Axioms for centrality: rank monotonicity for PageRank

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Why centrality
Why centrality
  ✤ Orienteering in the centrality jungle
Why centrality
  - Orienteering in the centrality jungle
  - Some important centrality indices
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Focus on rank monotonicity for PageRank
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Focus on rank monotonicity for PageRank

Conclusions
IR System
IR System
IR System

Document Repertoire
- pages retrieved by Google
- items on sale on Amazon
- members of facebook
- tweets posted by your friends
- photographs on Instagram
IR System
IR System

D
IR System

Set of Queries (query language)
- SE query
- product recommendation
- new-friend suggestion
- tweets to be shown
IR System

Result
- a selected subset $S \subseteq D$
- with a score (typically: a non-negative real number) assigned to every element of $S$
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**IMPORTANT**
Often $D$ is endowed with a graph structure
IR System: 1st simplification

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No selection, only scores
IR System: 1st simplification

The system can be formally represented as a function:
\[ c: Q \times D \rightarrow \mathbb{R} \]

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Scores depend only on the linkage structure on $D$

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Centrality

- The system, given a graph $G$ assigns a score to every node of $G$:

$$c_G: V_G \rightarrow \mathbb{R}$$
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- The system, given a graph $G$ assigns a score to every node of $G$:

$$c_G: V_G \rightarrow \mathbb{R}$$

- The nodes of $G$ are precisely our documents ($V_G = D$)
- This is what people refers to as a centrality index (or measure, or score, or just “centrality”)
Centrality in social sciences
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- First works by Bavelas at MIT (1946)
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This sparked countless works (Bavelas 1951; Katz 1953; Shaw 1954; Beauchamp 1965; Mackenzie 1966; Burgess 1969; Anthonisse 1971; Czapiel 1974…)}
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- Brought to CS through IR
- Key role in modern IR (=search engines)
Orienteering in the jungle of centrality indices
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- Path-based indices, based on the number of paths or shortest paths (geodesics) passing through a vertex [betweenness, Katz, ...]
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- *Spectral indices*, based on some linear-algebra construction [PageRank, Seeley, …]
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  (Actually, the first two families are largely the same, even if that wasn’t fully understood for a long time)
Path-based centralities
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- Katz’s index is a paradigmatic example
- Among them: betweenness (Anthonisse 1971), über-popular among social scientists
The path tribe: betweenness and Katz
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- Betweenness centrality:

\[ c_{\text{betw}}(x) = \sum_{y,z \neq x} \frac{\sigma_{yz}(x)}{\sigma_{yz}} \]
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Fraction of shortest paths from y to z passing through x
Betweenness centrality:

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  c_{Katz}(x) = \sum_{t=0}^{\infty} \alpha^t \Pi_x(t) = 1 \sum_{t=0}^{\infty} \alpha^t G^t
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  - Berge (1958) extends to general social graphs and develops the theory
  - A similar idea was proposed by Seeley to evaluate children popularity
The spectral tribe: Seeley index
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\[
c_{\text{Seeley}}(x) = \sum_{y \rightarrow x} \frac{c_{\text{Seeley}}(y)}{d^+(y)}
\]
The spectral tribe: Seeley index  
(Seeley 1949)

- Basic idea: in a group of children, a child is as popular as the sum of the popularities of the children who like him, but popularities are divided evenly among friends:

\[ c_{Seeley}(x) = \sum_{y \rightarrow x} \frac{c_{Seeley}(y)}{d^+(y)} \]

- In general it is a left dominant eigenvector of \( G_r \)
Spectral centralities (2)
In 1998, Page, Brin, Motwani and Winograd propose a spectral ranking for the web: PageRank
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- $G_r$ is Seeley’s matrix, $\alpha$ is the damping factor and $\nu$ the preference vector
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$G_r$ is Seeley’s matrix, $\alpha$ is the\textit{damping factor} and $\nu$ the\textit{preference vector}

This is just Katz’s index with $\ell_1$-normalization, i.e.,

$$(1 - \alpha)\nu \sum_{t \geq 0} \alpha^t G_r^t = (1 - \alpha)\nu(1 - \alpha G_r)^{-1}$$
The spectral tribe: PageRank
(Brin, Page, Motwani, Winograd 1999)
The recursive version of the definition (for uniform preference) is

\[ c_{pr}(x) = \alpha \sum_{y \rightarrow x} \frac{c_{pr}(x)}{d^+(x)} + (1 - \alpha) v_x \]
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\[ c_{pr}(x) = \alpha \sum_{y \rightarrow x} \frac{c_{pr}(x)}{d^+(x)} + (1 - \alpha)v_x \]

❖ ... or the dominant eigenvector of the Google matrix

\[ \alpha G_r + (1 - \alpha)1^Tv \]
Geometric centralities and neighbourhood functions
Define the *distance-count function* 

\[ D_G(x, t) = \# \{ z \mid d_G(z, x) = t \} \]
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\[ D_G(x,-) \text{ is the distance-count vector of } x \]
Geometric centralities and neighbourhood functions

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- BTW: in 1-to-1 correspondence with the better known “neighbourhood function”
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BTW: in 1-to-1 correspondence with the better known “neighbourhood function”

A geometric centrality is a function of the distance-count vector (i.e., two nodes with the same distance-count vector have the same centrality)
The geometric tribe: closeness and harmonic
(Bavelas 1946; B., Vigna 2013)
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❖ Closeness centrality:

\[ c_{\text{clos}}(x) = \frac{1}{\sum_y d(y, x)} \]
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- **Closeness centrality:**
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  c_{clos}(x) = \frac{1}{\sum_y d(y, x)}
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  - The summation is over all \( y \) such that \( d(y, x) < \infty \)
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- **Harmonic centrality:**

  \[ c_{\text{harm}}(x) = \sum_{y \neq x} \frac{1}{d(y, x)} \]
The geometric tribe: closeness and harmonic
(Bavelas 1946; B., Vigna 2013)

- **Closeness centrality:**
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- **Harmonic centrality:**
  \[ c_{\text{harm}}(x) = \sum_{y \neq x} \frac{1}{d(y, x)} \]

- Inspired by (Marchiori, Latora 2000), but may be dated back to (Harris 1954)
Making sense of centrality
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- Centrality indices can be studied
Making sense of centrality

- Centrality indices can be studied
  - individually (each single centrality index is a world in its own right)
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comparatively
Making sense of centrality

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  - *individually* (each single centrality index is a world in its own right)
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- Both kinds of studies can be based on
Making sense of centrality

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- Both kinds of studies can be based on
  - external source of *ground truth*
Making sense of centrality

- Centrality indices can be studied
  - individually (each single centrality index is a world in its own right)
  - comparatively

- Both kinds of studies can be based on
  - external source of ground truth
  - axioms (abstract desirable/undesirable properties)
Axioms for Centrality
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- Various attempts, with different flavours: (Sabidussi 1966), (Nieminien 1973), (Kitti 2012), (Brandes et al. 2012), (B. & Vigna 2014)
Axioms for Centrality

- Various attempts, with different flavours: (Sabidussi 1966), (Nieminen 1973), (Kitti 2012), (Brandes et al. 2012), (B. & Vigna 2014)

- Sometimes aimed at specific indices (e.g. PageRank): (Chien et al. 2004), (Altman and Tennenholtz 2005)
Orienteering in the jungle of axioms
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- Invariance properties
Orienteering in the jungle of axioms

- Invariance properties
- Score-dominance properties
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- Invariance properties
- Score-dominance properties
- Rank-dominance properties
Orienteering in the jungle of axioms

- Invariance properties
- Score-dominance properties
- Rank-dominance properties
- Many other axioms that still need a classification
Invariance properties
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- Two graphs $G$ and $G'$...
Invariance properties

- Two graphs $G$ and $G'$…
- ...and two nodes $x \in G$ and $x' \in G'$…
Invariance properties

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$$c_G(x) = c_{G'}(x')$$
Example: Invariance by isomorphism
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- If $G$ and $G'$ are isomorphic (via isomorphism $f: G \rightarrow G'$)
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- If $G$ and $G'$ are isomorphic (via isomorphism $f: G \rightarrow G'$)
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$$c_G(x) = c_{G'}(x')$$

This is so fundamental that it is often given for granted as part of the notion of *centrality*!
Example: Invariance by neighbours
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- Let $G$ be a graph and $x, x'$ be two nodes such that
Example: Invariance by neighbours

- Let $G$ be a graph and $x, x'$ be two nodes such that

- $N_G^-(x) = N_G^-(x')$ and $N_G^+(x) = N_G^+(x')$
Example: Invariance by neighbours

Let $G$ be a graph and $x, x'$ be two nodes such that

$N_G^-(x) = N_G^-(x')$ and $N_G^+(x) = N_G^+(x')$

$c_G(x) = c_G(x')$
Example: Invariance by neighbours

- Let $G$ be a graph and $x, x'$ be two nodes such that
- $N_G^-(x)=N_G^-(x')$ and $N_G^+(x)=N_G^+(x')$

$$c_G(x) = c_G(x')$$

“Two nodes with the same (in- and out-)neighbours have the same centrality”
Invariance by neighbours...
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- It is easy to verify that all geometric centralities are invariant by neighbours.
Invariance by neighbours...

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- Same for spectral centralities.
Example: Invariance by in-neighbours
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Example: Invariance by in-neighbours

- Let $G$ be a graph and $x, x'$ be two nodes such that
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Example: Invariance by in-neighbours

- Let $G$ be a graph and $x, x'$ be two nodes such that

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\[ c_G(x) = c_G(x') \]
Example: Invariance by in-neighbours

- Let $G$ be a graph and $x, x'$ be two nodes such that
- $N_G^-(x) = N_G^-(x')$

$$c_G(x) = c_G(x')$$

“Two nodes with the same in-neighbours have the same centrality”
Invariance by in-neighbours...

- A superficial observer *may* believe that geometric centralities satisfy invariance by in-neighbours
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\[ N_G^-(x) = N_G^-(x') \]
Invariance by in-neighbours...

- A superficial observer *may* believe that geometric centralities satisfy invariance by in-neighbours.

A shortest path from $z$ to $x$
Invariance by in-neighbours...

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\[ \forall z \notin \{x, x', z\} \quad d_G(z, x) = d_G(z, x') \]
A superficial observer *may* believe that geometric centralities satisfy invariance by in-neighbours...
A superficial observer *may* believe that geometric centralities satisfy invariance by in-neighbours

The difference (+1 in one position, -1 in another position) depends on the values of $d(x,x')$ and $d(x',x)$

$D_G(x, -)$ and $D_G(x', -)$ are almost the same...
Invariance by in-neighbours...
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- So, in general, geometric centralities are not invariant by in-neighbours
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- They are on symmetric (i.e. undirected) graphs, though.
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They are on symmetric (i.e. undirected) graphs, though.

But spectral centralities (e.g. PageRank) are invariant by in-neighbours.
Score-dominance properties
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Score-dominance properties

- Two graphs $G$ and $G'$...
- ...and two nodes $x \in G$ and $x' \in G'$...
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$$c_G(x) \geq c_{G'}(x')$$
Score-dominance properties

- Two graphs $G$ and $G'$...
- ...and two nodes $x \in G$ and $x' \in G'$...
- ...satisfying some constraints

\[ c_G(x) \geq c_{G'}(x') \]

- Sometimes $>$ is required (strict dominance)
Example: Dominance by in-neighbours (Schoch and Brandes, 2016)
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Example: Dominance by in-neighbours (Schoch and Brandes, 2016)

- Let $G$ be a graph and $x, x'$ be two nodes such that
- $\mathcal{N}^-_G(x) \subseteq \mathcal{N}^-_G(x')$
Example: Dominance by in-neighbours (Schoch and Brandes, 2016)

- Let $G$ be a graph and $x, x'$ be two nodes such that
- $N_G^{-}(x) \subseteq N_G^{-}(x')$

$$c_G(x) \leq c_G(x')$$
Example: Dominance by in-neighbours (Schoch and Brandes, 2016)

- Let $G$ be a graph and $x, x'$ be two nodes such that
- $N^-_G(x) \subseteq N^-_G(x')$

\[ c_G(x) \leq c_G(x') \]

- **Observe:** if a measure satisfies this property, it is also invariant by in-neighbours
Example: Dominance by in-neighbours
(Schoch and Brandes, 2016)

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  $$N_G^-(x) \subseteq N_G^-(x')$$

  $$c_G(x) \leq c_G(x')$$

- **Observe:** if a measure satisfies this property, it is also invariant by in-neighbours

  $$\Rightarrow$$ geometric centralities do not satisfy “dominance by in-neighbours”
Example: Score-dominance by arc addition
(a.k.a. score monotonicity)
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- If $G$ is a graph not containing the arc $x \rightarrow y$
Example: Score-dominance by arc addition
(a.k.a. score monotonicity)

- If $G$ is a graph not containing the arc $x \rightarrow y$
- And $G' = G \cup \{x \rightarrow y\}$
Example: Score-dominance by arc addition
(a.k.a. *score monotonicity*)

- If $G$ is a graph not containing the arc $x \rightarrow y$
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\[
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\]
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“Adding one arc towards $y$ (strictly) increases its score”

❖ The weak version (with $\geq$) also makes sense
Score monotonicity
(“Axioms for Centrality”, B. & Vigna 2014)

<table>
<thead>
<tr>
<th>Centrality Metric</th>
<th>General</th>
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</tr>
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<tbody>
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</tr>
<tr>
<td>Katz</td>
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</tr>
<tr>
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Score monotonicity
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PageRank satisfies score monotonicity
PageRank satisfies score monotonicity

- Proved by (Chien, Dwork, Kumar, Simon and Sivakumar 2003) for the case when all nodes have nonzero PageRank
PageRank satisfies score monotonicity

- Proved by (Chien, Dwork, Kumar, Simon and Sivakumar 2003) for the case when all nodes have nonzero PageRank
- Generalized in (B. and Vigna, 2014) to the case $r_x > 0$
Closeness does not satisfy score monotonicity
Closeness does not satisfy score monotonicity

\[ c_G(y) = \frac{1}{\sum_i d_G(t, y)} = \frac{1}{1} = 1 \]
Closeness does not satisfy score monotonicity

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Closeness does not satisfy score monotonicity

\[ c_G(y) = \frac{1}{\sum_t d_G(t, y)} = \frac{1}{1} = 1 \]

\[ c_G'(y) = \frac{1}{\sum_t d_{G'}(t, y)} = \frac{1}{1 + 1} = \frac{1}{2} \]
Closeness satisfies score monotonicity in the strongly connected case
Closeness satisfies score monotonicity in the strongly connected case

\[ c_{\text{clos}}(x) = \frac{1}{\sum_y d(y, x)} \]
Closeness satisfies score monotonicity in the strongly connected case

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- On strongly connected graphs
Closeness satisfies score monotonicity in the strongly connected case

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  - the summation includes all nodes
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\[ c_{\text{clos}}(x) = \frac{1}{\sum_y d(y, x)} \]

- On strongly connected graphs
  - the summation includes all nodes
  - the distances do not increase after adding the new arc
  - at least one distance strictly decreases
- So closeness centrality is score monotone on strongly connected graphs!
Rank-dominance properties
Rank-dominance properties
Rank-dominance properties

- Two graphs \( G \) and \( G' \)…
Rank-dominance properties

- Two graphs $G$ and $G'$…
- …and two nodes $x \in G$ and $x' \in G'$…
Rank-dominance properties

- Two graphs $G$ and $G'$...
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- ...satisfying some constraints
Two graphs $G$ and $G'$

...and two nodes $x \in G$ and $x' \in G'$

...satisfying some constraints

The rank of $x'$ in $G'$ is "not less" than the rank of $x$ in $G$
Rank-dominance properties

- Two graphs $G$ and $G'$...
- ...and two nodes $x \in G$ and $x' \in G'$...
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- The rank of $x'$ in $G'$ is "not less" than the rank of $x$ in $G$

Typically stated on two graphs with the same set of nodes, and for a single node.
Rank-dominance properties revised
(weak version)
Rank-dominance properties revised
(weak version)

- Two graphs $G$ and $G'$ with the same node set $V$ and node $x \in V$
Two graphs $G$ and $G'$ with the same node set $V$ and node $x \in V$

1. $\forall z. c_G(x) > c_G(z) \implies c_{G'}(x) > c_{G'}(z)$
Rank-dominance properties revised
(weak version)

- Two graphs $G$ and $G'$ with the same node set $V$ and node $x \in V$

  1. $\forall z. c_G(x) > c_G(z) \implies c_{G'}(x) > c_{G'}(z)$

  2. $\forall y. c_G(x) = c_G(y) \implies c_{G'}(x) \geq c_{G'}(y)$
Rank-dominance properties revised
(strict version)
Rank-dominance properties revised
(strict version)

- Two graphs $G$ and $G'$ with the same node set $V$ and node $x \in V$

  $$\forall z \cdot c_G(x) \geq c_G(z) \implies c_{G'}(x) > c_{G'}(z)$$
Example: Rank-dominance by arc addition
(a.k.a. rank monotonicity)
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(a.k.a. rank monotonicity)

- If $G$ is a graph not containing the arc $x \to y$
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- Then, for all $z$
  
  $\forall z. c_G(y) > c_G(z) \implies c_{G'}(y) > c_{G'}(z)$
  
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Example: Rank-dominance by arc addition
(a.k.a. rank monotonicity)

❖ If \( G \) is a graph not containing the arc \( x \rightarrow y \)
❖ And \( G' = G \cup \{x \rightarrow y\} \)
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  \[ \forall z . c_G(y) > c_G(z) \implies c_{G'}(y) > c_{G'}(z) \]
  \[ \forall z . c_G(y) = c_G(z) \implies c_{G'}(y) \geq c_{G'}(z) \]
❖ For the strict version, the last \( \geq \) should become a \( > \)
Rank monotonicity
(“Rank monotonicity in centrality measures.”, B. & Luongo & Vigna 2017)

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† provided that no node has null preference
# Rank monotonicity

(“Rank monotonicity in centrality measures.”, B. & Luongo & Vigna 2017)

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Rank monotonicity
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**Rank monotonicity**

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PageRank and rank monotonicity
Loose (non-strict) rank monotonicity
Loose (non-strict) rank monotonicity

- For PageRank, with $G' = G \cup \{ x \rightarrow y \}$

\[
\forall y. \ c_G(y) > c_G(z) \quad \implies \quad c_{G'}(y) > c_{G'}(z)
\]

\[
\forall y. \ c_G(y) = c_G(z) \quad \implies \quad c_{G'}(y) \geq c_{G'}(z)
\]

holds (Chien, Dwork, Kumar, Simon & Sivakumar 2004) for everywhere nonzero score
Loose (non-strict) rank monotonicity

- For PageRank, with $G' = G \cup \{x \rightarrow y\}$

\[\forall y. \ c_G(y) > c_G(z) \implies c_{G'}(y) > c_{G'}(z)\]
\[\forall y. \ c_G(y) = c_G(z) \implies c_{G'}(y) \geq c_{G'}(z)\]

holds (Chien, Dwork, Kumar, Simon & Sivakumar 2004) for everywhere nonzero score

- The strict version was proved in (B., Luongo, Vigna 2017)

\[\forall y. \ c_G(y) \geq c_G(z) \implies c_{G'}(y) > c_{G'}(z)\]

for everywhere nonzero preference
Loose vs. strict
Loose vs. strict

- The proof in (Chien, Dwork, Kumar, Simon & Sivakumar 2004) exploits the fact that the Google matrix is a regular Markov chain
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(B., Luongo, Vigna 2017) is based on some properties of M-matrices...
Loose vs. strict

- The proof in (Chien, Dwork, Kumar, Simon & Sivakumar 2004) exploits the fact that the Google matrix is a regular Markov chain
- (B., Luongo, Vigna 2017) is based on some properties of M-matrices...
- ...the results have wider applicability (e.g., Katz)
Damped spectral ranking
Damped spectral ranking

- Let $M$ be a nonnegative matrix, $0 < \alpha < 1/\rho(M)$, $\mathbf{v}$ a strictly positive vector
Damped spectral ranking

Let $M$ be a nonnegative matrix, $0 < \alpha < 1/\rho(M)$, $v$ a strictly positive vector.

Then, the centrality vector $r$ defined by

$$r = v(I - \alpha M)^{-1}$$

satisfies strict rank monotonicity, suitably generalised to matrices (see below).
Let $M$ be a nonnegative matrix, $0 < \alpha < 1/\rho(M)$, $v$ a strictly positive vector.

Then, the centrality vector $r$ defined by

$$r = v(I - \alpha M)^{-1}$$

defines strict rank monotonicity, suitably generalised to matrices (see below).

Applies to PageRank, Katz, …
Lemma (ext. Willoughby, 1977)

\[ C = (I - \alpha M)^{-1} \quad \mathbf{r} = \mathbf{v} \mathbf{C} \]
Lemma (ext. Willoughby, 1977)

\[ C = (I - \alpha M)^{-1} \quad r = vC \]
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Assume \( c_{yz} > 0 \) and let \( q = c_{yy}/c_{yz} \).
Lemma (ext. Willoughby, 1977)

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Assume \( c_{yz} > 0 \) and let \( q = c_{yy}/c_{yz} \)

- Then \( c_{wy}/c_{wz} \leq q \)
Lemma (ext. Willoughby, 1977)

\[ C = (I - \alpha M)^{-1} \quad r = vC \]

Assume \( c_{yz} > 0 \) and let \( q = c_{yy}/c_{yz} \)

- Then \( c_{wy} \leq q \cdot c_{wz} \)
Lemma (ext. Willoughby, 1977)

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Assume \( c_{yz} > 0 \) and let \( q = c_{yy}/c_{yz} \)

- Then \( c_{wy} \leq q c_{wz} \)

As a consequence if \( q < 1 \)...

\[ r_y = \sum_w v_w c_{wy} \leq \]
Lemma (ext. Willoughby, 1977)

\[ C = (I - \alpha M)^{-1} \quad r = vC \]

Assume \( c_{yz} > 0 \) and let \( q = \frac{c_{yy}}{c_{yz}} \)

- Then \( c_{wy} \leq q c_{wz} \)

As a consequence if \( q < 1 \)

\[
r_y = \sum_w v_w c_{wy} \leq \sum_w v_w \frac{c_{wz} c_{yy}}{c_{yz}}
\]
Lemma (ext. Willoughby, 1977)

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\[ r_y = \sum_w v_w c_{wy} \leq \sum_w v_w \frac{c_{wz} c_{yy}}{c_{yz}} \leq \sum_w v_w c_{wz} = r_z \]
Lemma (ext. Willoughby, 1977)

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Assume \( c_{yz} > 0 \) and let \( q = \frac{c_{yy}}{c_{yz}} \)

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As a consequence if \( q < 1 \)...

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Hence:
Lemma (ext. Willoughby, 1977)

\[ C = (I - \alpha M)^{-1} \quad \mathbf{r} = \mathbf{vC} \]

Assume \( c_{yz} > 0 \) and let \( q = \frac{c_{yy}}{c_{yz}} \)

- Then \( c_{wy} \leq q c_{wz} \)

As a consequence if \( q < 1 \)... \[
\begin{align*}
r_y &= \sum_w v_w c_{wy} \\
&\leq \sum_w v_w \frac{c_{wz} c_{yy}}{c_{yz}} \\
&\leq \sum_w v_w c_{wz} = r_z
\end{align*}
\]

Hence:

- If \( r_z \leq r_y \) then \( q \geq 1 \)
- If \( r_z < r_y \) then \( q > 1 \)
Lemma (ext. Willoughby, 1977)

\[ C = (I - \alpha M)^{-1} \quad r = vC \]

Assume \( c_{yz} > 0 \) and let \( q = c_{yy}/c_{yz} \)

\[ \text{As a consequence if } q < 1 \ldots \]

\[ r_y = \sum_w v_w c_{wy} \leq \sum_w v_w \frac{c_{wz}c_{yy}}{c_{yz}} \leq \sum_w v_w c_{wz} = r_z \]

Hence:

- If \( r_z \leq r_y \) then \( q \geq 1 \)
- If \( r_z < r_y \) then \( q > 1 \)
PageRank as a special case of damped spectral ranking
PageRank as a special case of damped spectral ranking

\[ \mathbf{r} = \mathbf{v}(I - \alpha \mathbf{M})^{-1} \]
PageRank as a special case of damped spectral ranking

\[ r = v(I - \alpha M)^{-1} \]

- In the case of PageRank, \( M = G_r \)
PageRank as a special case of
damped spectral ranking

\[ r = v(I - \alpha M)^{-1} \]

- In the case of PageRank, \( M = G_r \)
- When adding the arc \( x \rightarrow y \) we obtain a new matrix \( M' \)
  and
PageRank as a special case of damped spectral ranking

\[ \mathbf{r} = \mathbf{v}(I - \alpha \mathbf{M})^{-1} \]

- In the case of PageRank, \( \mathbf{M} = \mathbf{G}_r \)
- When adding the arc \( x \rightarrow y \) we obtain a new matrix \( \mathbf{M}' \) and

\[
\mathbf{M}' - \mathbf{M} = \begin{pmatrix}
0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0
\end{pmatrix} - \begin{pmatrix}
0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0
\end{pmatrix} \]
PageRank as a special case of
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- In the case of PageRank, \( \mathbf{M} = \mathbf{G}_r \)
- When adding the arc \( x \rightarrow y \) we obtain a new matrix \( \mathbf{M}' \) and

\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
\vdots & & & & & & \\
0 & -\frac{1}{d(d+1)} & \cdots & \frac{1}{d} & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0
\end{pmatrix}
\]

only row \( x \) is nonzero
PageRank as a special case of damped spectral ranking

\[ \mathbf{r} = \mathbf{v}(I - \alpha M)^{-1} \]

- In the case of PageRank, \( M = \mathcal{G}_r \)
- When adding the arc \( x \to y \) we obtain a new matrix \( M' \) and

\[
M' - M = \begin{pmatrix}
0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \frac{1}{d(d+1)} & \cdots & \frac{1}{d} & \cdots & 0 & 0 \\
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\end{pmatrix}
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PageRank as a special case of damped spectral ranking

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- In the case of PageRank, \( \mathbf{M} = \mathbf{G}_r \)
- When adding the arc \( x \rightarrow y \) we obtain a new matrix \( \mathbf{M}' \) and

```
M' - M =
```

“old” outneighbours of x

\[
\begin{pmatrix}
0 & 0 & \ldots & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & \frac{1}{d(d+1)} & \ldots & \frac{1}{d} & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & \ldots & 0 & 0 \\
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\end{pmatrix}
\]
PageRank as a special case of damped spectral ranking

\[ r = v(I - \alpha M)^{-1} \]

- In the case of PageRank, \( M = G_r \)
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\[
M' - M = \begin{pmatrix}
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\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
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- When adding the arc \( x \rightarrow y \) we obtain a new matrix \( \mathbf{M}' \) and

\[
\mathbf{M}' - \mathbf{M} = \begin{pmatrix}
0 & 0 & \ldots & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & \ldots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & -\frac{1}{d(d+1)} & \ldots & \frac{1}{d} & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & \ldots & 0 & 0
\end{pmatrix}
\]

only the \( y \)-th column is positive
PageRank as a special case of damped spectral ranking

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Rank monotonicity of PageRank (1)
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Rank monotonicity of PageRank (2)
We aim at proving that

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Let \( c_{yz} > 0 \) (the other case is easy), and \( q = c_{yy}/c_{yz} \)
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Let \( c_{yz} > 0 \) (the other case is easy), and \( q = c_{yy}/c_{yz} \)

\[ [\delta(1 - \alpha M)^{-1}]_y = \delta_y c_{yy} - \sum_{w \neq y} |\delta_w| c_{wy} \geq \delta_y q c_{yz} - \sum_{w \neq y} q |\delta_w| c_{wz} \]
Rank monotonicity of PageRank (2)

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- By the Lemma, \( r_z \leq r_y \implies q \geq 1 \)
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\[ \geq \delta_y c_{yz} - \sum_{w \neq y} |\delta_w| c_{wz} = [\delta(1 - \alpha M)^{-1}]_z \]
Rank monotonicity of PageRank (3)
We in fact proved only

\[ 0 < r_z \leq r_y \text{ implies } [\delta(I - \alpha M)^{-1}]_z \leq [\delta(I - \alpha M)^{-1}]_y \]
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The strict inequality requires more work…
Take-home messages
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❖ Centrality is **important** and ubiquitous
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❖ **A jungle of indices**: taxonomies (and generalizations) are of help
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❖ Apparently trivial properties fail to hold, or require *a lot of work* to be proved

❖ Beware, *it’s a wild world out there*
Thanks for your attention!