Complex network approach to evolving manifolds and simplicial complexes

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G. Timár, A. V. Goltsev, SND, J. F. F. Mendes, Mapping the structure of directed networks: Beyond the "Bow-tie" diagram, Phys. Rev. Lett. 118, 078301 (2017).

D. C. da Silva, G. Bianconi, R.A. da Costa, SND, J.F.F. Mendes, Complex network view of evolving manifolds, Phys. Rev. E **97**, 032316 (2018).

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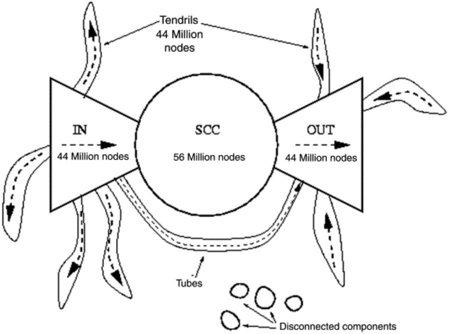
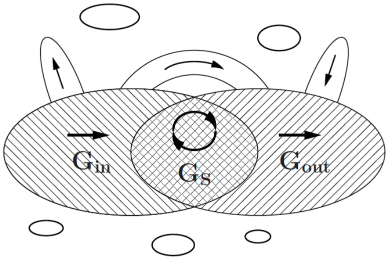
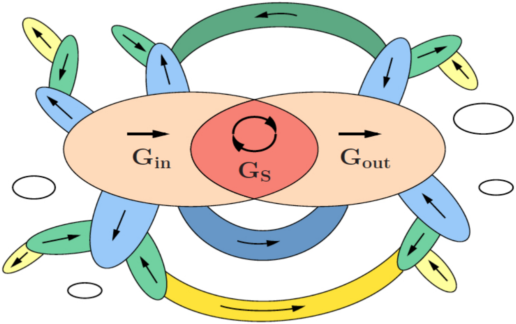
The bow-tie organization of the Web:



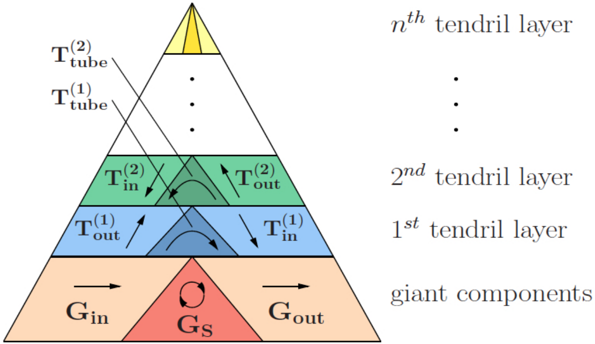
A. Broder, R. Kumar, F. Maghoul, P. Raghavan, S. Rajagopalan, R. Stata, A. Tomkins, J. Wiener (2000)

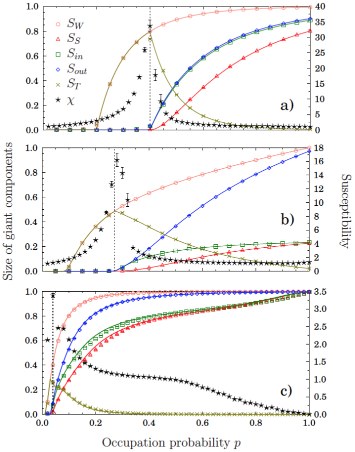


SND, J. F. F. Mendes, A. N. Samukhin (2001)



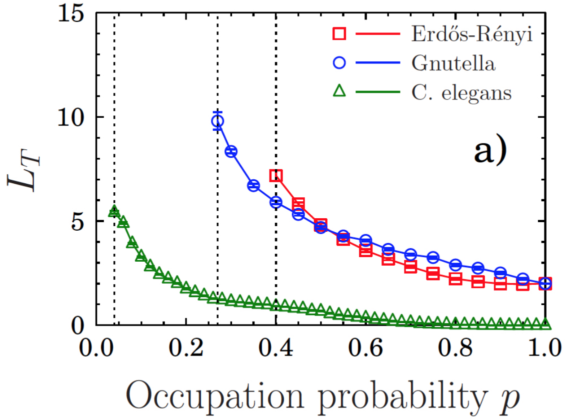
$\mathbf{G_{in}}$
$\mathbf{G_S}$
$\mathbf{G_{out}}$


$T^{(2)}_{tube}$
$T^{(1)}_{tube}$
$T^{(2)}_{in}$
$T^{(2)}_{out}$
$T^{(1)}_{out}$
$T^{(1)}_{in}$
G_{in}
G_S
G_{out}
n^{th} tendril layer
⋮
2^{nd} tendril layer
1^{st} tendril layer
giant components



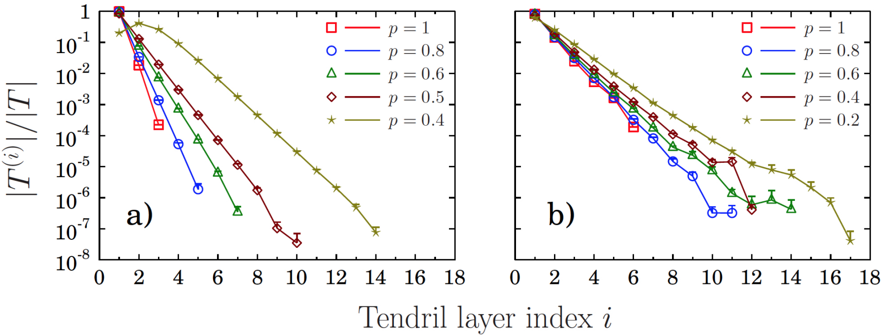
(a) directed ER ($N = 10000$, $\langle q_{tot} \rangle = 5$),
(b) the Gnutella P2P ($N = 62586$, $\langle q_{tot} \rangle = 4.726$),
(c) C. elegans ($N = 495$, $\langle q_{tot} \rangle = 32.073$).

Mean number of tendril layers L_T:



(Dashed lines correspond to p_c.)

Relative size of tendrils in layers:



(a) directed ER ($N = 10^6$, $\langle q_{tot} \rangle = 5$),
(b) the WWW ($N = 875713$, $\langle q_{tot} \rangle = 11.659$).

Manifolds & simplicial complexes:

Manifolds $\equiv$ topological spaces locally homeomorphic to Euclidean spaces
(e.g., surfaces)

d-simplex $\equiv d+1$-clique
[$d+1$ vertex and $(d+1)d/2$ edges]

d-simplicial complex $\equiv$ constructed only of d-simplexes

Simplicial complexes are the discrete versions of manifolds,
e.g., triangulation networks (sets of triangular faces) are the discrete versions of surfaces.

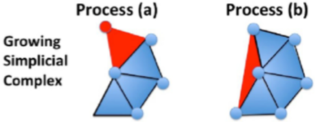
Closed manifolds vs. manifolds with a boundary

Triangulations are everywhere:

Triangulations are in the heart of modern civilization as the main method of treatment of surfaces in topography, engineering, hydrodynamics and aerodynamics, visualization techniques, and everywhere.

Particular planar graphs, strong constraint (!)

Manifolds with boundaries:



Wu, Menichetti, Rahmede, and Bianconi (2015)

Modelling closed manifolds using evolving simplicial complexes:

- local structure — ???

(degree distribution, degree–degreecorrelations)

- space dimension — ???

(Hausdorff and spectral dimensions)

- topology — ???

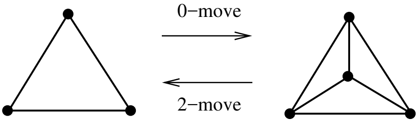
Topology:

Topology is about the properties of space that are preserved under continuous deformations.

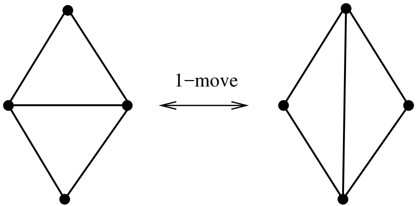
Triangular mesh operations—1:

Pachner moves:

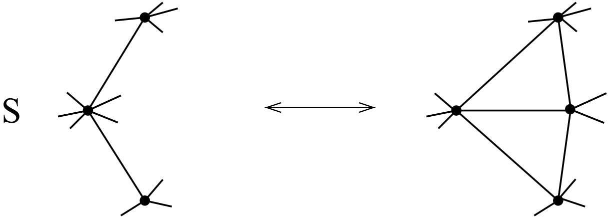
P1



P2



Triangular mesh operations—2:



Triangulations of closed surfaces:

Euler's formula for general polyhedra

$$\chi = F + N - E,$$

$$\chi = 2(1 - h),$$

If there are no boundaries, then

$$3F = 2E,$$

$$\chi = N - \frac{1}{3}E = N - \frac{1}{2}F.$$

Local curvature:

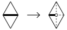
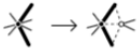
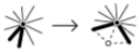
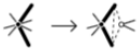
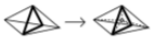
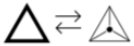
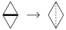
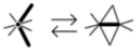
$$R_i = 1 - \frac{1}{6} q_i ,$$

assuming that all edges are equal.

Organization of models:

At each time step,
(i) an element or neighboring elements of the simplicial complex under consideration are chosen with some preference or, in the simplest particular case, without preference, i.e., uniformly at random. For triangulations, such elements are vertices, edges, and triangles.
Then,
(ii) a specific transformation from the set of operations that keep the simplicial complex intact is applied to this element.

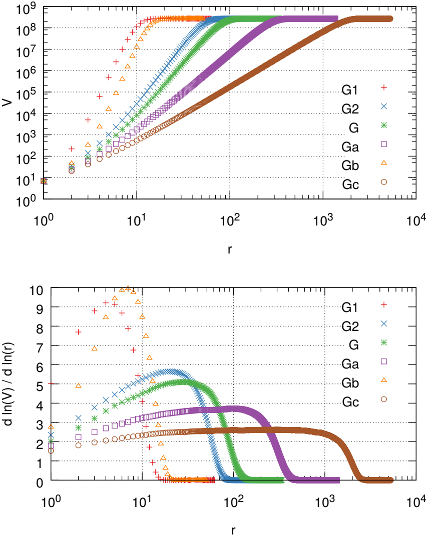
Zoo:

Model	Operation at each step of evolution	Scheme
Growing triangulations		
G1	(i) choose a triangle uniformly at random, (ii) attach a new vertex to all three vertices of this triangle (Pachner's 0-move). (This rule is closely related to the one governing the evolution of random Apollonian networks [32–34]. The difference is that the manifold is closed here.)	
G2	(i) choose an edge uniformly at random, (ii) exchange it for a new vertex attached to all four vertices of the two triangles sharing this edge.	
G	(i) choose a vertex uniformly at random and two its random edges, (ii) split them in the way shown in Fig. 3, the move from left to right.	
Ga	(i) choose a vertex uniformly at random and one of its edges at random; then, among the rest edges of the vertex, if the vertex degree is even, choose the opposite edge to the first, if the degree is odd, choose, with equal probability, one of the two most remote edges (here remote, relatively to the first); (ii) split them in the way shown in Fig. 3, the move from left to right.	
Gb	(i) choose a vertex uniformly at random and one of its edges at random; then, among the rest edges of the vertex, choose, with equal probability, one of the two closest (to the first) edges; (ii) split them in the way shown in Fig. 3, the move from left to right. (Gb is close to rule G1. The difference is that here a random triangle incident to a uniformly chosen vertex is selected.)	
Gc	(i) choose an edge uniformly at random and its end vertex with the highest number of connections (if the degrees of the ends coincide, then choose any one of them with equal probability), (ii) choose the second edge as in rule Ga, (iii) split the two chosen edges and the vertex in the way shown in Fig. 3, the move from left to right.	
G′	(i) choose a vertex uniformly at random and two its random edges except those belonging to the same triangle, (ii) split them in the way shown in Fig. 5, the move from left to right.	
Growing d-dimensional manifolds		
G1d	(i) choose a simplicial complex uniformly at random, and (ii) attach a new vertex to all $d+1$ vertices of this simplicial complex. (Note that G1d directly generalizes G1 to an arbitrary $d \geq 2$.)	
G2d	(i) choose a $(d-1)$-simplex uniformly at random, (ii) attach a new vertex to all $d+2$ vertices of the two d-simplices sharing the chosen $(d-1)$-simplex. (Note that G2d is defined only for $d \geq 3$, and so G2d does not generalize G2 directly.)	
Generation of holes in a growing triangulation		
GW	(i) at each step perform rule G2, and, in addition, (ii) at each θ-th step choose two triangles, excluding first- and second-neighboring ones, uniformly at random and merge them into a single triangle creating a hole in the manifold.	
Equilibrium triangulations		
E1	(i) choose a vertex of degree 3 uniformly at random and remove it, (ii) choose a triangle uniformly at random and attach a new vertex to all three vertices of this triangle.	
E2	(i) choose an edge uniformly at random, (ii) perform Pachner 1-move (flip) with this edge (see Fig. 1, P2).	
E3	(i) choose an edge uniformly at random and compress it into one vertex as in transformation S, Fig. 3, the move from left to right, (ii) make a step according to rule G.	

Be careful:

The elements of triangulations are faces and not edges and vertices.

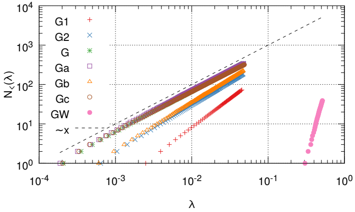
Hausdorff dimension:

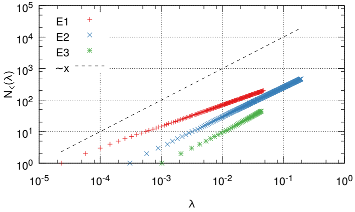


Spectral dimension:

$$\varphi_0(t) \sim t^{-d_S/2}, \qquad \rho(\lambda) \sim \lambda^{d_S/2-1}$$

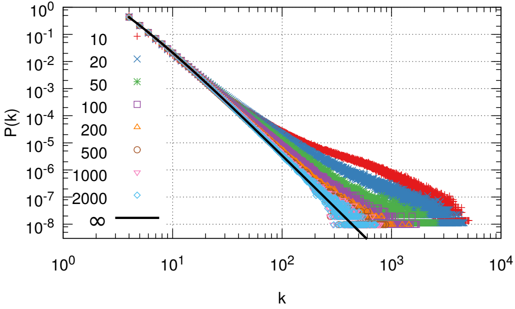
$$d_S \leq d_H$$





Evolving topology—

generation of wormholes:
at each step, perform G2 (creates a new face);
in addition, at each θ-th step merge two faces and eliminate them



Results:

model	exponent γ	Hausdorff dimension d_H	spectral dimension d_S
		Growing triangulations	
G1	3	∞	2.9(2)
G2	4	>5.7	2.4(2)
G	∞	>5.0	2.0(3)
Ga	∞	3.8(2)	2.0(4)
Gb	>4.5	∞	2.4(3)
Gc	∞	2.6(2)	2.0(3)
		Growing triangulation with increasing number of holes	
GW	$\lesssim 2.0$	∞	$\gtrsim 14$
		Equilibrium triangulations	
E1	∞	—	1.4(2)
E2	∞	—	1.9(4)
E3	3.0(2)	—	2.1(2)

E1 vs. equilibrium random trees

Model E1:
random addition and addition vertices of degree 3 with equal rates.

$$d_H \sim 2(?), \qquad d_S = 1.4(2).$$

Equilibrium random trees:

$$d_H = 2, \qquad d_S = 4/3.$$

Self-assembling systems:

Optimal nets for self-folding systems:

Self-assembling

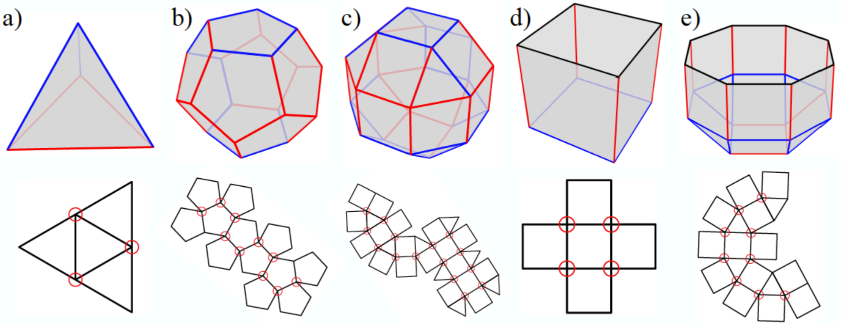
3D shells are synthesized from the self-folding of 2D templates of interconnected panels, called nets.

The yield is maximized following sequentially two design rules:
(i) maximum number of vertices with a single-edge cut and, e.g.,
(ii) minimum radius of gyration of the net.

A vertex with a single-edge cut is one whose the number of adjacent faces is the same in the net and in the shell.

Step (i) is NP hard

Shells and nets:



Difficult:

Previously:
a random search, scanning only a small subset of possible nets and thus missing the global optimum.

(a dodecahedron, E=30 edges, has more than 5 million possible cuts).

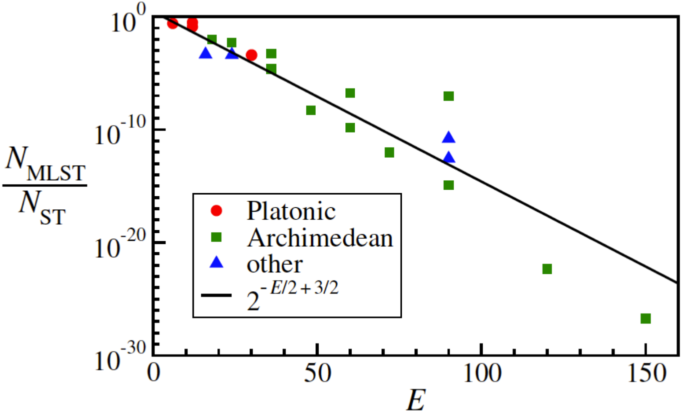
Our deterministic algorithm:
finds exactly the global minimum for solids with E up to 150 (desktop).
Its stochastic version:
approaches the global minimum for higher E.

Idea:

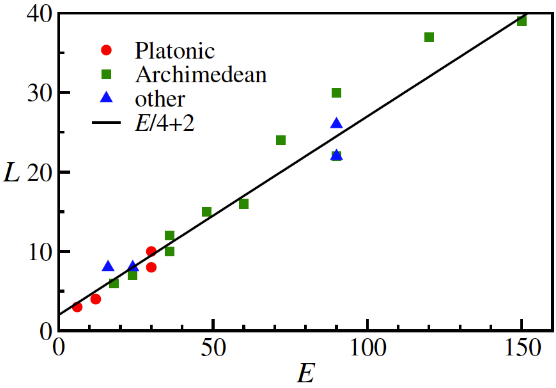
Definition:
A shell graph consists of the vertices and edges of the shell.

Design rule (i) corresponds to finding the set of maximum leaf spanning trees of the shell graph.

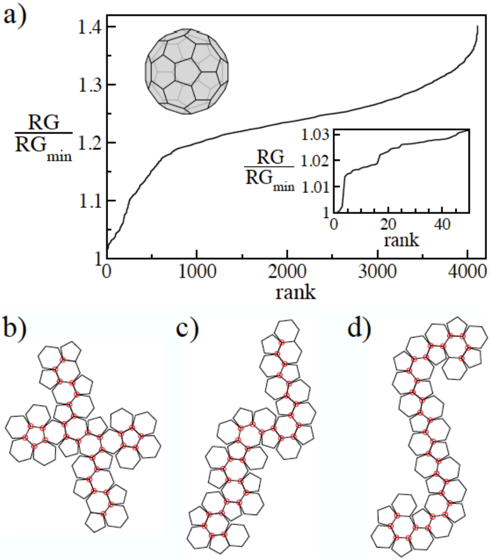
Fraction of spanning trees, N_{ST}, that are maximum leaf spanning trees, N_{MLST}, as a function of the number of shell edges, E



Number of leaves, L, in a MLST as a function of the number of shell edges, E



Soccer ball-1:



Soccer ball-2:

Truncated icosahedron (soccer ball): (a) spectrum of the radii of gyration for all the 4114 non-isomorphic MLSTs, ordered by increasing radius of gyration; the MLST with the (b) optimal (rank 1); (c) intermediate (rank 2057: $RG = RG_{min} \approx 1.23$); and (d) largest (rank 4114: $RG = RG_{min} \approx 1.40$) radius of gyration.

Conclusion:

$$\sim$$
$$\sim$$
$$\sim$$