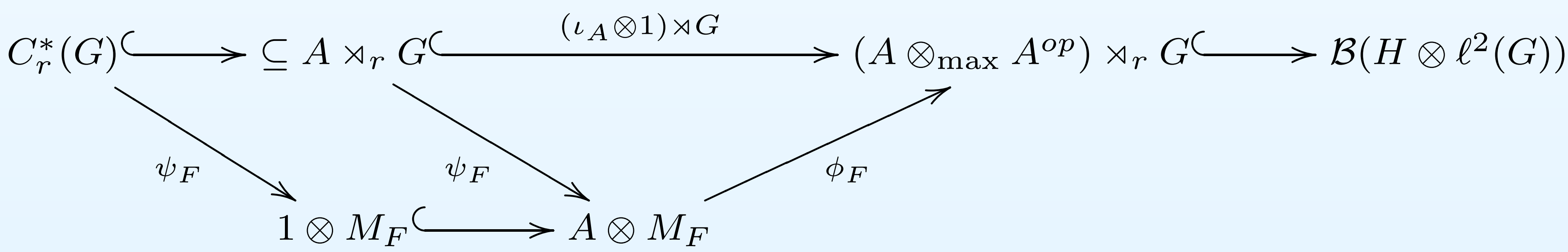
Proof: If $G$ is exact, restrict the diagram to $C_r^*(G) \subseteq A \rtimes_r G$:




Hence $C_r^*(G)$ (and therefore $G$) is exact!

# The Kaminker-Ozawa theorem

Corollary (cf Kaminker-Ozawa theorem) TFAE

(a) $G$ is exact.

(b) There exists a compact amenable $G$-space $X$.

(c) There exists a (strongly) amenable, unital $G$-algebra $A$.

(d) There exists a commutant amenable, unital $G$-algebra $A$.

# Injective $G$-algebras and the $G$-WEP

**Definition** Let $A$ be a $G$-algebra, $\iota : A \hookrightarrow A^{**}$ the canonical map.

(a) $A$ is called $G$-injective, if

$\forall \psi : A \hookrightarrow B$ ($G$-emb.) $\exists$ cond. exp. $P : B \to A$ s.t. $P \circ \psi = \mathrm{id}_A$ .

(b) $A$ has the $G$-WEP, if

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**(1)** $\ell^\infty(G, B)$ is $G$-injective for any injective $B$ (e.g., $B = \mathcal{B}(H)$).

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**(3)** If $A$ has the $G$-WEP (or $A$ is $G$-injective), then

$$A \rtimes_{\max} G = A \rtimes_{\mathrm{inj}} G.$$

## Injective $G$-algebras and the $G$-WEP

**Theorem (BEW)** Let $A$ be a $G$-algebra.

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(ii) $A^{**}$ is $G$-injective.

(ii)$\Rightarrow$ (i) Consider the inclusion $\iota : A^{**} \hookrightarrow \ell^\infty(G, A^{**})$. By $G$-injectivity of $A^{**}$ we obtain a $G$-conditional exp.

$$P : \ell^\infty(G, A^{**}) \to A^{**}$$

hence one of the equivalent characterizations of amenability.

## The $G$-injective envelope

For a $G$-algebra $A$ let $I_G(A)$ denote Hamana's $G$-injective envelope of $A$, i.e., If $A \hookrightarrow B$ and $B$ is $G$-injective, then

$$A \hookrightarrow I_G(A) \hookrightarrow B.$$

Note that $I_G(A)$ is always unital!

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Theorem (BEW) For a discrete group $G$ TFAE:

(i) $G$ is exact.

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Idea for (i) $\Rightarrow$ (ii): Use $G$-injectivity of $I_G(A)$ to the inclusion

$$A \hookrightarrow \ell^\infty(G) \otimes A$$

to obtain a ucp map $\quad \ell^\infty(G) \to Z(I_G(A))$.

## Ozawa's conjecture

Conjecture (Ozawa) For every exact C*-algebra $B$ there exists a nuclear C*-algebra $N(B)$ such that

$$B \subseteq N(B) \subseteq I(B) \quad (I(B) \text{ inj. env. of } B)$$

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Corllary (BEW-Suzuki) Suppose $G$ is exact and $A$ is nuclear.

Then Ozawa's conjecture holds for $A \rtimes_r G$.

Proof: $I_G(A)$ strongly amenable $\Rightarrow Z(I_G(A))$ amenable. Then

$$D := C^*(A \cup Z(I_G(A))) \subseteq I_G(A)$$

is nuclear and amenable. Hence $N(B) := D \rtimes_r G$ is nuclear, and

$$A \rtimes_r G \subseteq N(B) = D \rtimes_r G \subseteq I_G(A) \rtimes_r G \overset{Hamana}{\subseteq} I(A \rtimes_r G).$$

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My best wishes to Jean!
Thank you for your friendship!