A groupoid approach to classifiable C*-algebras

Xin Li
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Goal: Construct Cartan subalgebras in all classifiable C*-algebras.
What does “classifiable” mean?

**Classification programme for C*-algebras**

Classify all simple, nuclear C*-algebras by

\[ \text{Ell}(A) = (K_0(A) \ (\&\ldots),\ T(A),\ r_A,\ K_1(A)). \]
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- So: We need regularity (Toms-Winter): \( \mathcal{Z} \)-stability \( (A \cong \mathcal{Z} \otimes A) \)
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Big Classification Theorem

(Unital) Separable simple nuclear \( \mathcal{Z} \)-stable UCT C*-algebras are classified by \( \text{Ell}(\cdot) \).
What is a Cartan subalgebra?

Definition:
Let $A$ be a C*-algebra. $B \subseteq A$ is a Cartan subalgebra if
(i) $B$ is a maximal abelian sub-C*-algebra;
(ii) $B$ is regular: $N_A(B) := \{ n \in A : nBn^* \subseteq B$ and $n^*Bn \subseteq B \}$ generates $A$ as a C*-algebra;
(iii) $B$ contains an approximate identity of $A$;
(iv) There exists a faithful conditional expectation $P : A \twoheadrightarrow B$.

Examples:
• $D_n \subseteq M_n$• Given a topological dynamical system $\Gamma \curvearrowright X$, $C_0(X) \subseteq C_0(X) \rtimes_r \Gamma$ is Cartan $\iff \Gamma \curvearrowright X$ is topologically free: For all $e \neq g \in \Gamma$, $\{ x \in X : g.x \neq x \} \subseteq X$ dense.

Theorem (Kumjian, Renault): Every Cartan pair $(A, B)$ is of the form $(C_\ast r(G, \Sigma), C_0(G(0)))$, where $(G, \Sigma)$ is a twist over an étale locally compact Hausdorff effective groupoid.
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**Examples:**

- $D_n \subseteq M_n$ for all $n$.
- Given a topological dynamical system $\Gamma \curvearrowright X$, $C_0(X) \subseteq C_0(X)^r \curvearrowright_r \Gamma$ is Cartan if and only if $\Gamma \curvearrowright X$ is topologically free:
  
  For all $e \neq g \in \Gamma$, 
  $$\{ x \in X : g \cdot x \neq x \} \subseteq X \text{ dense.}$$

**Theorem (Kumjian, Renault):** Every Cartan pair $(A, B)$ is of the form $(C^*(r(G, \Sigma)), C_0(G(0)))$, where $(G, \Sigma)$ is a twist over an étale locally compact Hausdorff effective groupoid $G$. 


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**Theorem** (Kumjian, Renault): Every Cartan pair $(A, B)$ is of the form $(C^*_r(G, \Sigma), C_0(G^{(0)}))$, where $(G, \Sigma)$ is a twist over an étale locally compact Hausdorff effective groupoid $G$. 


Theorem (L): Every classifiable C*-algebra has a Cartan subalgebra. More precisely, given data $E$ which could possibly be the Elliott invariant of a classifiable C*-algebra (i.e., $E$ is weakly unperforated), there is a Cartan pair $(A, B)$, with $A$ classifiable, such that $\text{Ell}(A) \sim = E$.

In the purely infinite case (i.e., for Kirchberg algebras), this follows from work of Spielberg, Katsura, Exel-Pardo ...

In the stably finite case, there are partial results by Deeley-Putnam-Strung, Putnam, Austin-Mitra.

Idea of proof: Thomsen, Elliott, Gong-Lin-Niu, ... already constructed $\lim \rightarrow$ models exhausting the Elliott invariant. So we need to construct Cartan subalgebras in $\lim \rightarrow$ C*-algebras.
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Theorem (Barlak-L): Let \((A_n, B_n)\) be Cartan pairs with normalizers \(N_n := N_{A_n}(B_n)\) and faithful conditional expectations \(P_n : A_n ↠ B_n\). Let \(ϕ_n : A_n ↠ A_{n+1}\) be injective \(*\)-homomorphisms with \(ϕ_n(B_n) ⊆ B_{n+1}\), \(ϕ_n(N_n) ⊆ N_{n+1}\) and \(P_{n+1} \circ ϕ_n = ϕ_n \circ P_n\) for all \(n\).

Then \(\lim_{→} \{B_n; ϕ_n\}\) is a Cartan subalgebra of \(\lim_{→} \{A_n; ϕ_n\}\).

What does condition (1) mean?

Proposition (L): Let \((G_n, Σ_n)\) be twisted groupoid of \((A_n, B_n)\). \(ϕ_n : A_n ↠ A_{n+1}\) satisfies (1) if and only if there exists \((G_n, Σ_n)\) \(π ↹ (H, T)\) \(ι ↹ (G_{n+1}, Σ_{n+1})\), where \(ι\) has open image, and \(π\) is proper and fibrewise bijective, such that \(ϕ_n = ι^* \circ π^*\).
Theorem (Barlak-L): Let \((A_n, B_n)\) be Cartan pairs with normalizers \(N_n := N_{A_n}(B_n)\) and faithful conditional expectations \(P_n : A_n \twoheadrightarrow B_n\).
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Let $\varphi_n : A_n \hookrightarrow A_{n+1}$ be injective $*$-homomorphisms with

$$\varphi_n(B_n) \subseteq B_{n+1}, \ varphi_n(N_n) \subseteq N_{n+1} \ and \ P_{n+1} \circ \varphi_n = \varphi_n \circ P_n \quad (1)$$

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Constructing Cartan subalgebras in inductive limits

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**Theorem** (Barlak-L): Let \((A_n, B_n)\) be Cartan pairs with normalizers \(N_n := N_{A_n}(B_n)\) and faithful conditional expectations \(P_n : A_n \to B_n\). Let \(\varphi_n : A_n \hookrightarrow A_{n+1}\) be injective \(*\)-homomorphisms with

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Constructing Cartan subalgebras in the Jiang-Su algebra

The Jiang-Su algebra is given by

\[ Z = \lim_{n \to -\infty} \{ A_n; \phi_n \} \]

where

\[ A_n = \{ f \in C([0,1], M_{p_n} \otimes M_{q_n}) : f(0) \in M_{p_n} \otimes 1, f(1) \in 1 \otimes M_{q_n} \} \]

with \( p_n, q_n \in \mathbb{N} \), \( \gcd(p_n, q_n) = 1, p_n | p_n + 1, q_n | q_n + 1 \). \( \phi_n(f) = u_n^{+1} \cdot (f \circ \lambda_y) y \cdot u_n^{+1} \), with \( \lambda_y \in \{ t_2^2, 1_2^2, t_2^2+1 \} \), and \( u_n^{+1} \) is a path of unitaries.

\( A_n \) has a canonical Cartan subalgebra:

\[ \{ f \in A_n : f(t) \in D_{p_n} \otimes D_{q_n} \} \]

But: These Cartan subalgebras are not preserved by \( \phi_n \) because of \( u_n^{+1} \).

We only need \( u_n^{+1} \) to ensure the boundary conditions at \( t = 0, 1 \).

So only the permutation matrices \( u_n^{+1}(0) \) and \( u_n^{+1}(1) \) matter.

So we can modify the building blocks to \( \bar{A}_n \) so that the new connecting maps are given by \( \bar{\phi}_n(f) = (f \circ \lambda_y) y \).

Now, these new connecting maps preserve the Cartan subalgebras \( B_n := \{ f \in \bar{A}_n : f(t) \in D_{p_n} \otimes D_{q_n} \} \).

So we get a Cartan subalgebra \( B := \lim_{n \to -\infty} \{ B_n; \bar{\phi}_n \} \in Z \).
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- $\varphi_n(f) = u_{n+1}^* \cdot (f \circ \lambda_y)_y \cdot u_{n+1}$, with $\lambda_y \in \left\{ \frac{t}{2}, \frac{1}{2}, \frac{t+1}{2} \right\}$, and $u_{n+1}$ is a path of unitaries.
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2. $\varphi_n(f) = u_{n+1}^* \cdot (f \circ \lambda_y)y \cdot u_{n+1}$, with $\lambda_y \in \{\frac{t}{2}, \frac{1}{2}, \frac{t+1}{2}\}$, and $u_{n+1}$ is a path of unitaries.

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The Jiang-Su algebra is given by $\mathcal{Z} = \lim_{n \to -\infty} \{A_n; \varphi_n\}$, where

- $A_n = \{f \in C([0, 1], M_{p_n} \otimes M_{q_n}): f(0) \in M_{p_n} \otimes 1, f(1) \in 1 \otimes M_{q_n}\}$, with $p_n, q_n \in \mathbb{N}$: $\gcd(p_n, q_n) = 1$, $p_n \mid p_{n+1}$, $q_n \mid q_{n+1}$ ...;

- $\varphi_n(f) = u_{n+1}^* \cdot (f \circ \lambda_y)_y \cdot u_{n+1}$, with $\lambda_y \in \{\frac{t}{2}, \frac{1}{2}, \frac{t+1}{2}\}$, and $u_{n+1}$ is a path of unitaries.

- $A_n$ has a canonical Cartan subalgebra: $\{f \in A_n: f(t) \in D_{p_n} \otimes D_{q_n}\}$.

But: These Cartan subalgebras are not preserved by $\varphi_n$ because of $u_{n+1}$.

- We only need $u_{n+1}$ to ensure the boundary conditions at $t = 0, 1$.

So only the permutation matrices $u_{n+1}(0)$ and $u_{n+1}(1)$ matter.
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- So we can modify the building blocks to \( \bar{A}_n \) so that the new connecting maps are given by \( \bar{\varphi}_n(f) = (f \circ \lambda_y)_y \).
Constructing Cartan subalgebras in the Jiang-Su algebra

The Jiang-Su algebra is given by $\mathcal{Z} = \lim_{\to} \{ A_n; \varphi_n \}$, where

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- We only need $u_{n+1}$ to ensure the boundary conditions at $t = 0, 1$.
  So only the permutation matrices $u_{n+1}(0)$ and $u_{n+1}(1)$ matter.

- So we can modify the building blocks to $\tilde{A}_n$ so that the new connecting maps are given by $\tilde{\varphi}_n(f) = (f \circ \lambda_y)_y$. Now, these new connecting maps perserve the Cartan subalgebras $B_n := \{ f \in \tilde{A}_n : f(t) \in D_{p_n} \otimes D_{q_n} \}$.
Constructing Cartan subalgebras in the Jiang-Su algebra

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  So we get a Cartan subalgebra \( B := \lim_{\leftarrow} \{ B_n; \tilde{\varphi}_n \} \) in \( \mathcal{Z} \cong \lim_{\leftarrow} \{ \tilde{A}_n; \tilde{\varphi}_n \} \).
How does the groupoid look like?

Our Cartan subalgebra is given by

$$B := \lim_{\to} \{ B_n; \bar{\phi}_n \}.$$ 

Define

$$X_n := \text{Spec}(B_n)$$

and

$$X := \text{Spec}(B).$$

Then

$$X \sim = \lim_{\leftarrow} \{ X_n; \pi_n \},$$

where

$$\pi_n: X_n + 1 \to X_n$$

induces

$$\bar{\phi}_n: B_n \to B_n + 1,$$

i.e.,

$$\bar{\phi}_n = (\pi_n)^*.$$ 

So

$$X = \{ (x_n)_{n \in \mathbb{N}} \in \prod X_n: \pi_n(x_{n+1}) = x_n \},$$

and

$$X_n$$

are bipartite graphs of the form

Our groupoid $$G$$ with

$$(C^* r(G), C(G(0))) \sim = (Z, B)$$

is then given by the "tail equivalence relation" on

$$X = G(0).$$
How does the groupoid look like?

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- Define $X_n := \text{Spec}(B_n)$ and $X := \text{Spec}(B)$. Then $X \sim = \lim \left\{ X_n; \pi_n \right\}$, where $\pi_n: X_n + 1 \to X_n$ induces $\varphi_n: B_n \to B_n + 1$, i.e., $\varphi_n = (\pi_n)^\ast$. 

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- Our groupoid $G$ with $(C^\ast_r(G), C(G(0))) \sim = (\mathbb{Z}, B)$ is then given by the "tail equivalence relation" on $X = G(0)$. 

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How does the groupoid look like?

- Our Cartan subalgebra is given by $B := \lim \{ B_n; \varphi_n \}$.
- Define $X_n := \text{Spec} (B_n)$ and $X := \text{Spec} (B)$. Then $X \cong \lim \{ X_n; \pi_n \}$, where $\pi_n : X_{n+1} \to X_n$ induces $\varphi_n : B_n \to B_{n+1}$, i.e., $\varphi_n = (\pi_n)^*$. 
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- Our groupoid $G$ with $(C^*_r(G), C(G^{(0)})) \cong (\mathcal{Z}, B)$ is then given by the “tail equivalence relation” on $X = G^{(0)}$. 
What is the unit space?

Our unit space $X$ is compact, Hausdorff, metrizable and connected. $\dim X = 1$ implies that $\dim X = 1$.

Theorem (L): Under a certain condition on the unitaries $u_n + 1$, $X$ is locally path-connected.

In this case, $X$ is a one-dimensional Peano continuum.

Computing Čech-homology of $X$, we get: $X \sim \text{ShHawaiian earring.}$
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**Theorem** (L): Under a certain condition on the unitaries \( u_{n+1} \), \( X \sim_{\text{Sh}} \) Hawaiian earring.
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**Theorem (L):** Under a certain condition on the unitaries $u_{n+1}$, $X \sim_{\text{Sh}}$ Hawaiian earring.

**Theorem (L):** Under stronger conditions on the unitaries $u_{n+1}$, $X \cong$ Menger curve.
What exactly is the unit space?

**Theorem (L):** Under a certain condition on the unitaries $u_{n+1}$, $X \sim_{\text{Sh}} \text{Hawaiian earring}$.

**Theorem (L):** Under stronger conditions on the unitaries $u_{n+1}$, $X \cong \text{Menger curve}$.
How many Cartan subalgebras have we constructed?

We have constructed Cartan subalgebras of $\mathbb{Z}$ whose groupoids are given by "tail equivalence relations" on the Menger curve.

Theorem (L): Our construction yields $2^{\aleph_0}$ many different Cartan subalgebras of $\mathbb{Z}$ whose spectra are all homeomorphic to the Menger curve.
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The End

Thank you very much!