GROUPOIDS AND OPERATOR ALGEBRAS
Applications to Analysis, Geometry and Dynamics

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Weak Cartan subalgebras of C*-algebras

Ruy Exel & David Pitts

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1986 - Based on Renault’s work, Kumjian introduces the notion of C*-diagonals and relates it to twisted principal groupoids.

2008 - Based on Kumjian’s work, Renault finds the “right definition” of Cartan subalgebras in the context of C*-algebras, proving that they correspond exactly to Hausdorff, essentially principal groupoids.

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2011 - First example of a non-Hausdorff, essentially principal groupoid G, such that $C_0(G^{(0)})$ is not maximal commutative in $C^*_{\mathrm{red}}(G)$. It is also a counter-example for many well known results, once the Hausdorff hypotheses is droped.

2019 - In collaboration with D. Pitts, we found a generalization of Renault's Theorem for non-Hausdorff groupoids, which is the subject of this talk.

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(iv) there exists a faithful conditional expectation $E : B \to A$.

Theorem. (Renault 2008) The separable Cartan pairs "$A \subseteq B$" are precisely

$$C_0(G^{(0)}) \subseteq C^*_{\mathrm{red}}(G, \Sigma),$$

where (G, Σ) is a second countable, Hausdorff, essentially principal, twisted, étale groupoid.

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- In Renault's Theorem, there is a close connection between hypotheses (iv) (existence of conditional expectation) on the one hand, and Hausdorffness of the groupoid on the other.
- Therefore one would hope to prove a generalization where both of these are removed.
- However there is a non-Hausdorff groupoid G satisfying all of the above hypotheses (except for Hausdorffness), such $C_0(G^{(0)})$ is not maximal commutative in $C^*_{\mathrm{red}}(G)$. This dashes the above hope!

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(i) We say that x is a <u>free point</u> if φ_x admits a unique extension to a state on B.

(ii) We say that "$A \subseteq B$" is a <u>topologically free inclusion</u> if the subset $F \subseteq X$ consisting of free points is dense in X.

- This is inspired by the fact that, for "$C_0(G^{(0)}) \subseteq C^*_{\mathrm{red}}(G, \Sigma)$", a point x in $G^{(0)}$ is free iff the isotropy group $G(x)$ is trivial.

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- If x is free, then it is free relative to every $b \in B$, but x may be free only relative to some b.

- Topologically the F_b may be very badly behaved. There is an example in which $X = [0, 1]$, and F_b consists of the irrational numbers in $[0, 1]$.

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- Smoothness is one of the two main hypotheses in our generalization. It replaces maximal commutativity.

- Every separable smooth inclusion is topologically free.

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Proposition. The set

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Definition. We shall call it the <u>gray ideal</u>.

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- Regarding the inclusion "$C_0(G^{(0)}) \subseteq C^*(G,\Sigma)$", and assuming that G is Hausdorff, the gray ideal is the kernel of the left regular representation.

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- The vanishing of the gray ideal will be our replacement for the existence of a conditional expectation.

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- Alternatively, $C^*_{\text{ess}}(G, \Sigma)$ may be defined as the completion of $C_c(G, \Sigma)$ under the smallest C*-seminorm coinciding with the uniform norm on $C_c(G^{(0)})$.
- One must prove that such a smallest C*-seminorm does indeed exist.

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Definition. A regular inclusion of C*-algebras "$A \subseteq B$" is said to be <u>weak Cartan</u> if it is smooth and its gray ideal vanishes.

Theorem. (E., Pitts) The weak Cartan inclusions, with B separable, are precisely

$$C_0(G^{(0)}) \subseteq C^*_{\text{ess}}(G, \Sigma),$$

where (G, Σ) is a (not necessarily Hausdorff) second countable, topologically free, twisted, étale groupoid.

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- In fact one can get smoothness with MUCH LESS than maximal commutativity!

Theorem. Suppose that $N \subseteq N_B(A)$ is a *-semigroup such that $\overline{\mathrm{span}}(NA) = B$, and

$$(NA) \cap A' \subseteq A,$$

then every n in N is smooth, and consequently "$A \subseteq B$" is a smooth inclusion.

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Warning: the above twist is nontrivial!

VANISHING OF THE GRAY IDEAL

One of the outstanding problems in this subject is to determine when is the gray ideal of the reduced groupoid C*-algebra trivial, meaning that

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- As already noticed, this is the case for Hausdorff groupoids.
- The vanishing of the gray ideal depends on the twist!

VANISHING OF THE GRAY IDEAL

One of the outstanding problems in this subject is to determine when is the gray ideal of the reduced groupoid C*-algebra trivial, meaning that

$$C^*_{\mathrm{ess}}(G, \Sigma) \overset{(?)}{=} C^*_{\mathrm{red}}(G, \Sigma)$$

- As already noticed, this is the case for Hausdorff groupoids.
- The vanishing of the gray ideal depends on the twist!
- More precisely, there is a twisted groupoid (G, Σ), such that the gray ideal for the inclusion

  $$C_0(G^{(0)}) \subseteq C^*_{\mathrm{red}}(G, \Sigma)$$

  is zero but it is nonzero for the untwisted version

  $$C_0(G^{(0)}) \subseteq C^*_{\mathrm{red}}(G).$$

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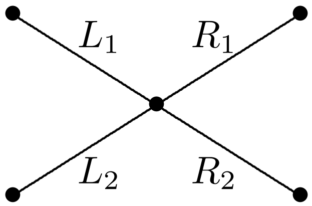

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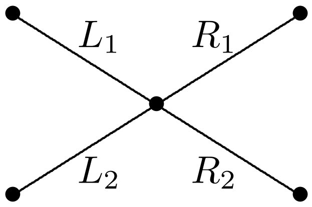

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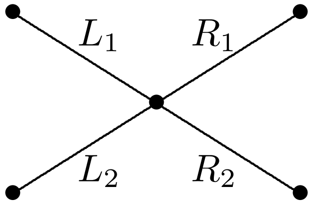

- ▶ The twist is nontrivial and "$C_0(G^{(0)}) \subseteq C^*_{\text{red}}(G, \Sigma)$" has a zero gray ideal.
- ▶ However, for the same groupoid, the untwisted inclusion "$C_0(G^{(0)}) \subseteq C^*_{\text{red}}(G)$" has a nonzero gray ideal!

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- If x is a trivial point for n then, not only is x fixed by β_n, but the germ of β_n at x is trivial.

- It is easy to show that if ψ is a state on B extending φ_x then, for every normalizer n, one has that

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- For the inclusion "$C_0(G^{(0)}) \subseteq C^*_{\text{red}}(G, \Sigma)$", where (G, Σ) is any twisted, étale groupoid, and for N being the set of all normalizers supported on bisections, there is a unique N-canonical state ψ on $C^*_{\text{red}}(G, \Sigma)$ relative to any given x, namely

$$\psi(f) = f(x), \quad \forall\, f \in C^*_{\text{red}}(G, \Sigma).$$

Theorem. Let "$A \subseteq B$" be a regular inclusion, and let $N \subseteq N_B(A)$ be a *-semigroup such that $\overline{\operatorname{span}}(NA) = B$.

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(i.e. μ is a C*-norm on $C_c(G, \Sigma)$ in between the full and reduced norms), the isomorphism carrying $C_0(G^{(0)})$ onto A.

THANK YOU JEAN!

For building the infrastructure of this beautiful playground where we've been having so much fun!