From Fractal Weyl Laws to spectral questions on sparse directed graphs

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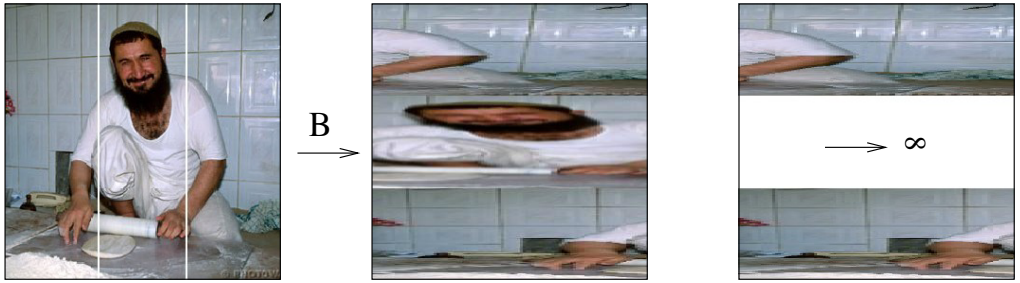
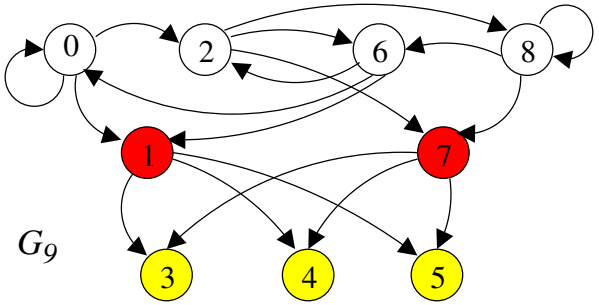




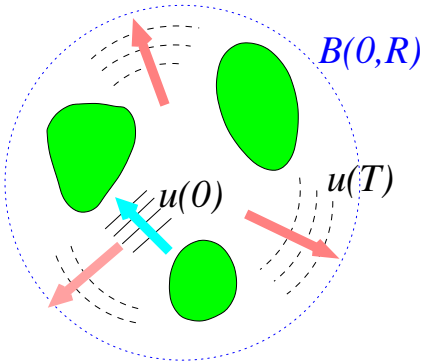
Wave scattering and decay

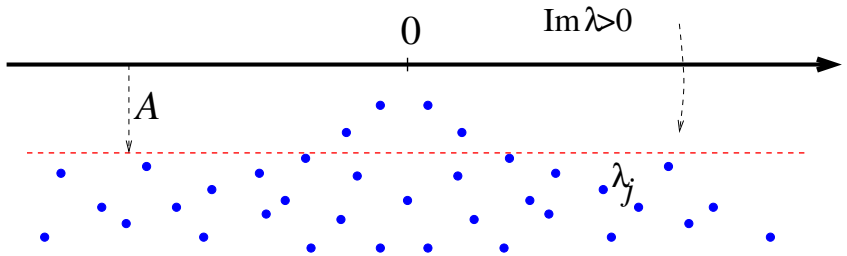
Wave scattering problem: wave equation $(\partial_t^2 - \Delta_\Omega)u = 0$ outside a set of obstacles, $\Omega = \mathbb{R}^d \setminus \mathcal{O}$.

- Long time behaviour of $u(t)$? $\rightsquigarrow$ consider the “spectrum" of the Laplacian $-\Delta_\Omega$ on L^2.

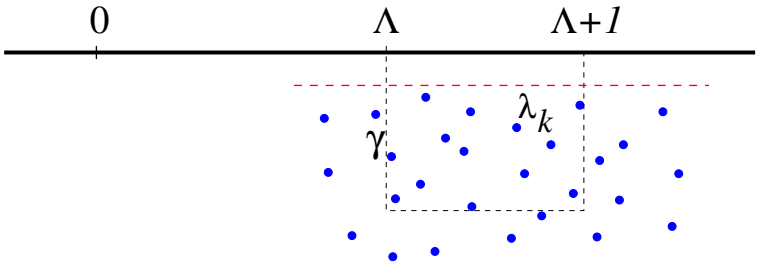
Continuous real spectrum, but complex-valued resonances govern the long time evolution:

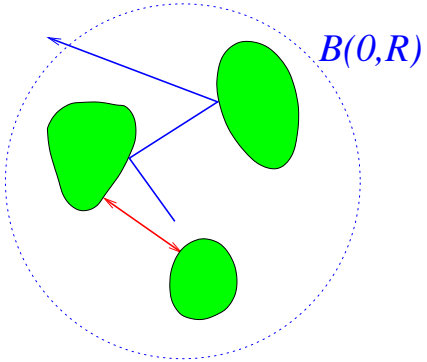
$$\text{inside } B(0,R), \quad u(t) = \sum_{\operatorname{Im}\lambda_j > -A} e^{-it\lambda_j} \langle v_j, \partial_t u(0)\rangle\, v_j + \mathcal{O}(e^{-At}), \quad t \to \infty.$$





High frequency resonances. Quantum open chaos

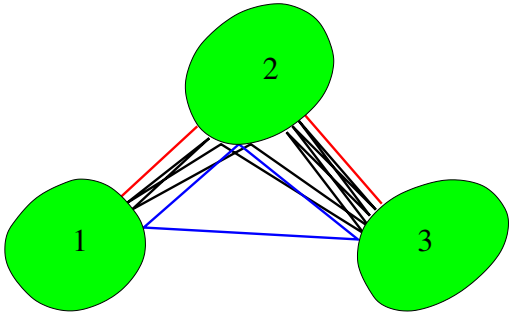




High frequency waves: need to understand the distribution of resonances in the regime $\mathrm{Re}\,\lambda_j \gg 1$.

- Counting resonances: $\mathcal{N}(\Lambda,\gamma) \sim$? when $\Lambda \gg 1$?
- Is there a resonance gap $\mathrm{Im}\,\lambda_j \leq -\alpha$?

High frequency $\Longrightarrow$ ray dynamics. Long times $\rightarrow$ trapped rays are crucial.



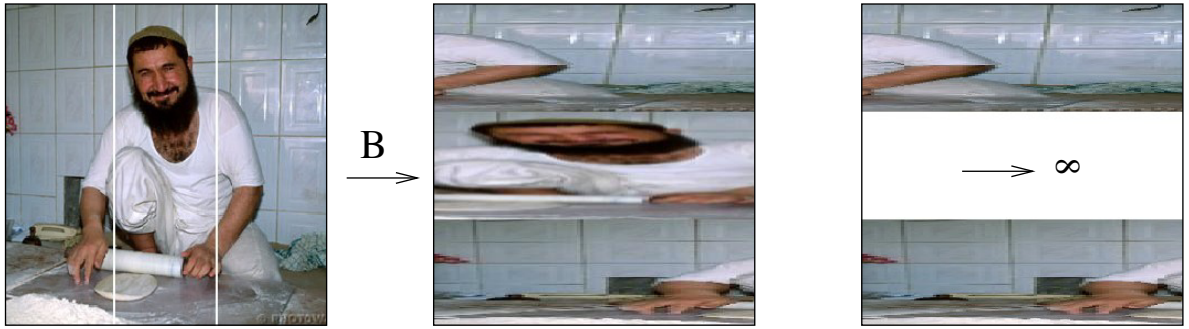
For $n \geq 3$ convex obstacles, the trapped rays form a *fractal chaotic repeller*.
[SJÖSTRAND'90, ZWORSKI'99] conjecture a fractal Weyl law

$$\mathcal{N}(\Lambda,\gamma) \sim C\Lambda^{\nu},$$

with $\nu > 0$ related with the dimension of the trapped set.
They proved the upper bound.

A chaotic toy model: the open baker's map

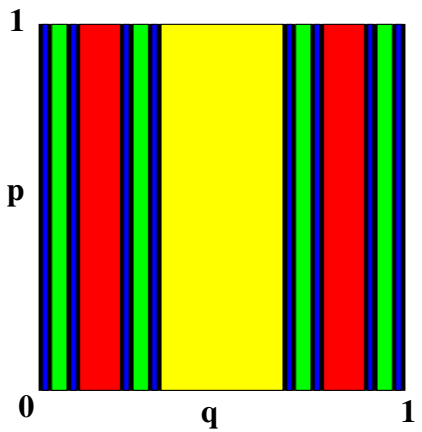
Difficult to compute the resonances of $-\Delta_\Omega$ numerically at high frequency.
$\Longrightarrow$ construct toy models: discrete-time dynamics



An example of chaotic map: the **open baker's map** on $\mathbb{T}^2 \ni (q,p)$:

$$(q,p) \mapsto B(q,p) = \begin{cases} (3q \bmod 1, \frac{1}{3}(p+[3q])), & q \in [0,1/3) \cup [2/3,1) \\ \infty \quad \text{(hole)}, & q \in [1/3,2/3) \end{cases} .$$

In base 3: $(p,q) \equiv \ldots \epsilon_2' \epsilon_1' \bullet \epsilon_1 \epsilon_2 \ldots \mapsto \ldots \epsilon_2' \epsilon_1' \epsilon_1 \bullet \epsilon_2 \epsilon_3 \ldots$ if $\epsilon_1 \in \{0,2\}$.



(each color: points escaping at a given time).

Trapped set:
$\Gamma_- = \{(p,q), B^n(p,q)$ exists for all $n \geq 1\}$
$= \{\ldots \epsilon_2' \epsilon_1' \bullet \epsilon_1 \epsilon_2 \ldots, \ \epsilon_k \neq 1\}$

$\Gamma_- = [0,1] \times Can$, with $\nu = \dim(Can) = \frac{\log 2}{\log 3}$.

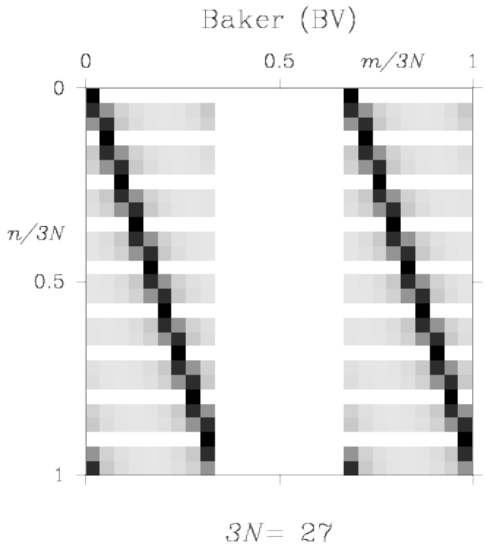
Quantum open baker

One can set up a quantum mechanics associated with the *phase space* $\mathbb{T}^2$: to each $N \in \mathbb{N}^*$ the quantum space $\mathcal{H}_N \equiv \mathbb{C}^N$ is generated by the basis of *position states* $\{\mathbf{q}_0, \ldots, \mathbf{q}_{N-1}\}$ localized at positions $q_j = \frac{j}{N}$ connected with *momentum states* $\{\mathbf{p}_0, \ldots, \mathbf{p}_{N-1}\}$ through the discrete Fourier transform:

$$\boldsymbol{p}_k = F_N{}^* \boldsymbol{q}_k = \sum_{j=0}^{N-1} (F_N^*)_{jk}\, \boldsymbol{q}_j, \qquad (F_N)_{kj} = \frac{\mathrm{e}^{-2i\pi \frac{j\,k}{N}}}{\sqrt{N}}$$

The open map $B : \mathbb{T}^2 \to \mathbb{T}^2$ can be quantized (when $3|N$) into a subunitary matrix $M_N : \mathcal{H}_N \to \mathcal{H}_N$ [Balazs-Voros'89]:

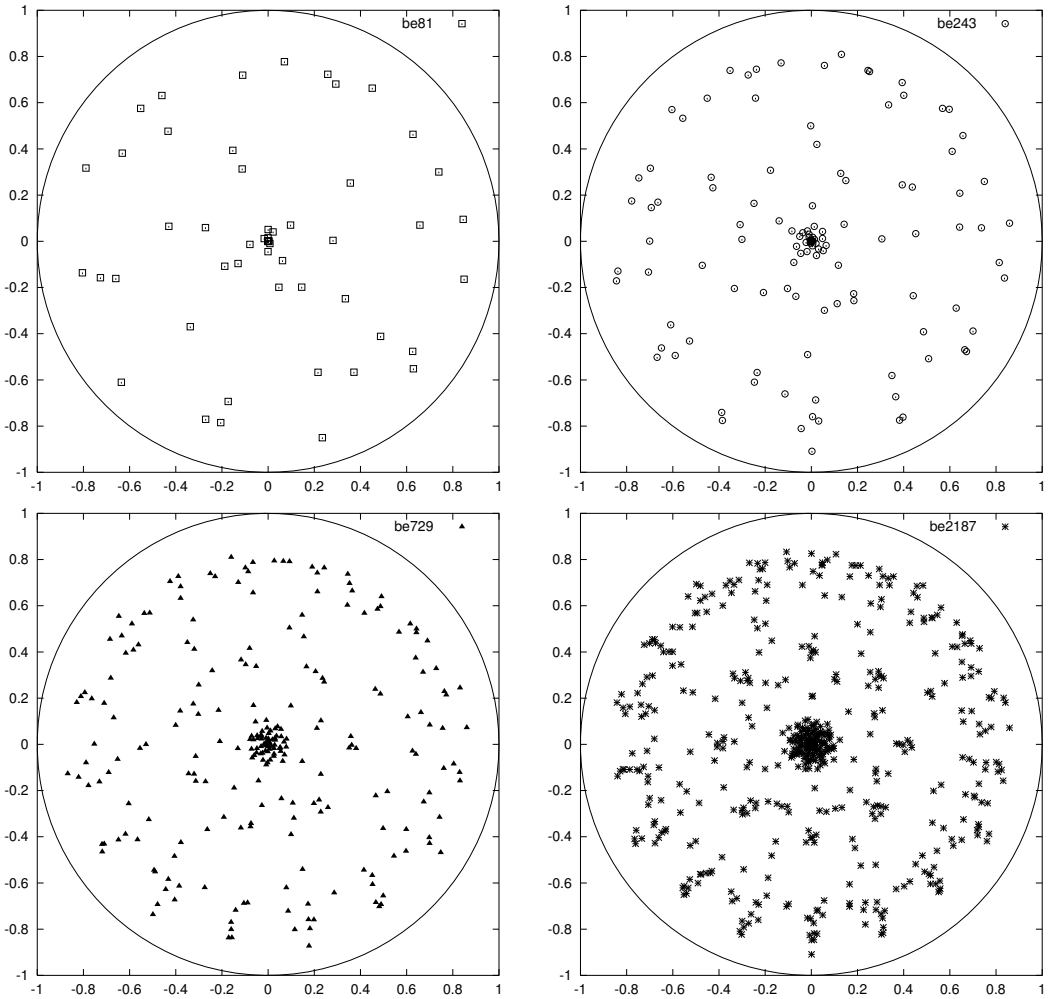
$$B_N = F_N^* \begin{pmatrix} F_{N/3} & & \\ & 0 & \\ & & F_{N/3} \end{pmatrix}$$



Like in the classical map, the central position states $\{1/3 \leq q_j < 2/3\}$ are killed by B_N.

Spectrum of the open baker

We expect the eigenvalues $(z_{j,N})_{j=1,\dots,N}$ of B_N to have a similar distribution as the resonances of the Laplacian.
Large-N regime $\Longleftrightarrow$ High-frequency regime.



$$\{z_{j,N}\} \Longleftrightarrow \{e^{-i\lambda_k},\ \mathrm{Re}\,\lambda_k \approx N\}$$

Although $\mathrm{rank}(B_N) = 2N/3$, we observe that most of the eigenvalues are very small.

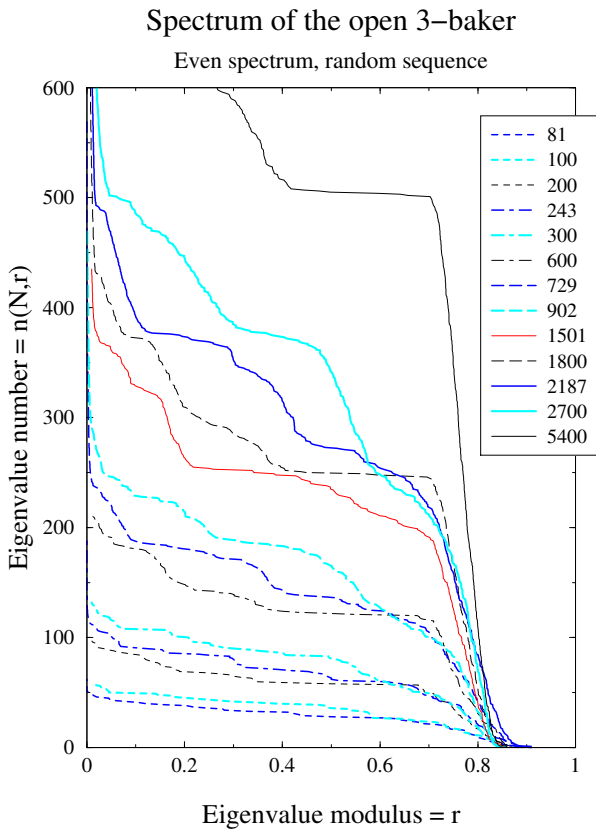
We count eigenvalues in annuli $\{1 \geq |z| \geq r\}$:

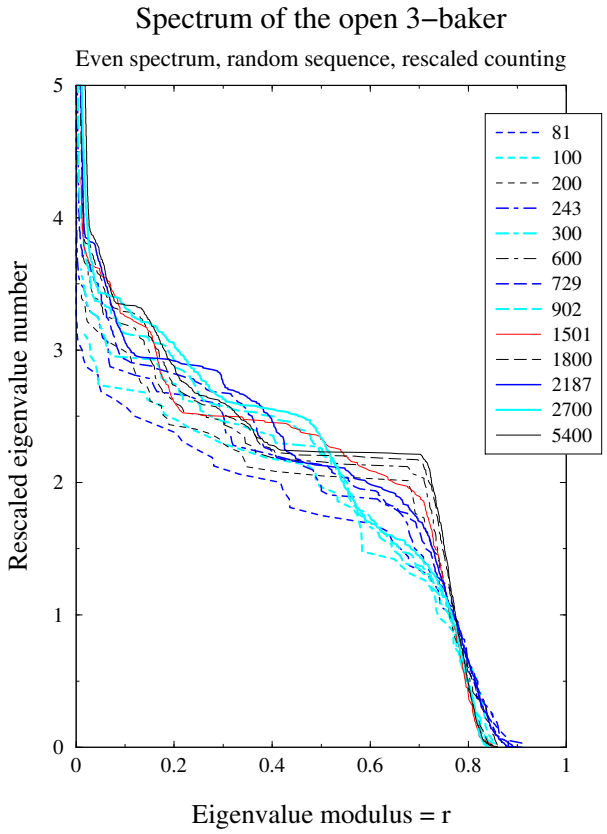
$$\mathcal{N}(N,r) = \#\{|z_{j,N}| \geq r\}.$$

Do we have $\mathcal{N}(N,r) \sim C(r)N^{\nu}$ as $N \to \infty$?
(Fractal Weyl Law)

Fractal Weyl law for the open baker

We know the fractal dimension of the trapped set: $\nu = \frac{\log 2}{\log 3}$.
Left: plot of $\mathcal{N}(N, r)$ as function of r, for several N. Right: plot of $\mathcal{N}(N, r)/N^{\nu}$.

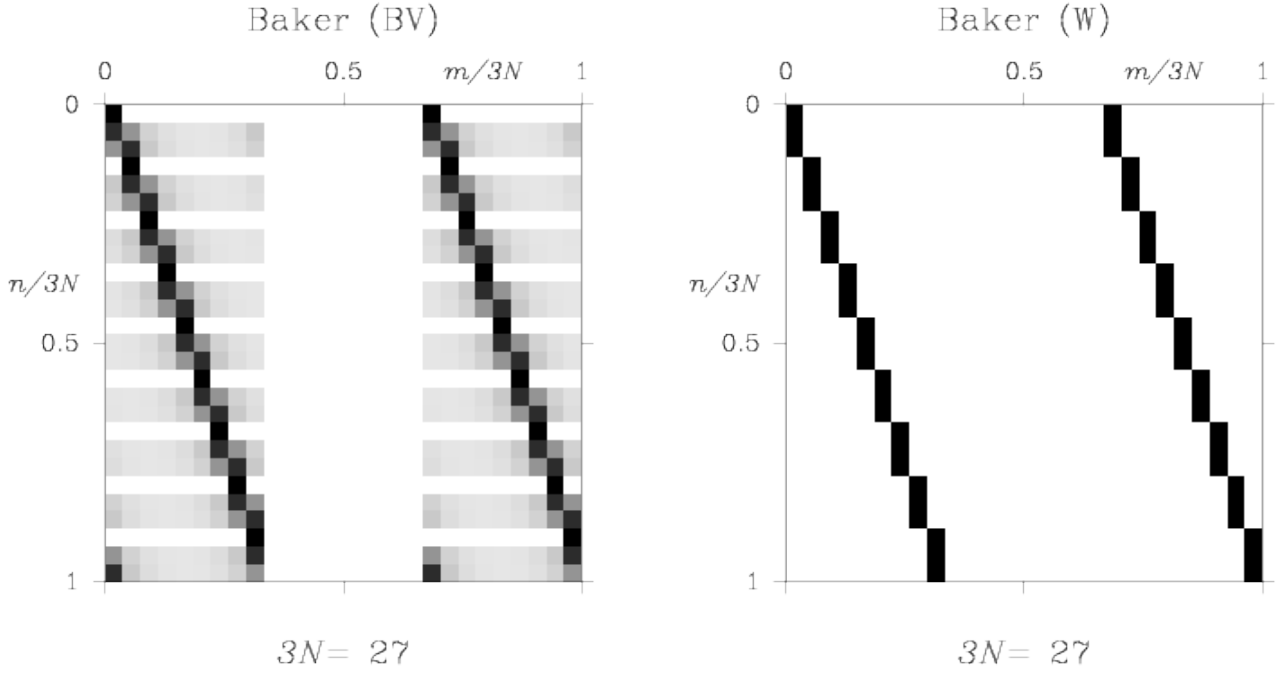




A FWL has been numerically observed on various quantized chaotic maps [SCHOMERUS-TWORZYDŁO'04, N-ZWORSKI'05, N-RUBIN'07, SHEPELYANSKY'08, KOPP-SCHOMERUS'10].

A toy model of the toy model

Even for a simple matrix like B_N, we have no proof of the FWL.
(Only the upper bound could be proved, for a "smoothed" map.)



• $(B_N)_{jk}$ is concentrated around the “lines" discretizing the graph of $q \mapsto 3q \bmod 1$ on $[0, 1/3) \cup [2/3, 1)$.

$\Longrightarrow$ Toy^2 model: replace B_N by its skeleton matrix S_N, keeping only the values along the “lines" $\{j = 3k + \epsilon,\ \epsilon = 0, 1, 2\}$ [N-Zworski'05].

Explicit spectrum of S_{3^K}

$\oplus$ For dimensions $N = 3^K$, the spectrum of the matrix S_N can be computed explicitly thanks to a tensor product decomposition.

$$S_9 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & \omega^2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & \omega & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \omega^2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \omega & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \omega^2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \omega \end{pmatrix}, \ \omega = e^{2\pi i/3}.$$

In base 3, $j \in \{0, \ldots, N-1\}$ is written as $j \equiv \epsilon_1 \epsilon_2 \cdots \epsilon_K$, with $\epsilon_i \in \{0, 1, 2\}$. $\mathbf{q}_j \in \mathbb{C}^N$ is represented by $e_{\epsilon_1} \otimes e_{\epsilon_2} \otimes \cdots \otimes e_{\epsilon_K} \in (\mathbb{C}^3)^{\otimes K}$

S_N acts nicely on this tensor product structure:

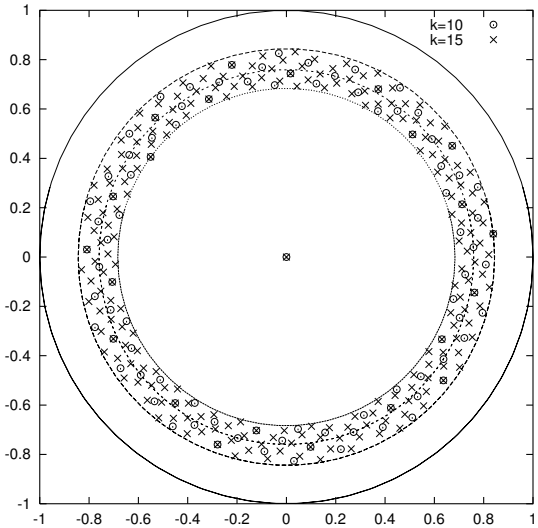
$S_N(v_1 \otimes v_2 \otimes \cdots v_K) = v_2 \otimes v_3 \cdots v_K \otimes \Omega_3 v_1$, with $\Omega_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & \omega^2 \\ 1 & 0 & \omega \end{pmatrix}$

$\Longrightarrow (S_N)^K v_1 \otimes v_2 \otimes \cdots v_K = \Omega_3 v_1 \otimes \Omega_3 v_2 \otimes \cdots \Omega_3 v_K.$

Spectrum of S_{3^K}

$$(S_N)^K = (\Omega_3)^{\otimes K}, \qquad \Omega_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & \omega^2 \\ 1 & 0 & \omega \end{pmatrix}$$

$$\mathrm{Spec}(\Omega_3) = \{0, \lambda_-, \lambda_+\} \Longrightarrow \mathrm{Spec}(S_N) = \{\lambda_+^{\ell/K}\lambda_-^{1-\ell/K}e^{2i\pi n/K}\} \cup \{0\}.$$



Large multiplicities, eigenvalues asymptotically concentrate near a circle.

Since $\mathrm{rank}\,\Omega_3 = 2$, the nontrivial spectrum of S_N has dimension $2^K = N^{\log 2/\log 3}$
$\Longrightarrow S_{3^K}$ satisfies a **fractal Weyl law**.

$\oplus$ First rigorous example of fractal Weyl law.
[N.-Zworski'05]

$\ominus$ The spectrum is very regular, strongly depends on $\mathrm{Spec}(\Omega_3)$.
For some modified versions of S_N, the corresponding Ω_3 may accidentally have a larger kernel ($\mathrm{rank}\,\Omega_3 = 1 \Longrightarrow \mathrm{rank}\,S_N = 1$).

What can we learn from the sole topology of the skeleton S_N?

Deriving the FWL from the topology of S_N

The topology of S_N is represented by a matrix A_N.

$$A_9 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

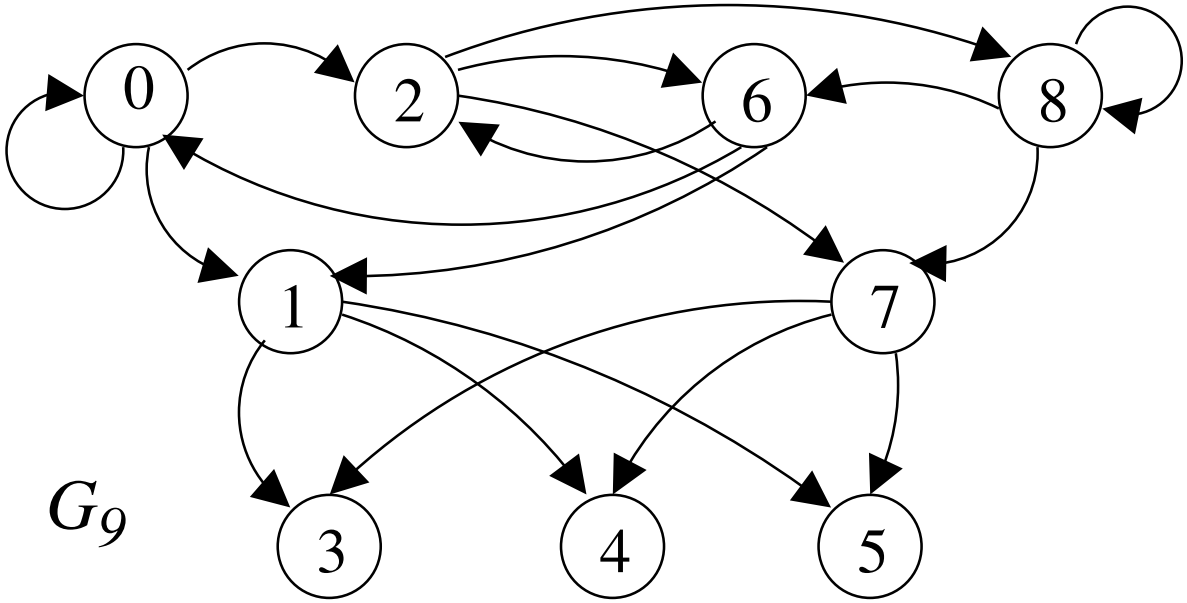
For $N = 3^K$, the tensor product structure shows that $(A_N)^K = |\Omega_3|^{\otimes K}$, with $|\Omega_3| = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$.

$\Longrightarrow$ all columns indexed by $j \equiv \epsilon_1 \cdots \epsilon_K$ with some $\epsilon_\ell = 1$ are null.
$(A_N)^K$ has exactly $3^K - 2^K$ null columns $\Longrightarrow \dim(\mathrm{gker}(S_N)) \geq 3^K - 2^K$.

- topological property: applies to S_N as well.

We may view A_N as the adjacency matrix of a directed graph G_N with $V = \{j \in \{0 \ldots, N-1\}\}$ and $E = \{(kj),\ A_{jk} = 1\}$.

A_N as a directed graph



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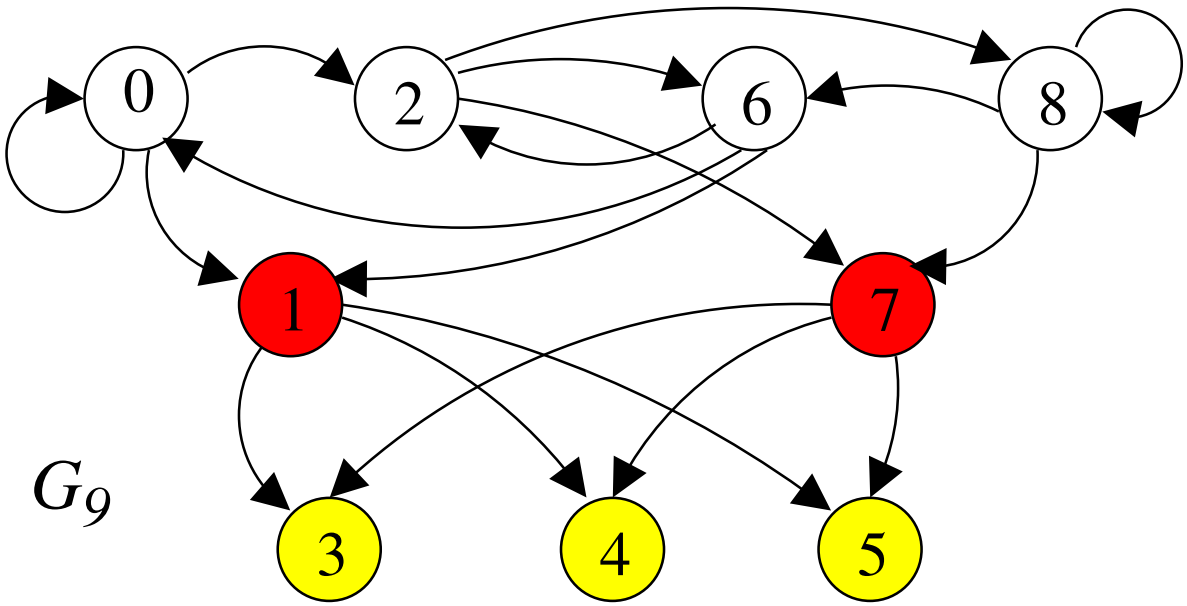
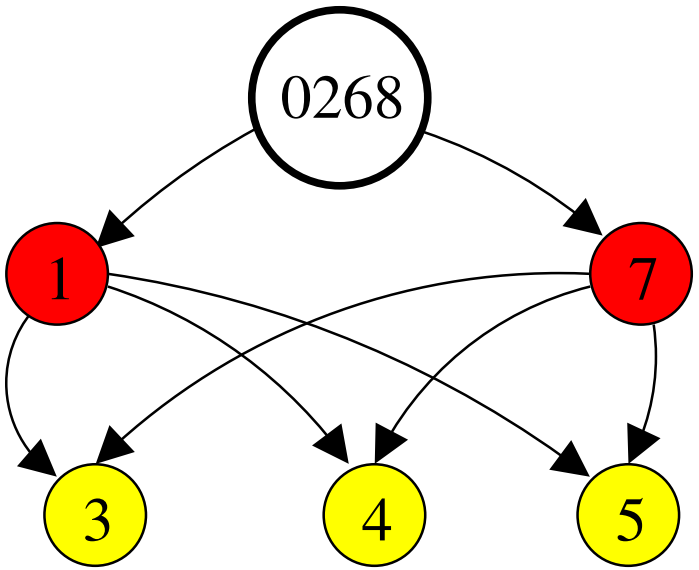


Yellow vertices have no image; red vertices have for only images the yellow ones. The remaining vertices belong to the same strongly connected component (s.c.c.). They are the non-null columns of $(A_9)^2$.

Def: $H \subset G$ is strongly connected iff for all pair $v, w \in H$, there is a path $v \to w$ and a path $w \to v$, and H is maximal.

A_N as a directed graph

$\tilde{G}_9$



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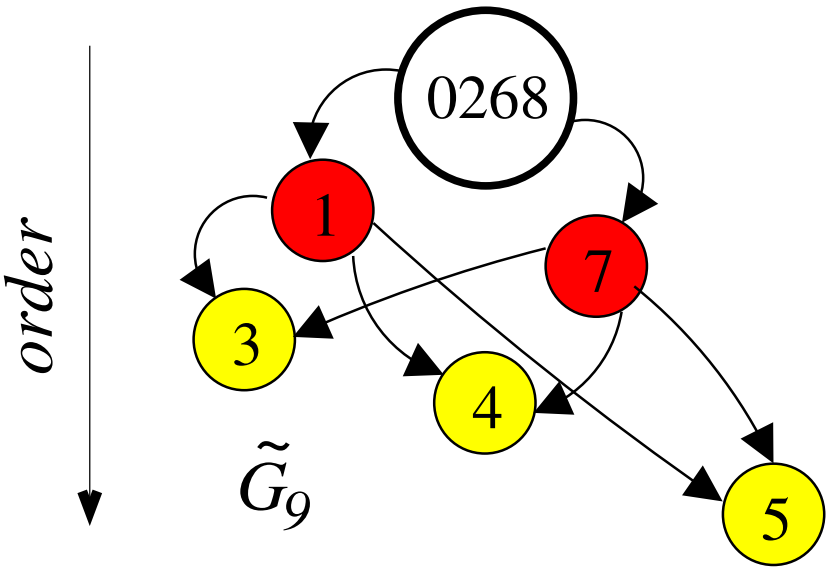
Fact: if we contract each s.c.c. to a point, the *reduced graph* $\tilde{G}$ we obtain is *acyclic*. One can *order* its vertices such that $\tilde{v} < \tilde{w}$ if $\tilde{v}\tilde{w} \in \tilde{E}$.

A_N as a directed graph



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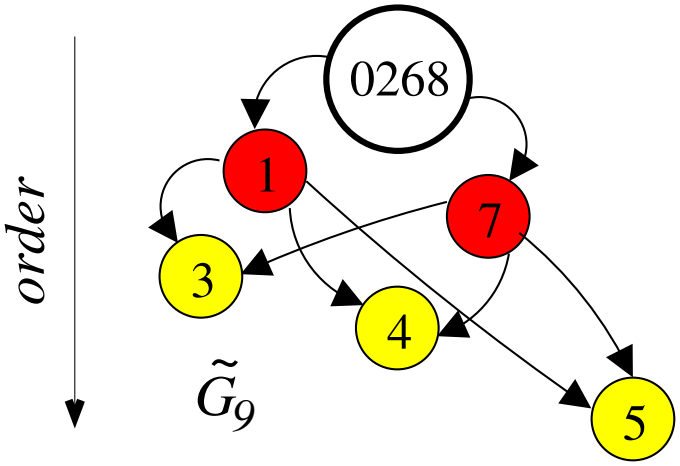
Ordered reduced graph $\equiv$ Jordan structure

If we *permute* the indices of the reduced graph $\tilde{G}_N$ according to this order, the (reduced) adjacency matrix becomes lower triangular, with the nonzero diagonal elements representing the s.c.c.



$$\tilde{P}\tilde{A}\tilde{P}^{-1} = \begin{array}{c} s.c.c. \\ 1 \\ 7 \\ 3 \\ 4 \\ 5 \end{array} \begin{pmatrix} 1 & & & & & \\ 1 & 0 & & & & \\ 1 & & 0 & & & \\ & 1 & 1 & 0 & & \\ & 1 & 1 & & 0 & \\ & 1 & 1 & & & 0 \end{pmatrix}$$

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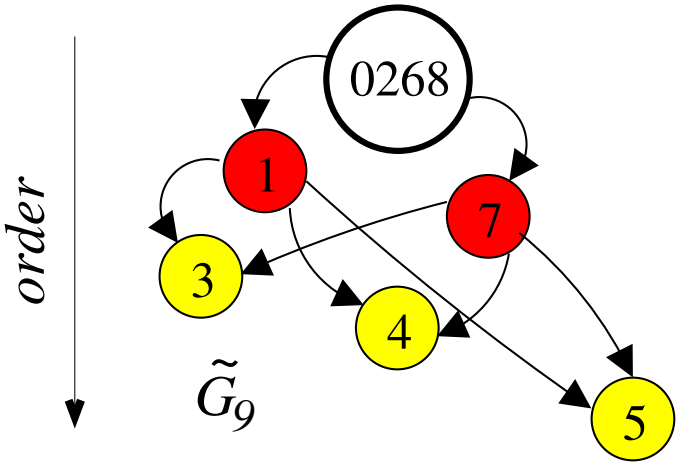


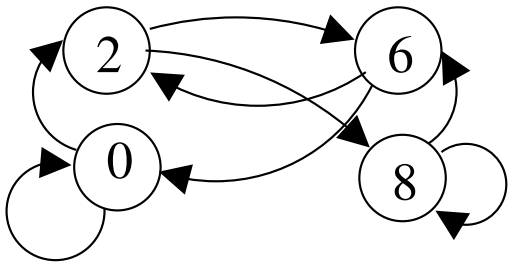
$$PAP^{-1} = \begin{pmatrix} \square & & & & & \\ * & 0 & & & & \\ * & & 0 & & & \\ & 1 & 1 & 0 & & \\ & 1 & 1 & & 0 & \\ & 1 & 1 & & & 0 \end{pmatrix}$$

Restoring the s.c.c., we obtain for $A(G_N)$ a **block-lower triangular matrix**, where each diagonal block represents a s.c.c.

The same permutation P applies to S_N.

$\Longrightarrow$ the nontrivial spectrum of S_N is given by the spectrum of the diagonal block (= spectrum of the s.c.c.).

The strongly connected component of S_{3^K}



$$A_9^{scc} = \begin{array}{c} 0 \\ 2 \\ 6 \\ 8 \end{array} \begin{pmatrix} 1 & & 1 & \\ 1 & & 1 & \\ & 1 & & 1 \\ & 1 & & 1 \end{pmatrix}, \quad S_9^{scc} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & & 1 & \\ 1 & & \omega & \\ & 1 & & 1 \\ & 1 & & \omega \end{pmatrix}$$

The only s.c.c. in G_9 is the *De Bruijn graph* $D_B(2,2)$ (alphabet of 2 symbols, words of length 2): we recover the rule $\epsilon_1\epsilon_2 \to \epsilon_2\epsilon_3$, with $\epsilon_i \in \{0,2\}$.

- Similar structure for $N = 3^K$: G_N has a unique s.c.c., $D_B(2,K)$.
- Nontrivial spectrum depends on $\{\lambda_-, \lambda_+\}$, eigenvalues of $\tilde{\Omega}_3 = \left(\begin{smallmatrix} 1 & 1 \\ 1 & \omega \end{smallmatrix}\right)$.

Questions:

1. what happens if we modify the nontrivial entries of S_{3^K}?
2. what happens if we take $N \neq 3^K$?
3. what happens for other types of directed graphs?

1. Inserting random phases: a new random matrix model

To make the spectrum more "generic", we randomize our skeleton S_N: replace its nonzero entries by *independent random numbers*.
Ex: z_{jk} uniformly distributed random phases.

$$R_9 = \begin{pmatrix} z_{0,0} & 0 & 0 & 0 & 0 & 0 & z_{0,6} & 0 & 0 \\ z_{1,0} & 0 & 0 & 0 & 0 & 0 & z_{1,6} & 0 & 0 \\ z_{2,0} & 0 & 0 & 0 & 0 & 0 & z_{2,6} & 0 & 0 \\ 0 & z_{3,1} & 0 & 0 & 0 & 0 & 0 & z_{3,7} & 0 \\ 0 & z_{4,1} & 0 & 0 & 0 & 0 & 0 & z_{4,7} & 0 \\ 0 & z_{5,1} & 0 & 0 & 0 & 0 & 0 & z_{5,7} & 0 \\ 0 & 0 & z_{6,2} & 0 & 0 & 0 & 0 & 0 & z_{6,8} \\ 0 & 0 & z_{7,2} & 0 & 0 & 0 & 0 & 0 & z_{7,8} \\ 0 & 0 & z_{8,2} & 0 & 0 & 0 & 0 & 0 & z_{8,8} \end{pmatrix}$$

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The nontrivial spectrum reduces to R_9^{scc}, equal to a random matrix $\widetilde{R}_4$.
(Generalizes to $N = 3^K \rightsquigarrow \widetilde{R}_{2K}$).

$\rightarrow$ (New) *random matrix problem*: how does the spectrum of $\widetilde{R}_M$ look like?

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To make the spectrum more "generic", we randomize our skeleton S_N: replace its nonzero entries by *independent random numbers*.
Ex: z_{jk} uniformly distributed random phases.

$$R_9 = \begin{pmatrix} z_{0,0} & 0 & 0 & 0 & 0 & 0 & z_{0,6} & 0 & 0 \\ z_{1,0} & 0 & 0 & 0 & 0 & 0 & z_{1,6} & 0 & 0 \\ z_{2,0} & 0 & 0 & 0 & 0 & 0 & z_{2,6} & 0 & 0 \\ 0 & z_{3,1} & 0 & 0 & 0 & 0 & 0 & z_{3,7} & 0 \\ 0 & z_{4,1} & 0 & 0 & 0 & 0 & 0 & z_{4,7} & 0 \\ 0 & z_{5,1} & 0 & 0 & 0 & 0 & 0 & z_{5,7} & 0 \\ 0 & 0 & z_{6,2} & 0 & 0 & 0 & 0 & 0 & z_{6,8} \\ 0 & 0 & z_{7,2} & 0 & 0 & 0 & 0 & 0 & z_{7,8} \\ 0 & 0 & z_{8,2} & 0 & 0 & 0 & 0 & 0 & z_{8,8} \end{pmatrix} \rightsquigarrow R_9^{scc} = \widetilde{R}_4 \stackrel{\text{def}}{=} \begin{pmatrix} z_{0,0} & & z_{0,6} & \\ z_{2,0} & & z_{2,6} & \\ & z_{6,2} & & z_{6,8} \\ & z_{8,2} & & z_{8,8} \end{pmatrix}$$

The nontrivial spectrum reduces to R_9^{scc}, equal to a random matrix $\widetilde{R}_4$.
(Generalizes to $N = 3^K \rightsquigarrow \widetilde{R}_{2K}$).

$\rightarrow$ (New) *random matrix problem*: how does the spectrum of $\widetilde{R}_M$ look like?

$\oplus$ for any even M, $\widetilde{R}_M \widetilde{R}_M^*$ is block-diagonal, with *independent* 2×2 blocks
$\rightarrow$ statistics of the singular values of $\widetilde{R}_M$ is easy.

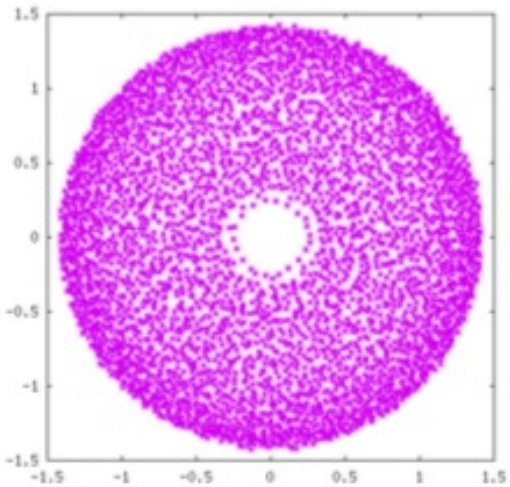
Theorem ([Gendron-N-Sabri])
$\exists C > 0, \forall r > 0$, *w.h.p. as* $M \to \infty$, $\#\{\mathrm{Spec}(\widetilde{R}_M) \cap \{|z| \leq r\}\} \leq C\, r\, M$.

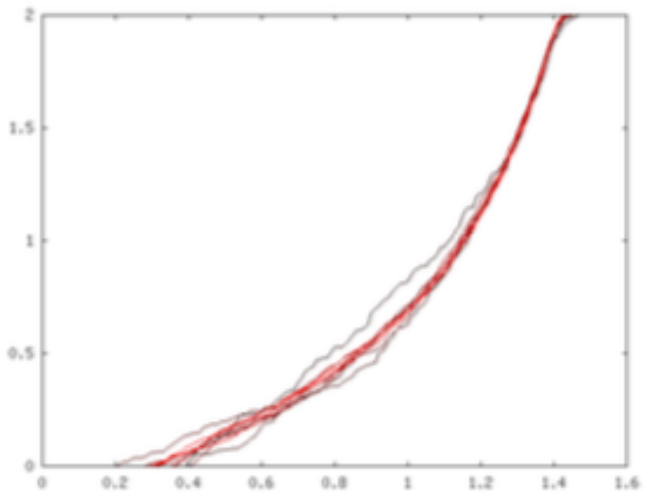
$\Longrightarrow$ Most of the 2^K eigenvalues of $R_{3^K}^{scc}$ are of order 1. We have obtained, w.h.p., a *lower bound* for the Fractal Weyl Law of R_{3^K}.

How does the spectrum of R_{3K} look like?

Does the empirical measure $\nu_M = \frac{1}{M}\sum_{z\in \mathrm{Spec}\,\widetilde{R}_M}\delta_z$ converge to a limit when $M \to \infty$?

Numerically, ν_M seems to be distributed according to a smooth density, vanishing near the origin.





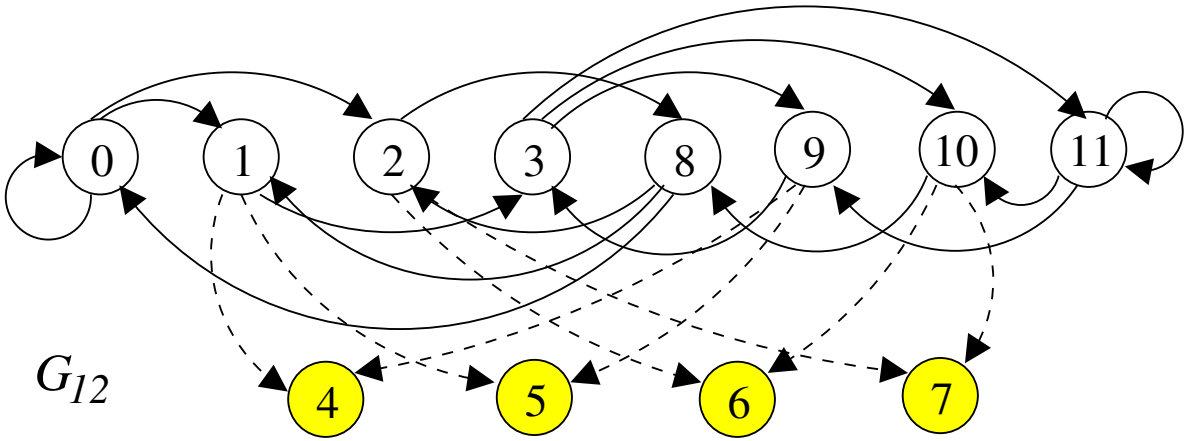
Left: the spectrum of a single realization of $\widetilde{R}_M$ for $M = 4096$.

Right: rescaled counting function for matrices $\widetilde{R}_M$ of sizes $M = 100 - 1000$. Observe the sharp spectral radius at $r \approx \sqrt{2}$.

(Plots by Q.Gendron).

Random model R_N for $N \neq 3^K$

If we now consider random matrices R_N for values $N \neq 3^K$, the structure of the graph G_N may be more complicated.



Ex: for $N = 12$, the complement of the "hole" forms a s.c.c..

$$R_{12}^{s.c.c.} = \begin{pmatrix} * & & & & & & & & & & & \\ * & & & & & & & & & & & \\ * & & & & & & & & & & & \\ & * & & & & & * & & & & & \\ & & * & & & & & * & & & & \\ & & & & & & & & & & & \\ & & & & * & & & & & & * & \\ & & & & * & & & & & & * & \\ & & & & * & & & & & & * & \end{pmatrix}$$

The reduced matrix $R_{12}^{s.c.c.}$ is less 'regular' than the De Bruijn type matrix, but it keeps a similar shape.
Notice that $R^{s.c.c.} R^{s.c.c.*}$ is still block diagonal, but now with blocks of different sizes.

Random model R_N for $N \neq 3^K$ (2)

For arbitrary N (divisible by 3), [TRIAS'12] showed that

1. the nontrivial spectrum of R_N has size $\leq 5\,N^{\nu} \Longrightarrow$ upper bound for the FWL.
2. there is always a "large" s.c.c. component of size $\asymp N^{\nu}$ (plus possibly some extra "small" s.c.c.).

Combining these facts with the $\widetilde{R}\widetilde{R}^*$ trick, one can hope to prove the following

Conjecture

For any N divisible by 3, the nontrivial spectrum of R_N has size $\asymp N^{\nu}$. Besides, w.h.p. this nontrivial spectrum does not have a macroscopic accumulation near the origin.

As a consequence, the matrices R_N would satisfy a FWL, in the following sense:

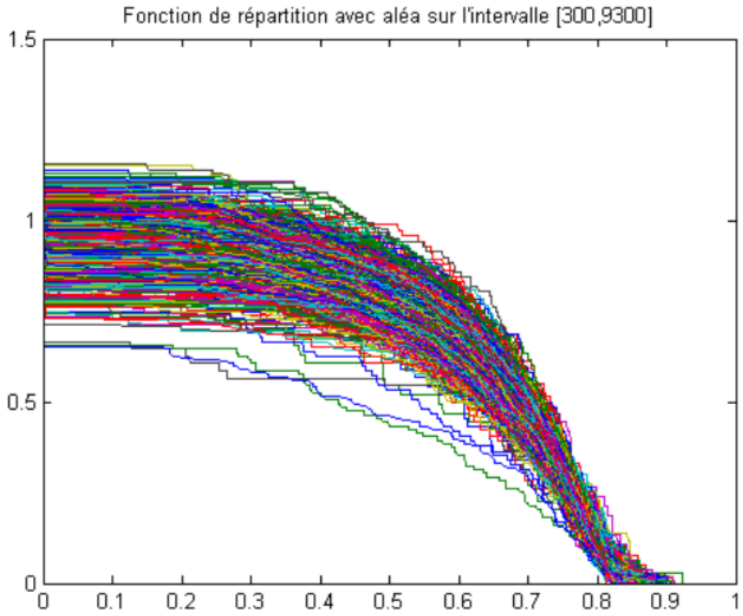
$$\forall r > 0, \quad \mathcal{N}(R_N, r) \stackrel{\text{def}}{=} \#\,\mathrm{Spec}(R_N) \cap \{|z| \geq r\} \asymp N^{\nu}, \qquad N \to \infty.$$

What Ansatz could we have?

A limiting spectral density?

To recover a more precise version of the FWL, $\mathcal{N}(R_N, r) \sim C(r)\, N^{\nu}$, we would need to better understand the empirical measure of the reduced matrices $\widetilde{R}_M$.

[TRIAS'12] computed the rescaled counting functions $\frac{\mathcal{N}(\widetilde{R}_M, r)}{M}$ for all values $300 \leq M \leq 9300$:



The curves have a similar shape, up to a global prefactor approximately varying in the interval $[0.7, 1.2]$.

Apart from this prefactor, is there an asymptotic curve $C(r)$?

More general directed graphs

- Above we have studied random matrices R_N constructed above a *deterministic* skeleton $G_N \rightsquigarrow$ deterministic s.c.c. $\widetilde{G}_M$ with $M \asymp N^{\nu} \rightsquigarrow$ upper bound for the FWL.
- To get a more precise FWL, we would need to better understand the spectral distribution of the random matrices $\widetilde{R}_M$.

How about considering the graph $\widetilde{G}_M$ to be random as well?

More general directed graphs

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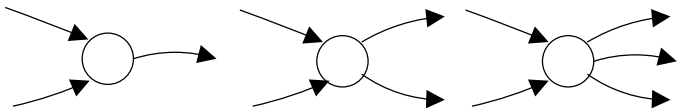
How about considering the graph $\widetilde{G}_M$ to be random as well?

The reduced graphs $\widetilde{G}_M$ we encoutered had the following property:



For any vertex j, $d_j^- = 2$ and $d_j^+ \in \{1, 2, 3\}$ (De Bruijn graph: $d_j^+ = d_j^- = 2$).

• For a given set of degrees $\mathbf{d} = (d_j^{\pm})_{j=1,\ldots,M}$ (satisfying $\sum_j d_j^+ = \sum_j d_j^-$), one can consider the family $\mathcal{G}_{\mathbf{d}}$ of all directed graphs on M vertices with these degrees: a class of random directed graphs $\widetilde{G} \in \mathcal{G}_{\mathbf{d}}$.

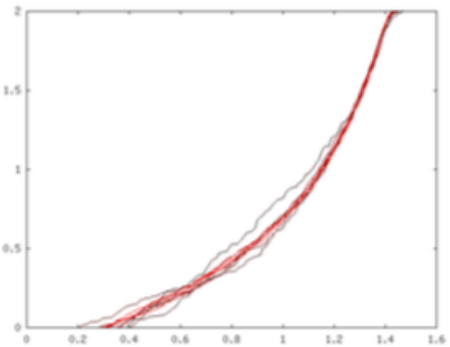
• equip each edge with a random phase $\rightsquigarrow$ obtain an ensemble of random matrices $\widetilde{R} \in \mathcal{R}_{\mathbf{d}}$.

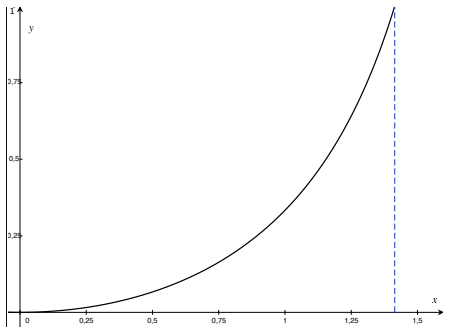
Spectrum of random sparse directed graphs

What do we know about spectral behaviour of the random graphs in $\mathcal{G}_{\mathbf{d}}$ and the matrices in $\mathcal{R}_{\mathbf{d}}$ when $M \to \infty$

• [COOPER-FRIEZE'04]: If all vertices have $d_j^+ \geq 2$, then w.h.p. as $M \to \infty$, $G \in \mathcal{G}_{\mathbf{d}}$ is strongly connected. If $d_j^{\pm} = 1$, then $\widetilde{G}$ contains a s.c.c. of size $M(1 - o(1))$.

• The spectrum of the adjacency matrices $\{A_{\widetilde{G}},\ \widetilde{G} \in \mathcal{G}_{\mathbf{d}}\}$ has been studied. [BORDENAVE-CHAFAÏ'12] conjecture that for the ensemble of random d-regular digraphs ($d_j^+ = d_j^- = d$), the empirical measure of $A_{\widetilde{G}}$ converges to $\frac{1}{\pi}\frac{d^2(d-1)}{(d^2-|z|^2)^2}\mathbb{1}_{|z|<\sqrt{d}}dx\,dy$ (complex Kesten-McKay measure).





$d = 2$: the induced counting function seems compatible with our numerics for $\widetilde{R}_M$ (De Bruijn graphs), including spectral edge at $\sqrt{2}$.

Would adding phases to the De Brujin graph make the spectrum of $\widetilde{R}_M$ more "typical"?

Conclusion & Perspectives

- The class of random matrices R_N has a large kernel because the corresponding directed graph G_N has small strongly connected components. $\Longrightarrow$ upper bound for the FWL obtained from the topology of G_N.
- The spectrum of the s.c.c. $\widetilde{R}_M$ did not have a macroscopic kernel, nor an accumulation of spectrum near the origin. Proved only in the case of the De Buijn graph. $\Longrightarrow$ lower bound for the FWL. Asymptotic spectral density?
- Larger scope: spectrum of random matrices $\widetilde{R}$ living on a random graph $\widetilde{G} \in \mathcal{G}_{\mathbf{d}}$ with specified (bounded) degrees. The random phases should make the analysis easier than for the adjacency matrices. [COSTE'17] studied the statistics of the second largest eigenvalue of $A_{\widetilde{G}}$ in ensembles $\widetilde{G} \in \mathcal{G}_{\mathbf{d}}$; this could give the spectral radius for $\widetilde{R}$.
- Our matrices A_N were discretizing $q \mapsto 3q$ on $[0, 1/3) \cup [2/3, 1)$. One could discretize any piecewise smooth *open expanding maps* on $[0, 1]$, and study the corresponding random matrices.