*(joint work with Sebastiano Vigna et al.)*

# Axioms for centrality: rank monotonicity for PageRank

**Paolo Boldi**
Università degli Studi di Milano

# TOC

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  - Orienteering in the centrality jungle

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- Focus on rank monotonicity for PageRank
- Conclusions

# IR System

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*Document Repertoire*

- pages retrieved by Google
- items on sale on Amazon
- members of facebook
- tweets posted by your friends
- photographs on Instagram

*D*

# IR System

$D$

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$Q$

*Set of Queries* (query language)

- SE query
- product recommendation
- new-friend suggestion
- tweets to be shown

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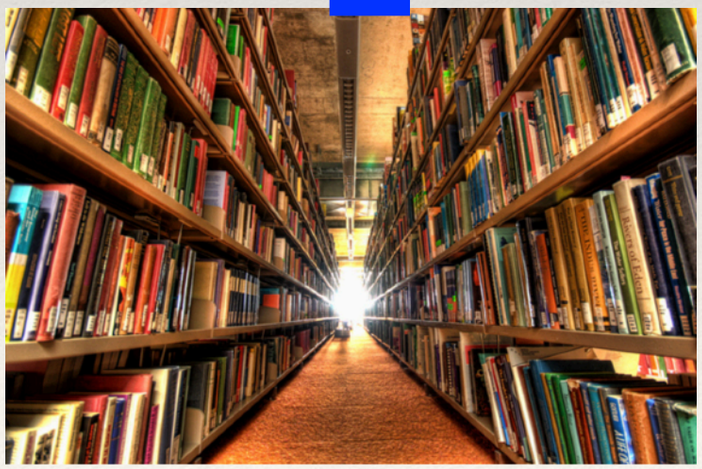

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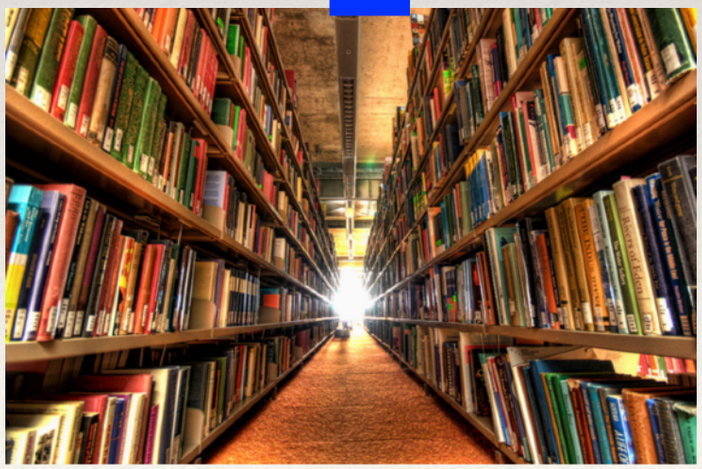
- a selected subset $S \subseteq D$
- with a score (typically: a non-negative real number) assigned to every element of $S$



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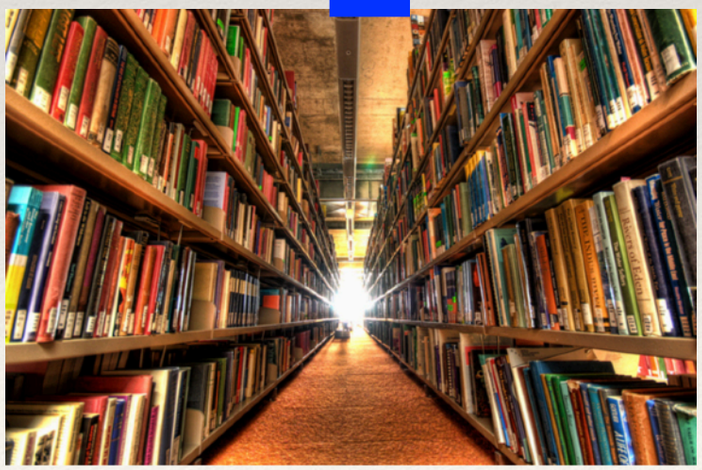

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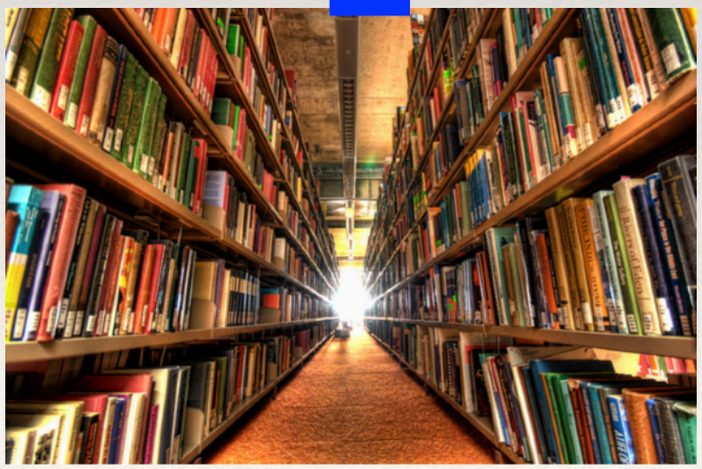

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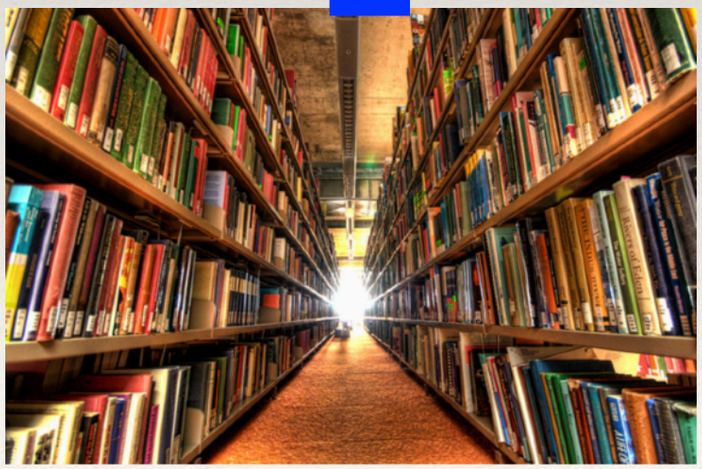

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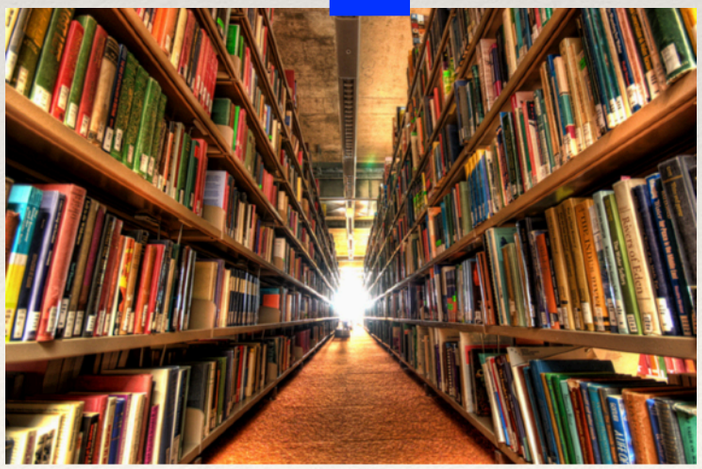

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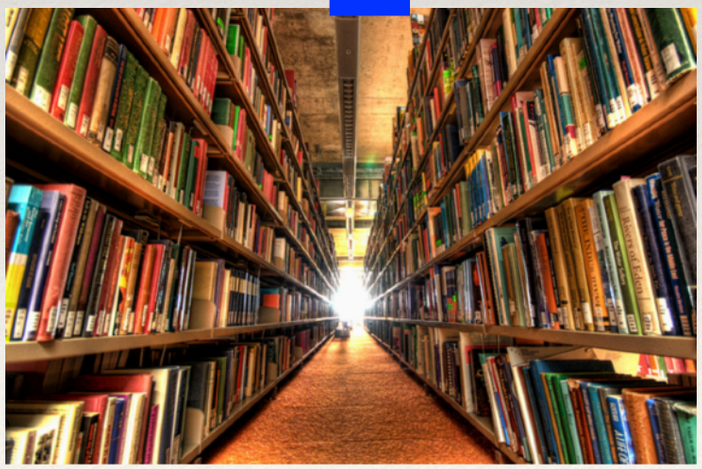

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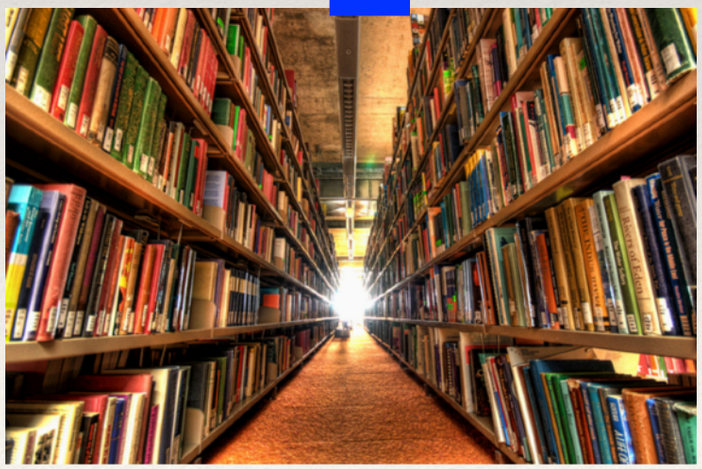

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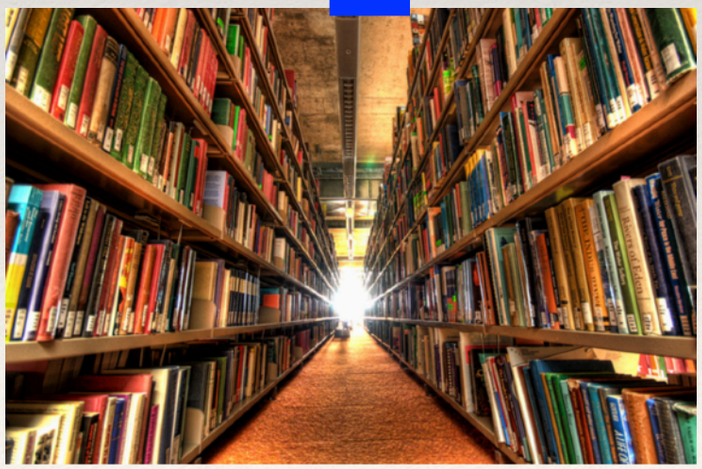

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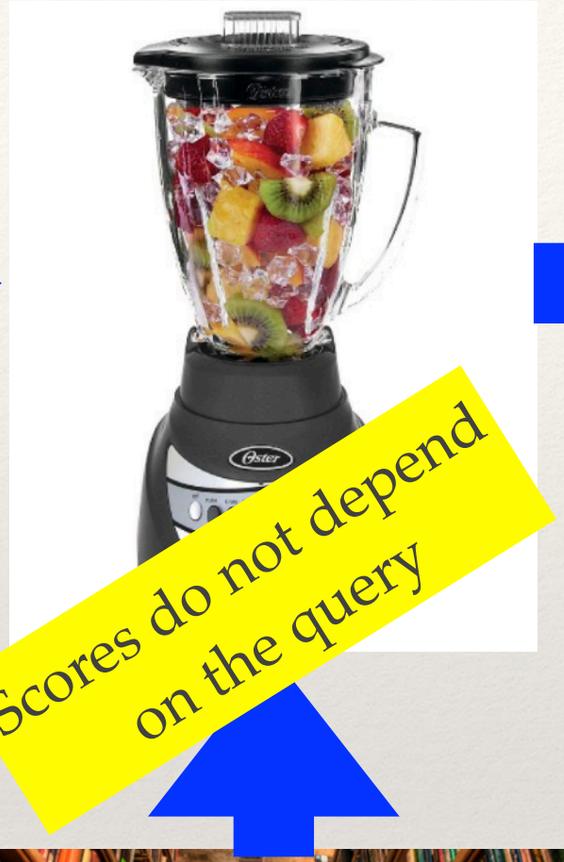


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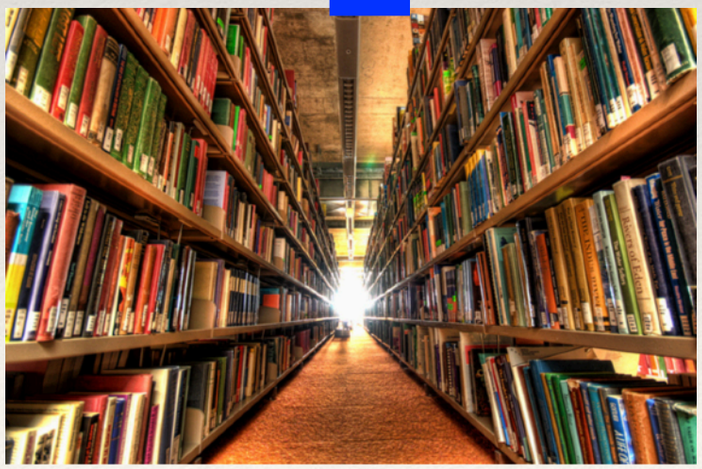

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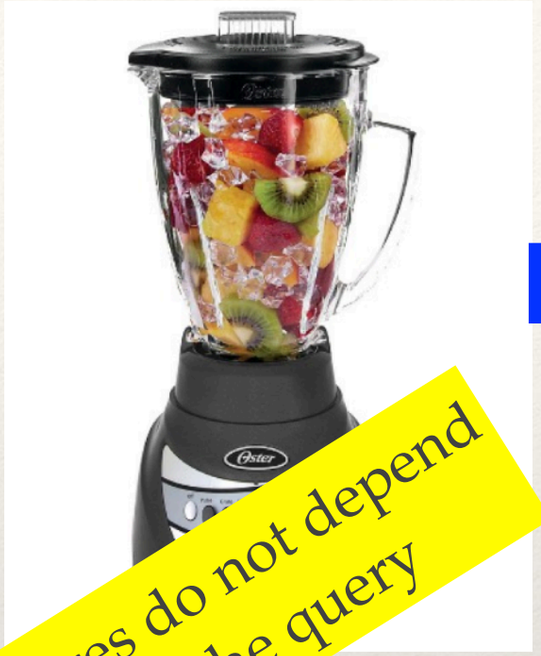

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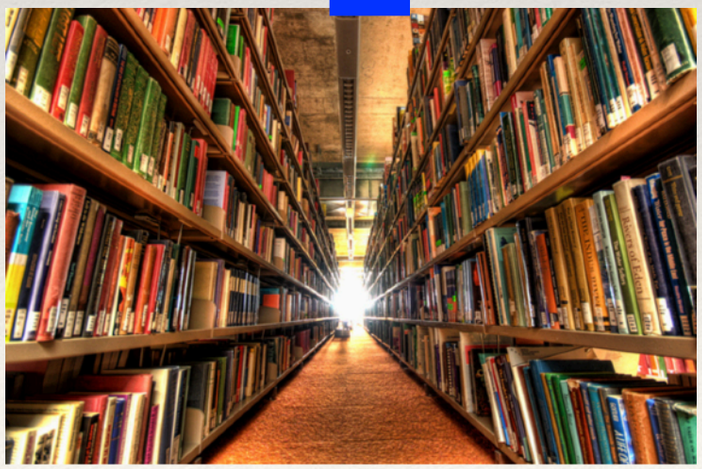

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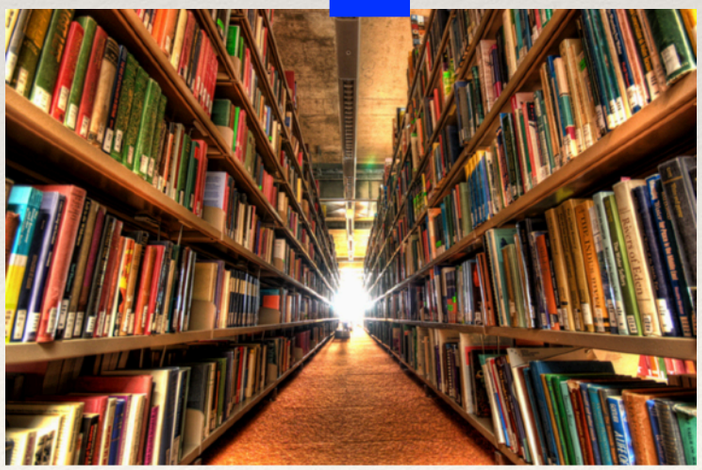

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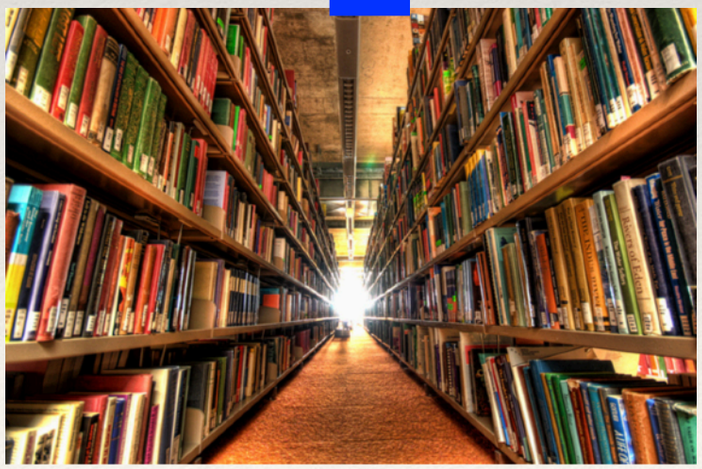

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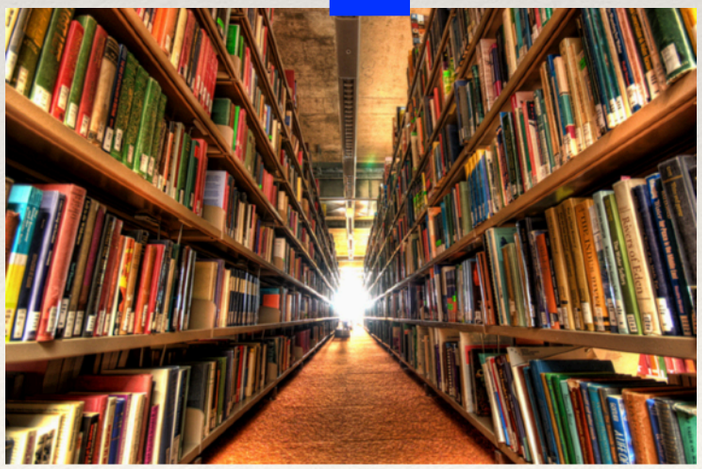

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- This is what people refers to as a *centrality index* (or measure, or score, or just "centrality")

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- Brought to CS through IR
- Key role in modern IR (=search engines)

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- (Actually, the first two families are largely the same, even if that wasn't fully understood for a long time)

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Fraction of shortest paths from $y$ to $z$ passing through $x$

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- A similar idea was proposed by Seeley to evaluate children popularity

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- In general it is a left dominant eigenvector of $G_r$

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- This is just Katz's index with $\ell_1$-normalization, i.e.,

$$(1 - \alpha)\boldsymbol{v}\sum_{t\geq 0} \alpha^t G_r^t = (1 - \alpha)\boldsymbol{v}(1 - \alpha G_r)^{-1}$$

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- … or the dominant eigenvector of the Google matrix

$$\alpha G_r + (1-\alpha)\mathbf{1}^T \mathbf{v}$$

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- BTW: in 1-to-1 correspondence with the better known "neighbourhood function"
- A geometric centrality is a function of the distance-count vector (i.e., two nodes with the same distance-count vector have the same centrality)

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- **Harmonic centrality**:

$$c_{\text{harm}}(x) = \sum_{y \neq x} \frac{1}{d(y,x)}$$

- Inspired by (Marchiori, Latora 2000), but may be dated back to (Harris 1954)

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  - external source of *ground truth*

# Making sense of centrality

- Centrality indices can be studied
  - *individually* (each single centrality index is a world in its own right)
  - *comparatively*
- Both kinds of studies can be based on
  - external source of *ground truth*
  - *axioms* (abstract desirable/undesirable properties)

# Axioms for Centrality

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- Various attempts, with different flavours: (Sabidussi 1966), (Nieminen 1973), (Kitti 2012), (Brandes *et al*. 2012), (B. & Vigna 2014)
- Sometimes aimed at specific indices (e.g. PageRank): (Chien *et al*. 2004), (Altman and Tennenholtz 2005)

# Orienteering in the jungle of axioms

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- Invariance properties

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# Orienteering in the jungle of axioms

- Invariance properties
- Score-dominance properties
- Rank-dominance properties
- Many other axioms that still need a classification

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This is **so fundamental** that
it is often given for granted as
part of the notion of *centrality*!

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- Same for **spectral centralities**

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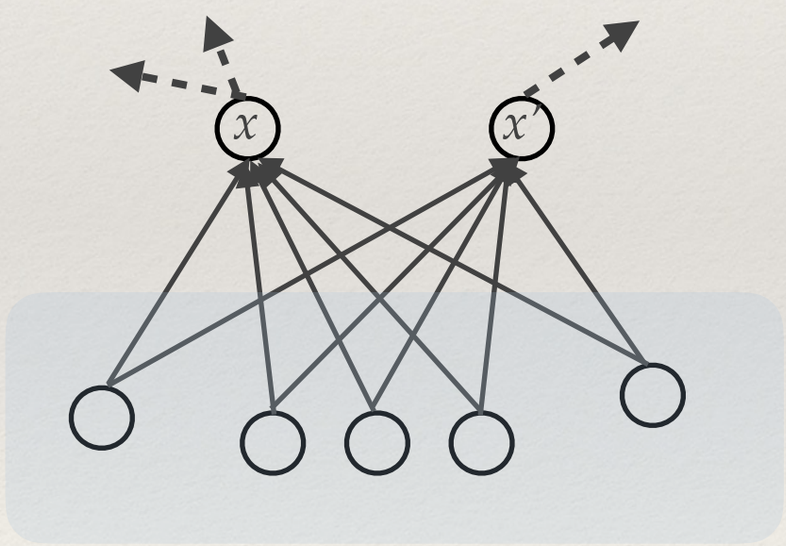


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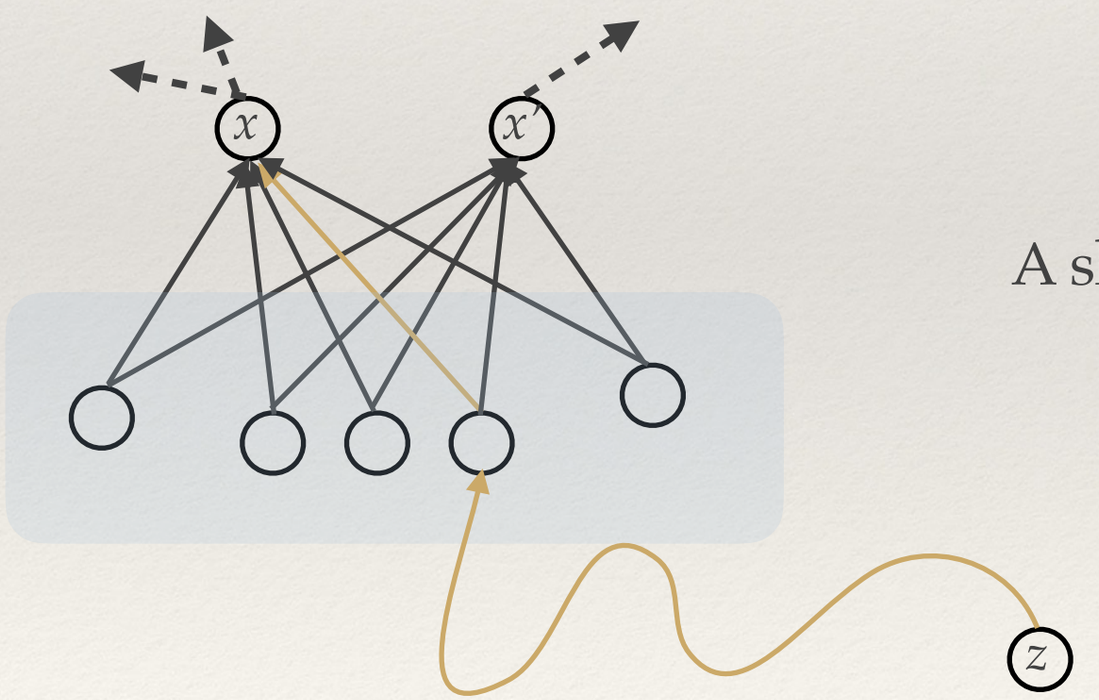


A shortest path from $z$ to $x$

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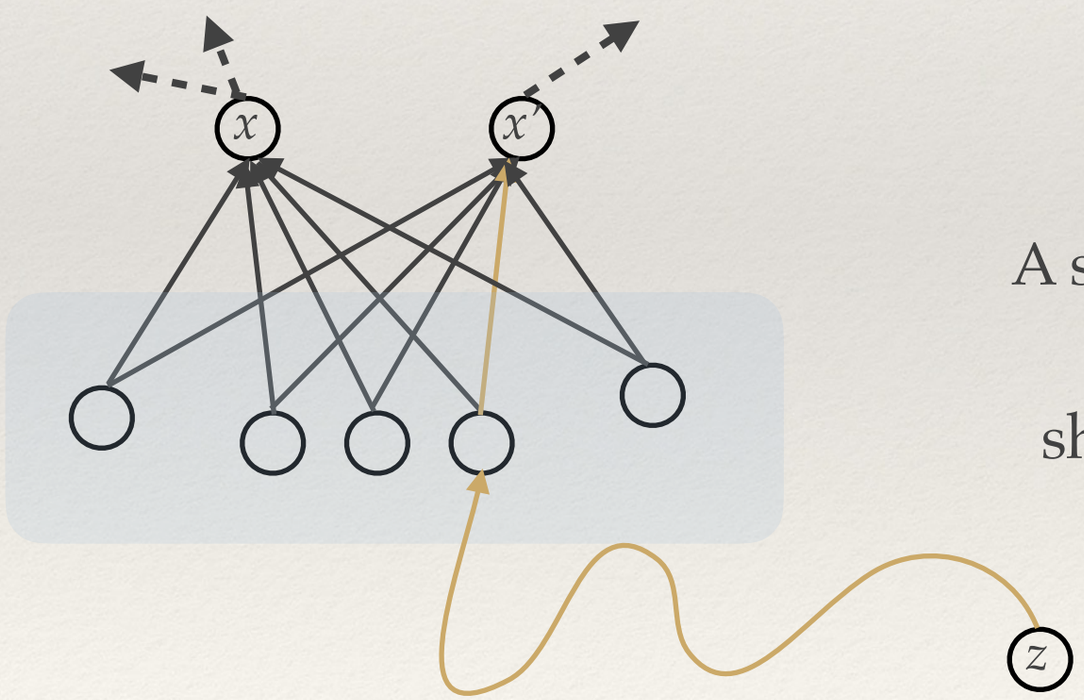


A shortest path from $z$ to $x$ can be turned into a shortest path from $z$ to $x'$

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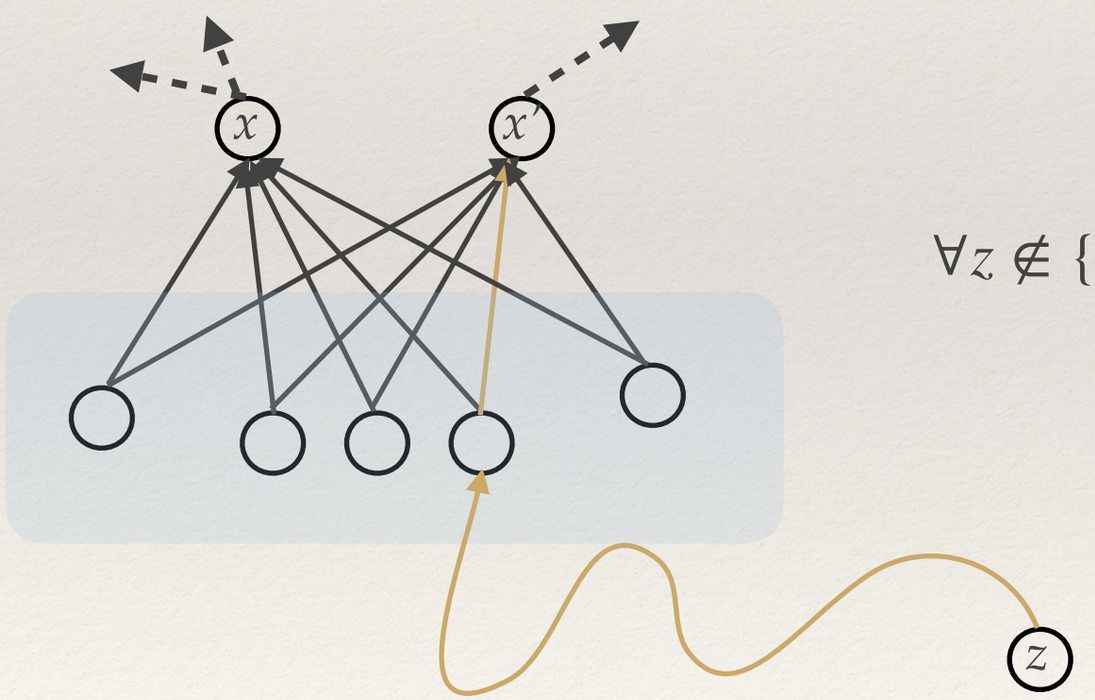


$$\forall z \notin \{x, x'\} \qquad d_G(z, x) = d_G(z, x')$$

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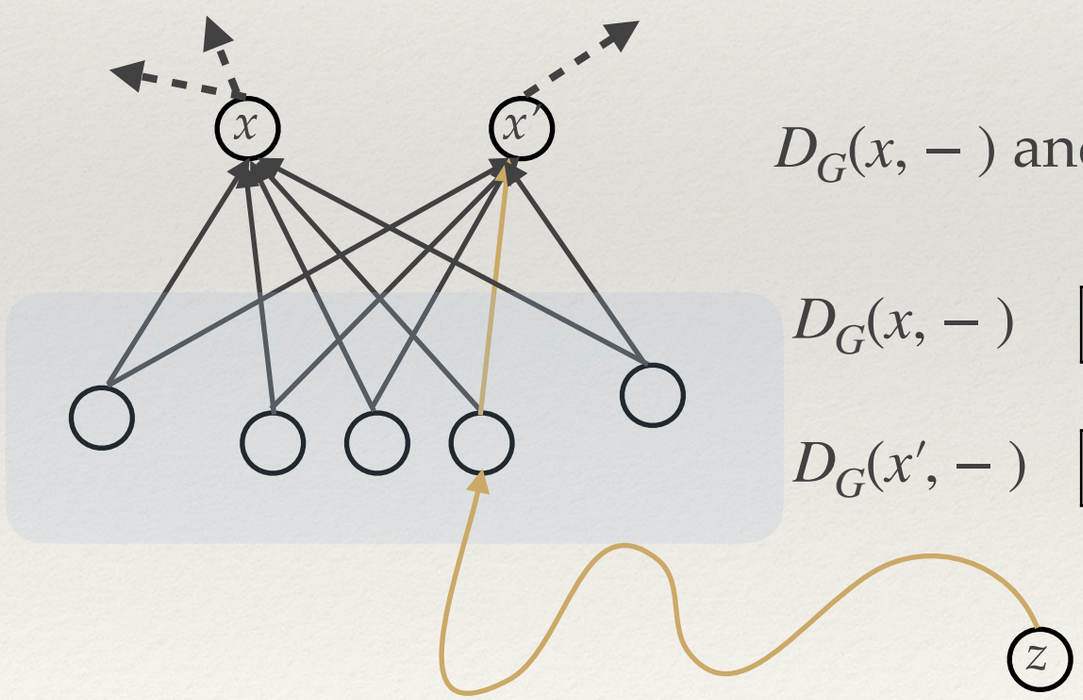


$D_G(x, -)$ and $D_G(x', -)$ are almost the same…

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | … |
|---|---|---|---|---|---|---|---|---|---|---|
| $D_G(x, -)$ | 5 | 0 | 3 | 7 | 2 | 0 | 0 | 4 | 1 | … |
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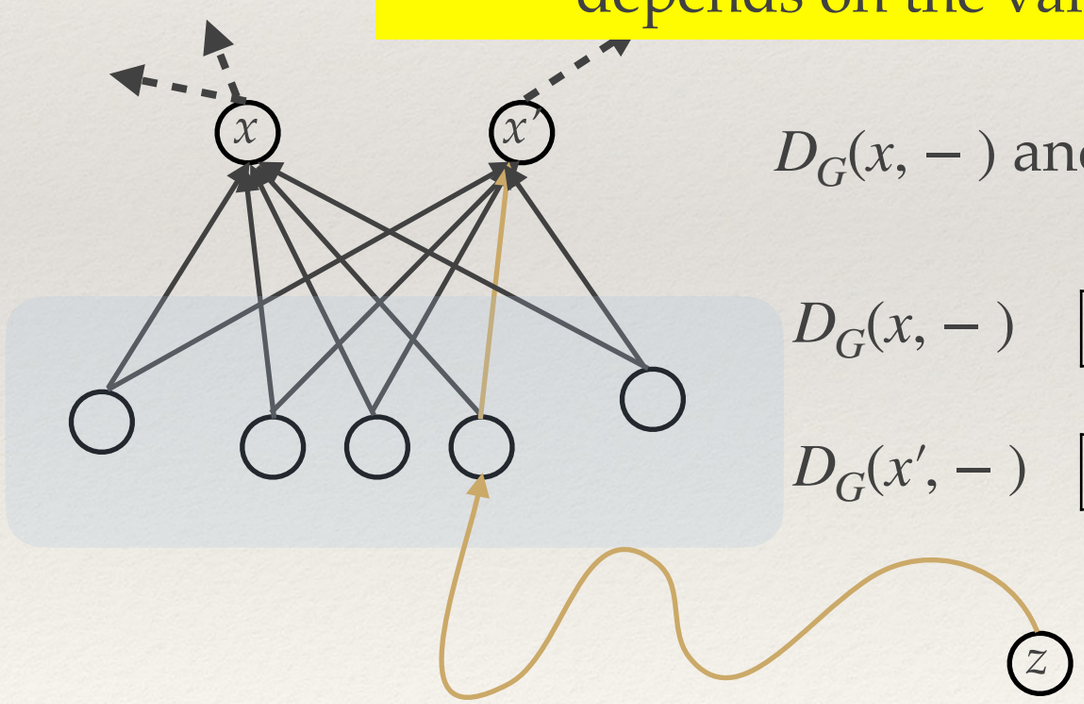
The difference (+1 in one position, -1 in another position) depends on the values of $d(x,x')$ and $d(x',x)$

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- So, in general, **geometric centralities** *are not invariant by in-neighbours*
  - They are on symmetric (i.e. undirected) graphs, though
- But **spectral centralities** (e.g. PageRank) *are invariant by in-neighbours*

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- Sometimes $>$ is required (*strict dominance*)

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- **Observe:** if a measure satisfies this property, it is also invariant by in-neighbours
- $\Rightarrow$ geometric centralities do not satisfy "dominance by in-neighbours"

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- The weak version (with $\geq$) also makes sense

# Score monotonicity
# ("Axioms for Centrality", B. & Vigna 2014)

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|---|---|---|
| Seeley | no | yes |
| PageRank | yes | yes |
| betweenness | no | no |
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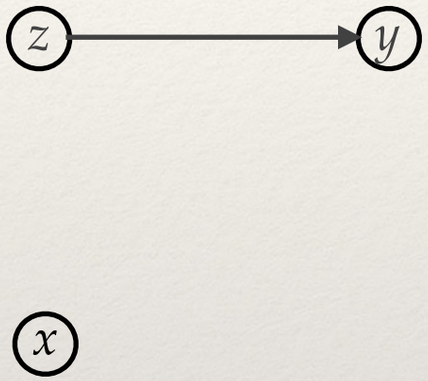
# Closeness does not satisfy score monotonicity



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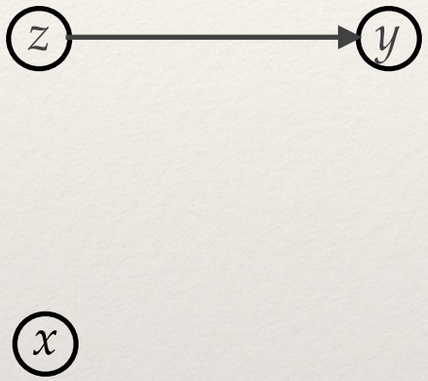


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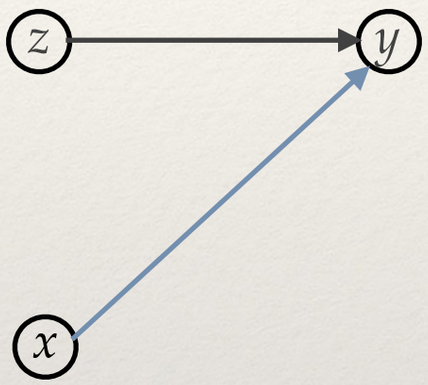


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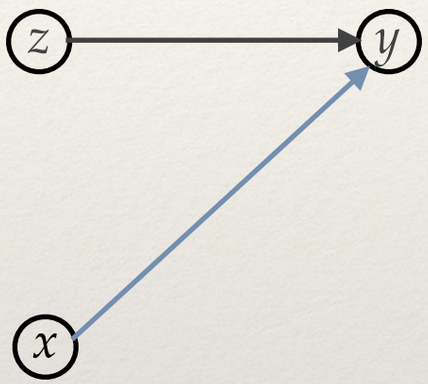


$$c_G(y) = \frac{1}{\sum_t d_G(t, y)} = \frac{1}{1} = 1$$

$$c_{G'}(y) = \frac{1}{\sum_t d_{G'}(t, y)} = \frac{1}{1+1} = \frac{1}{2}$$

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- So closeness centrality *is* score monotone on strongly connected graphs!

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Typically stated on two graphs with the same set of nodes, and for a single node

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# PageRank and rank monotonicity

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- The strict version was proved in (B., Luongo, Vigna 2017)

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- …the results have wider applicability (e.g., Katz)

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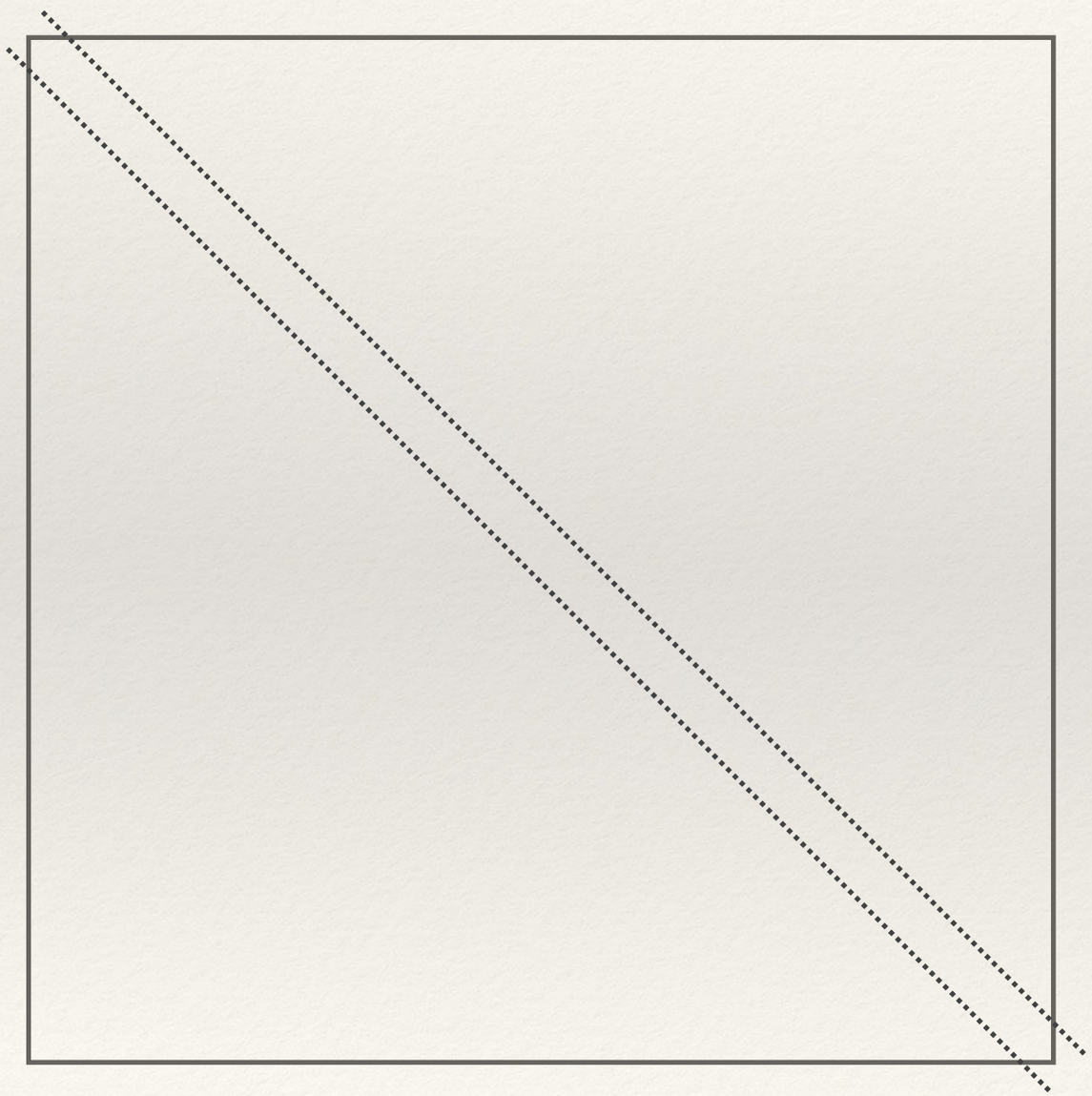
  satisfies strict rank monotonicity, suitably generalised to matrices (see below)
- Applies to PageRank, Katz, …

# Lemma (ext. Willoughby, 1977)

$$C = (I - \alpha M)^{-1} \quad \mathbf{r} = \mathbf{v}C$$



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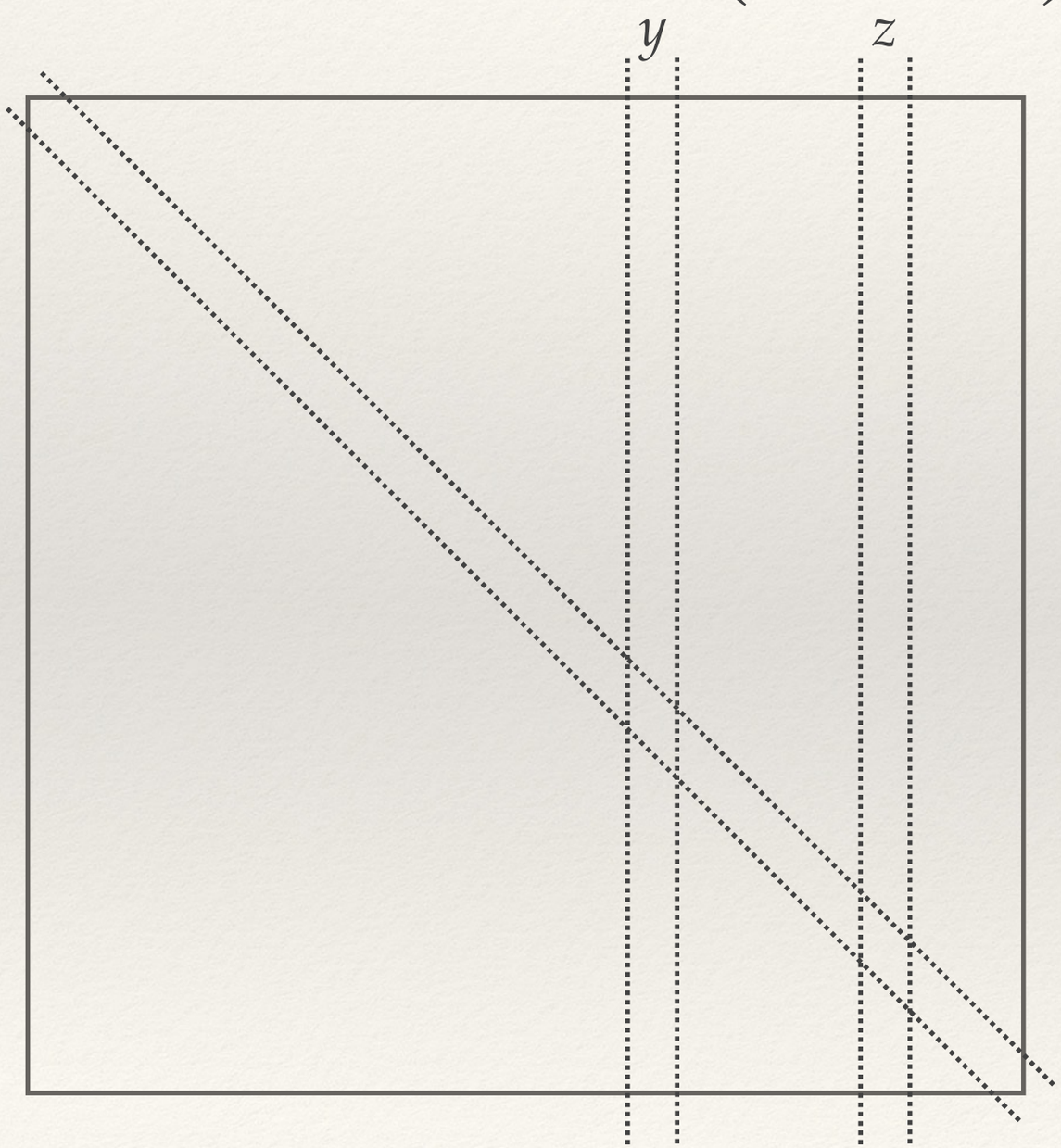

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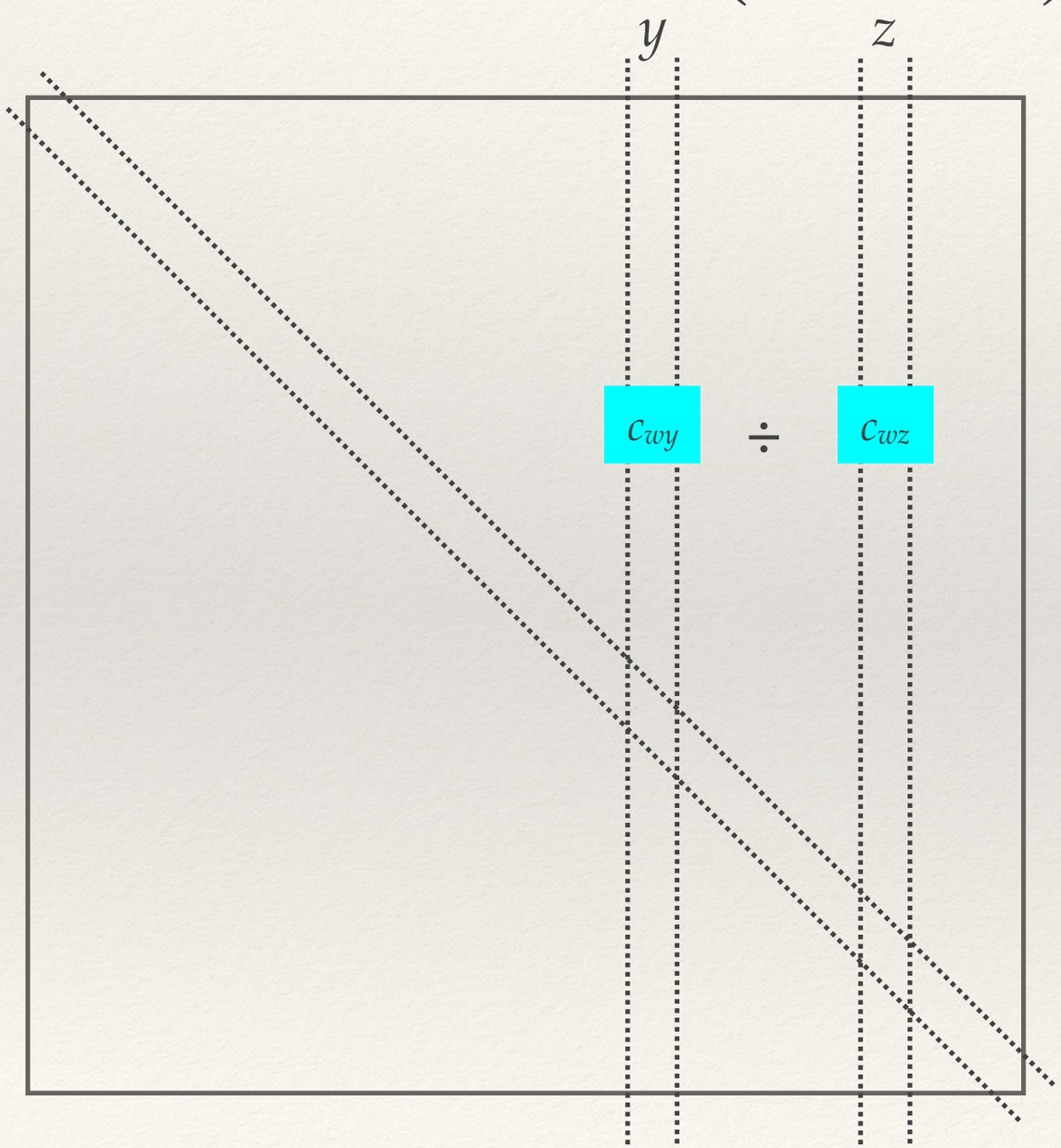

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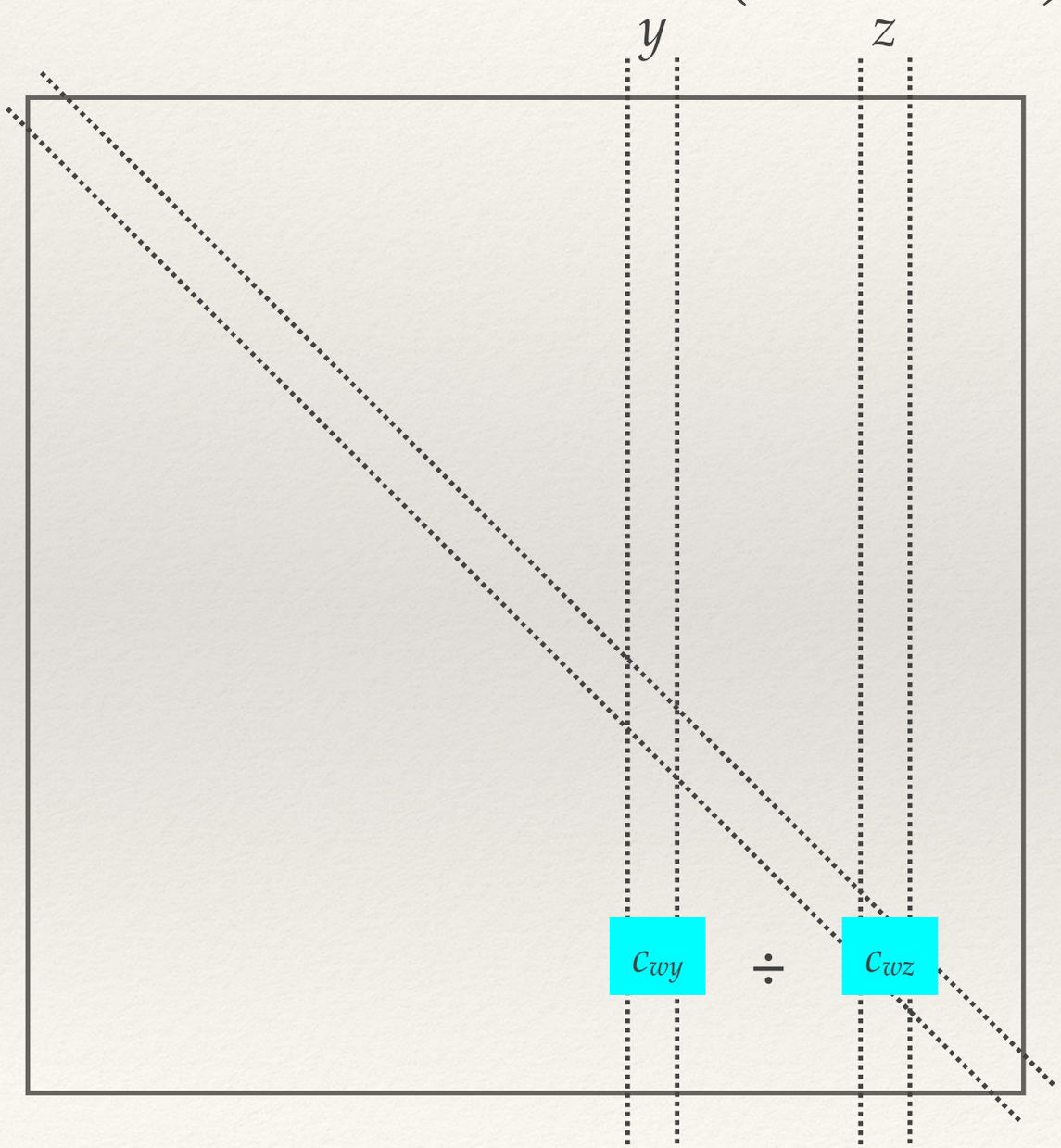

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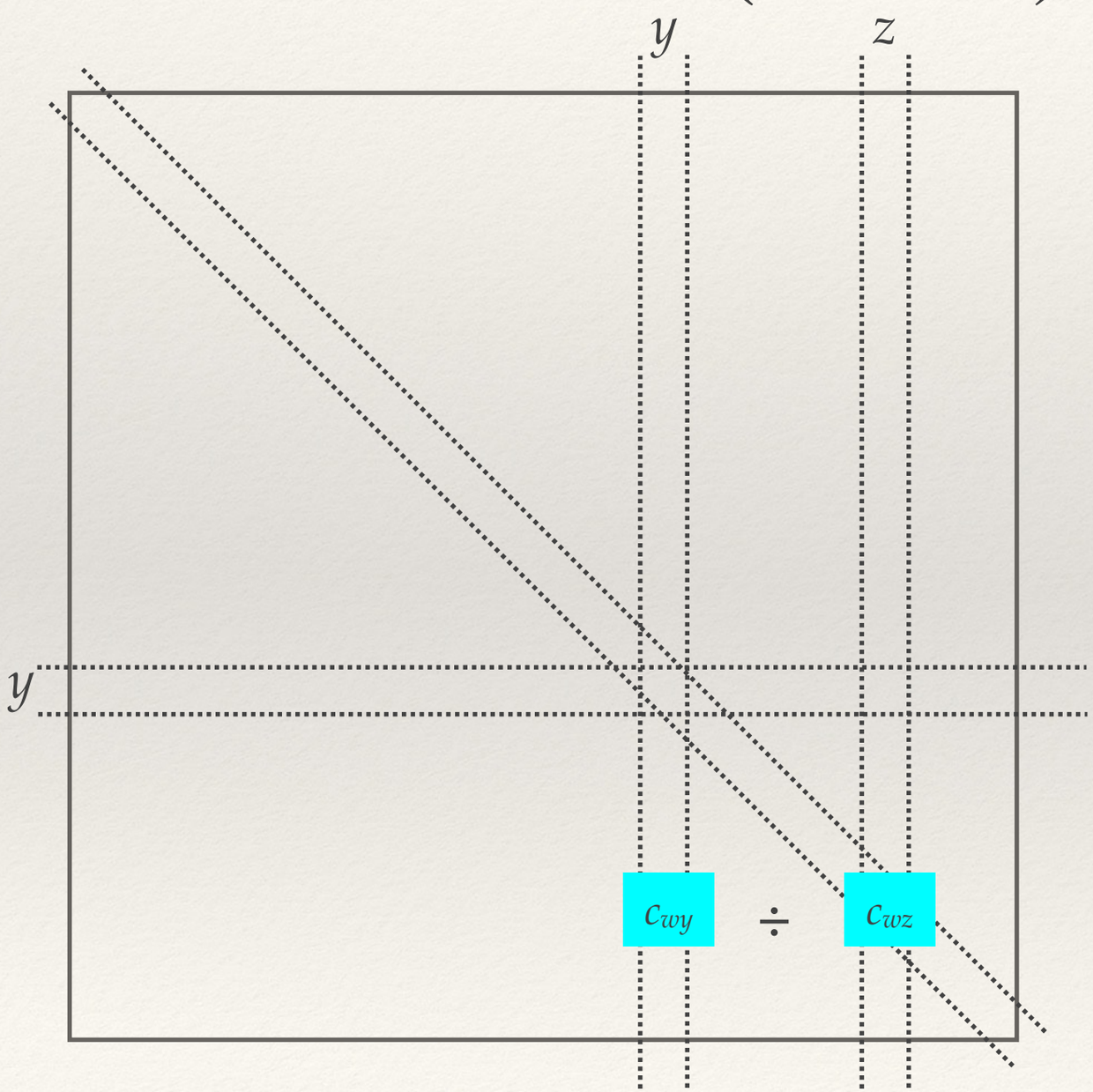

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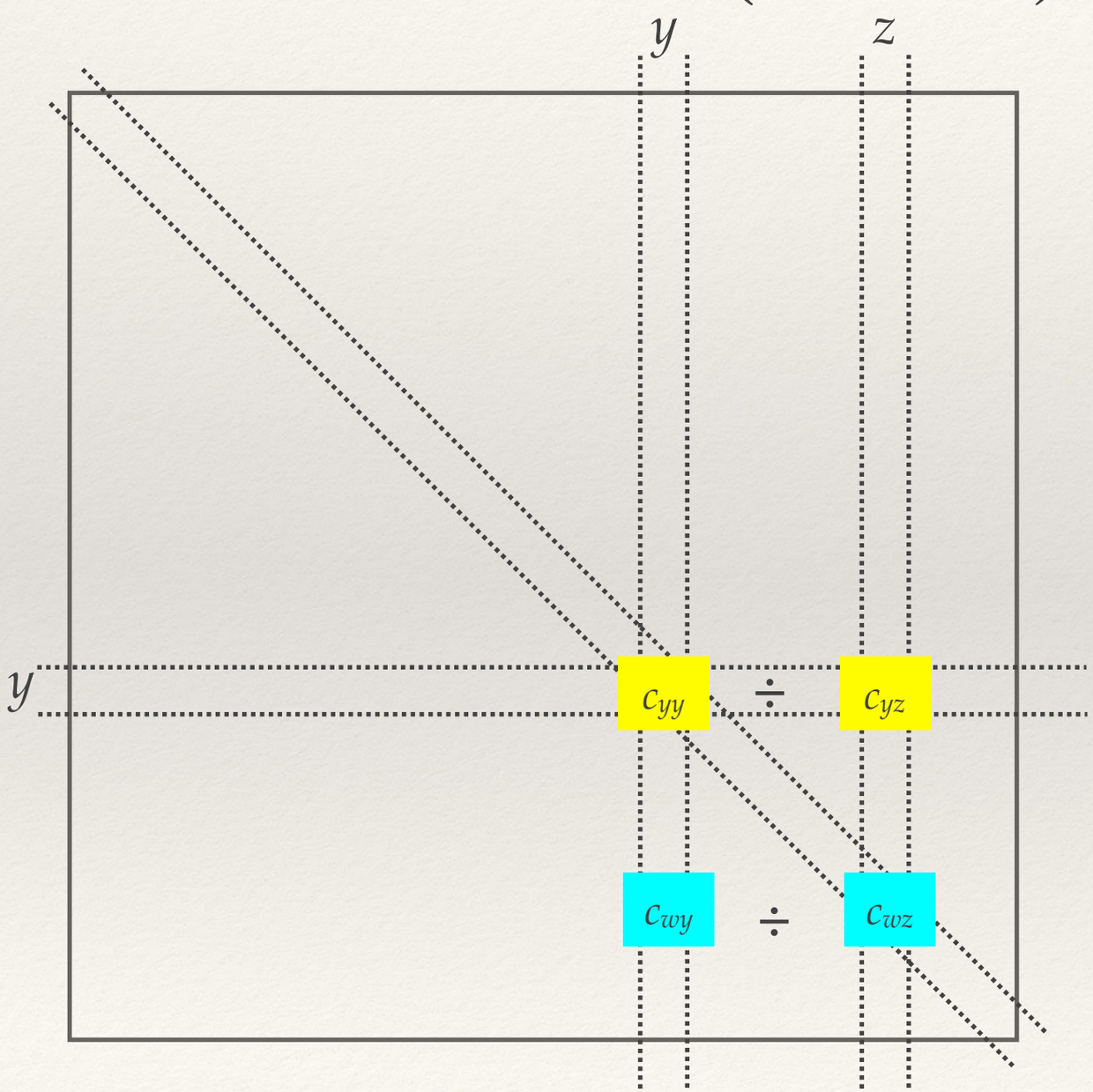

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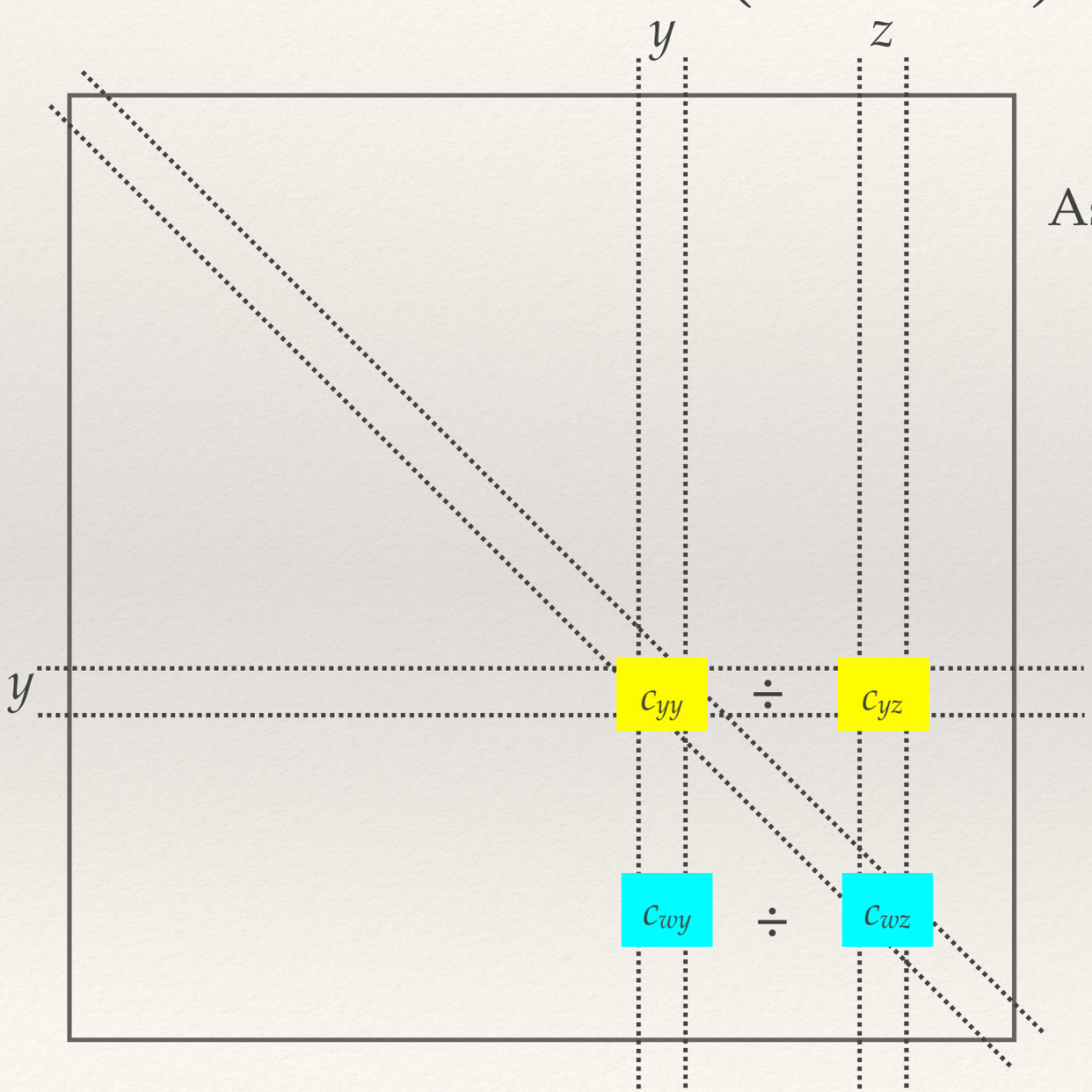


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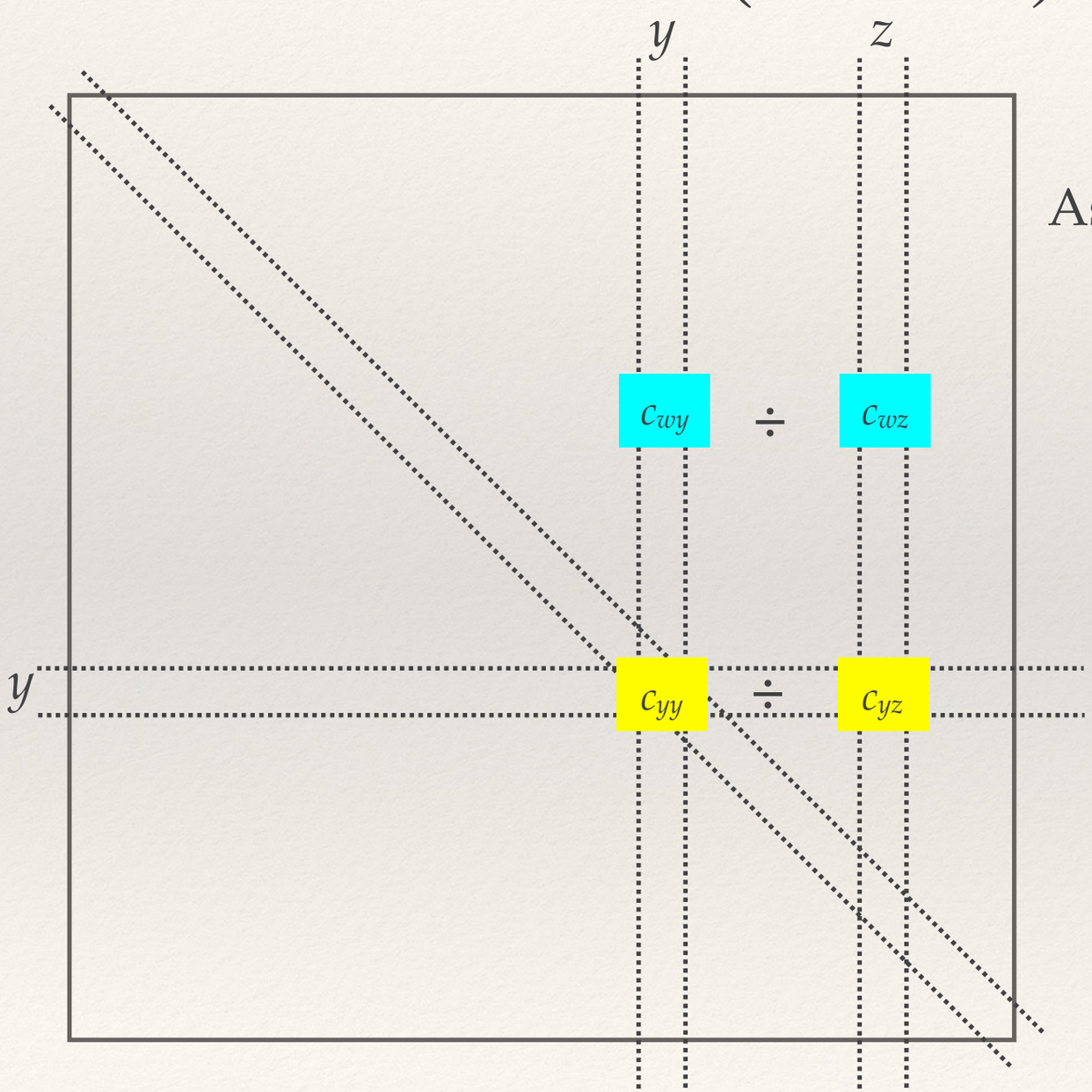


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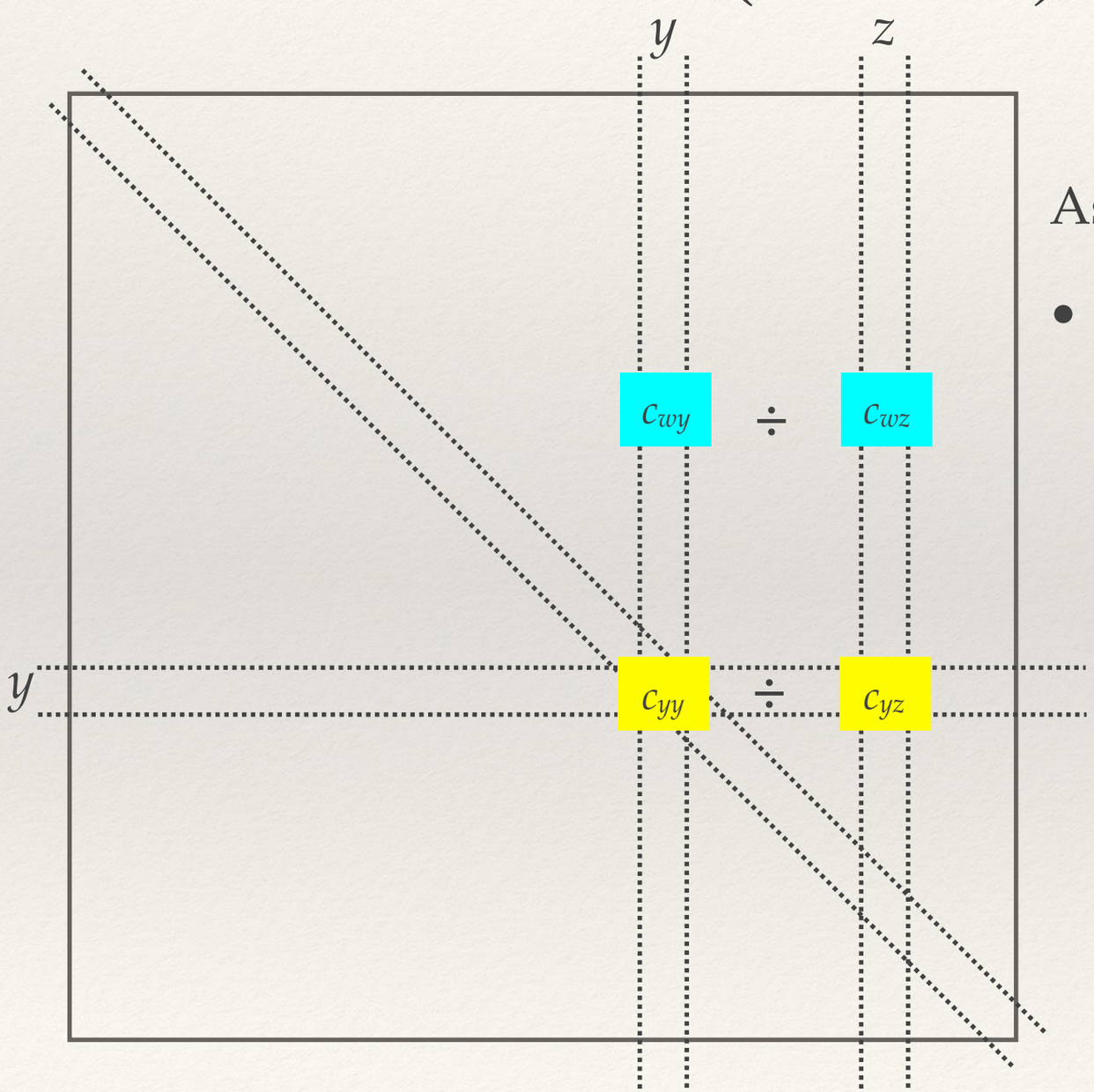


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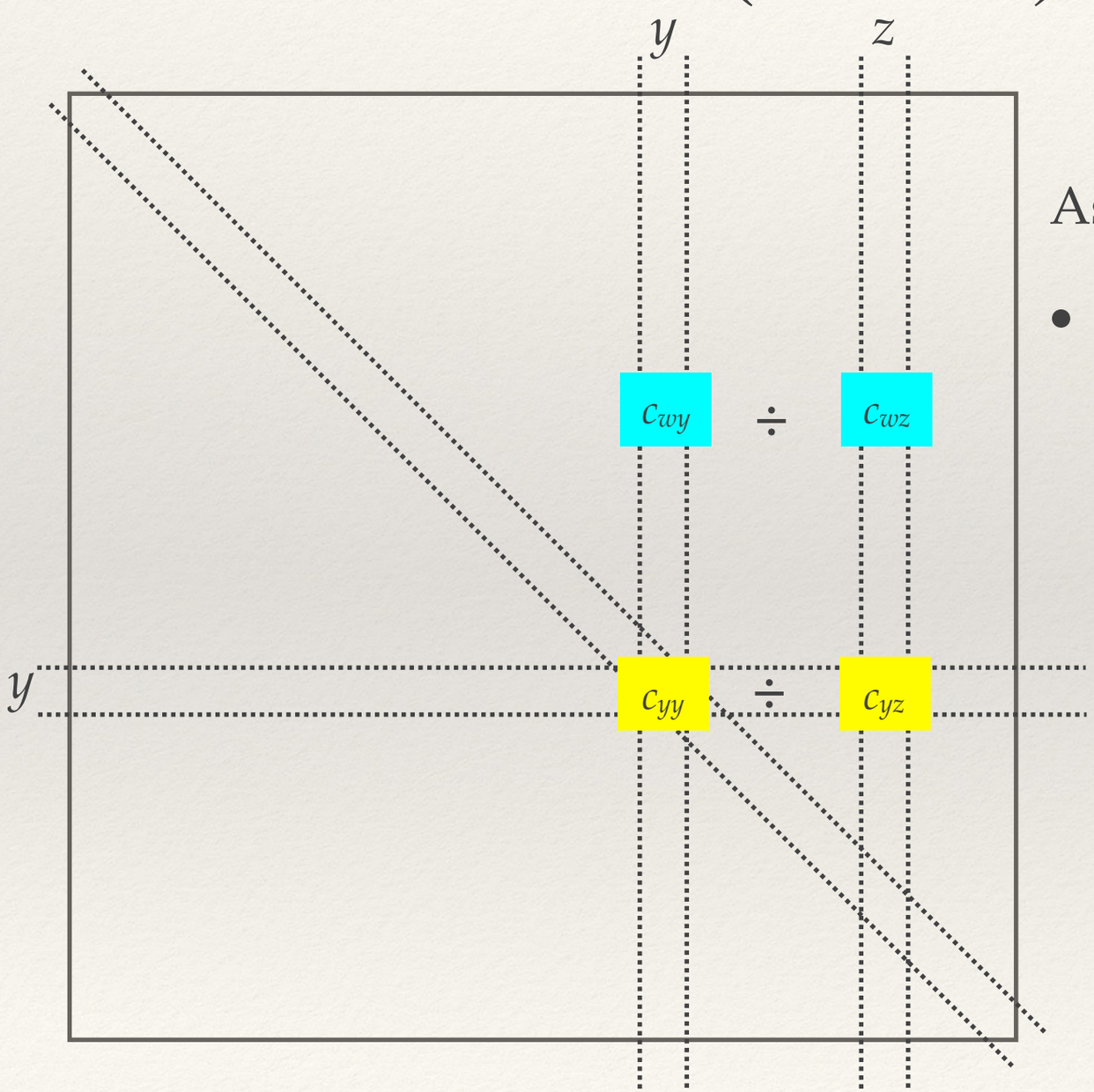


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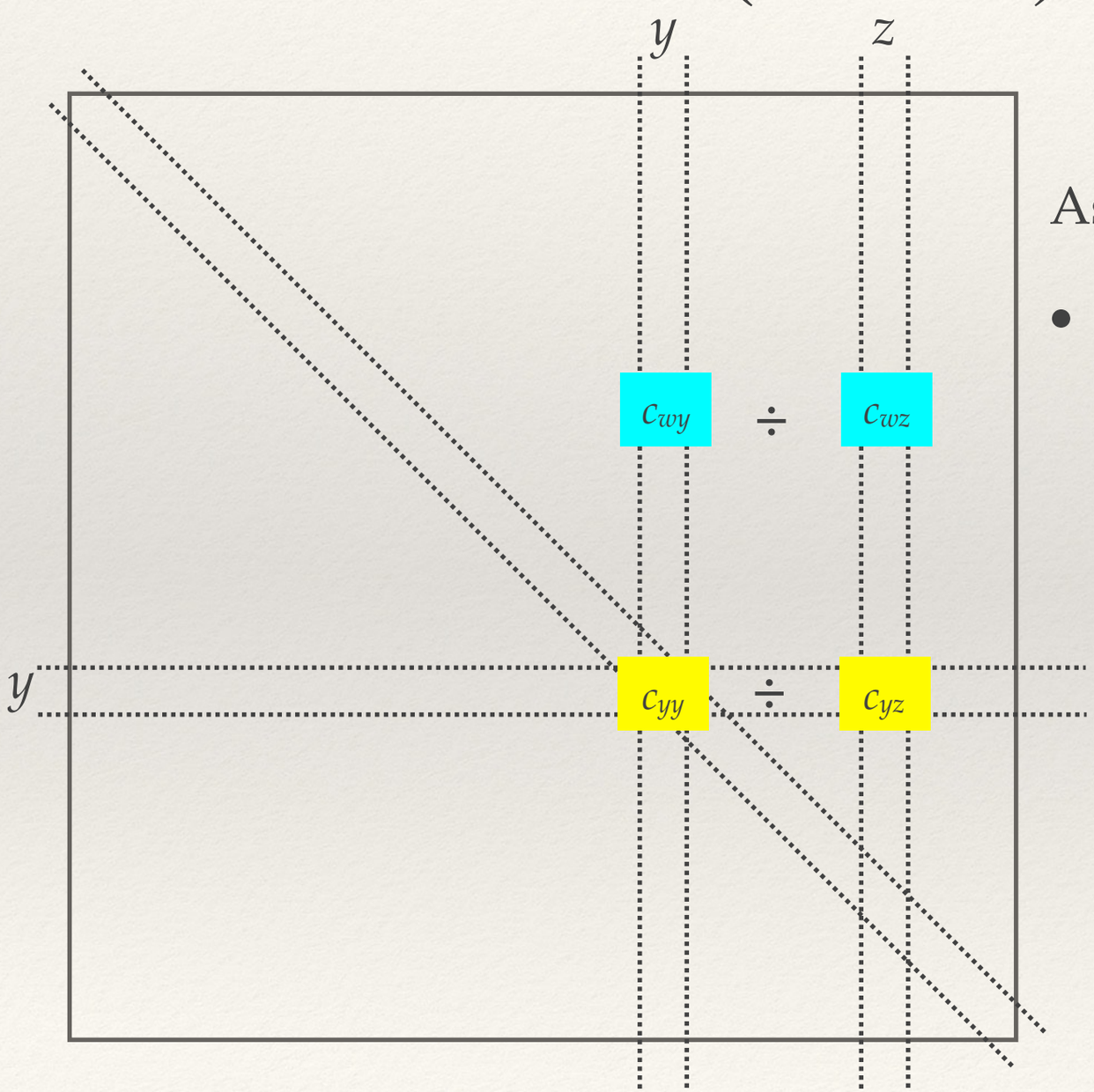


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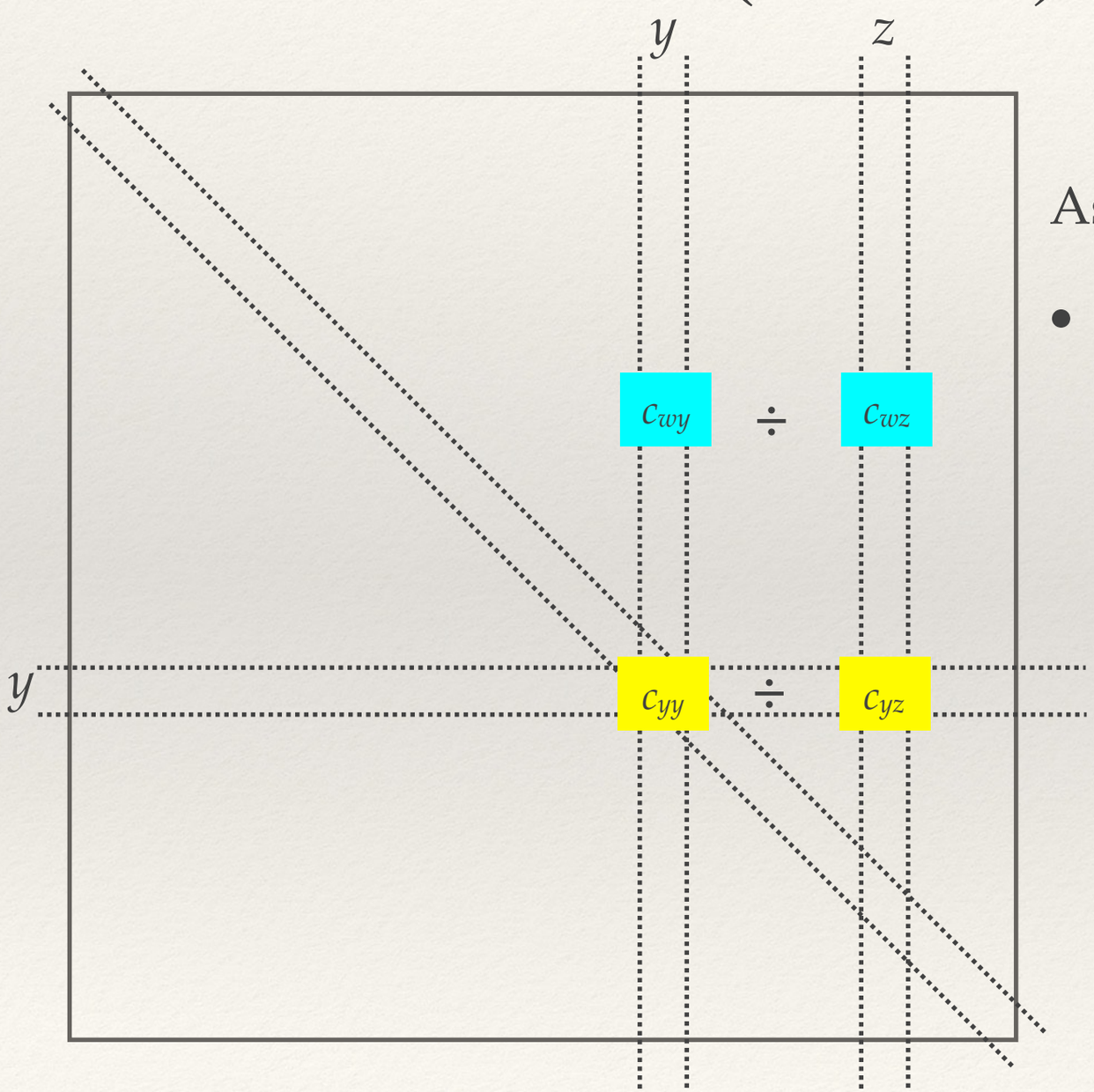


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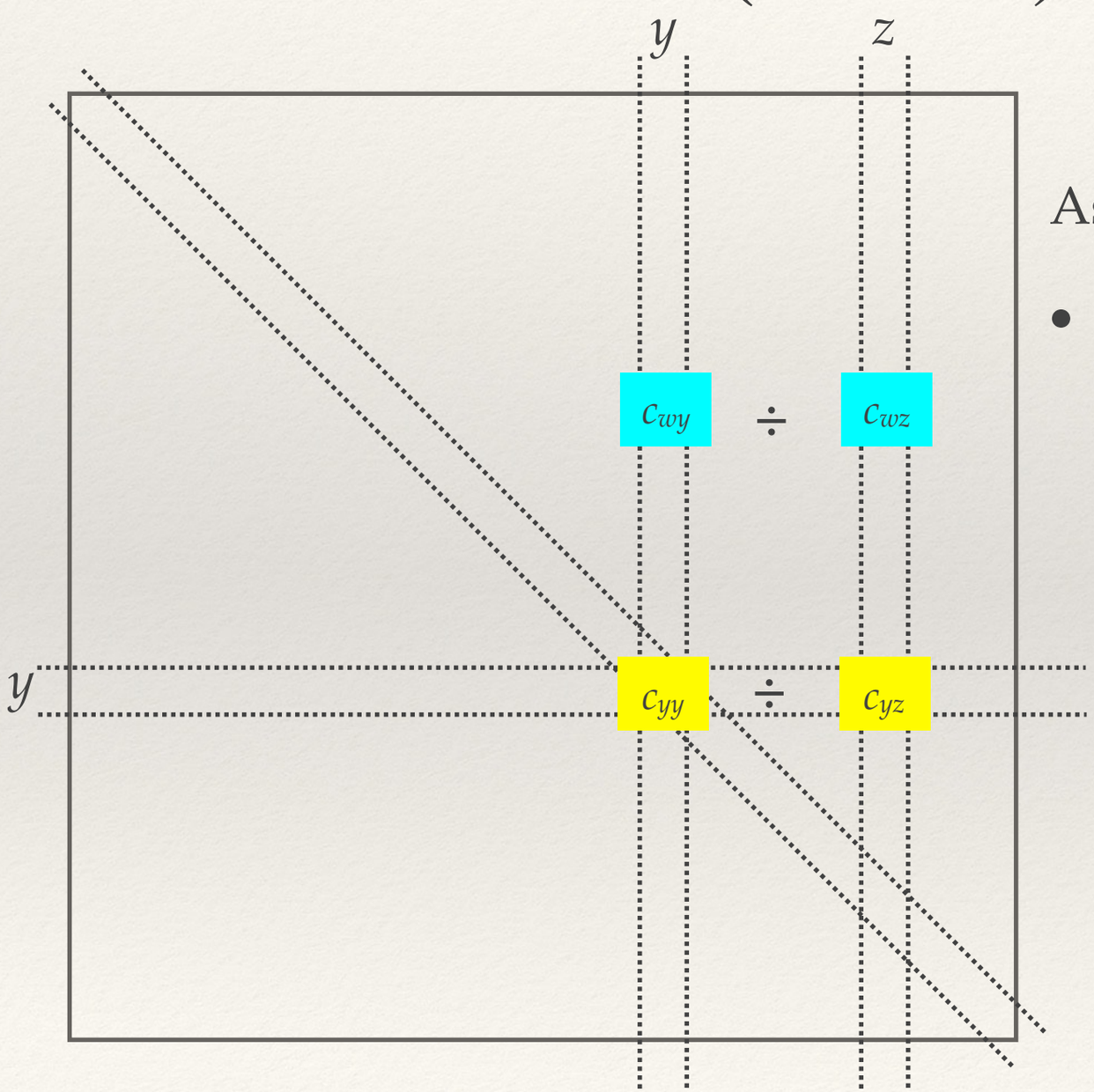


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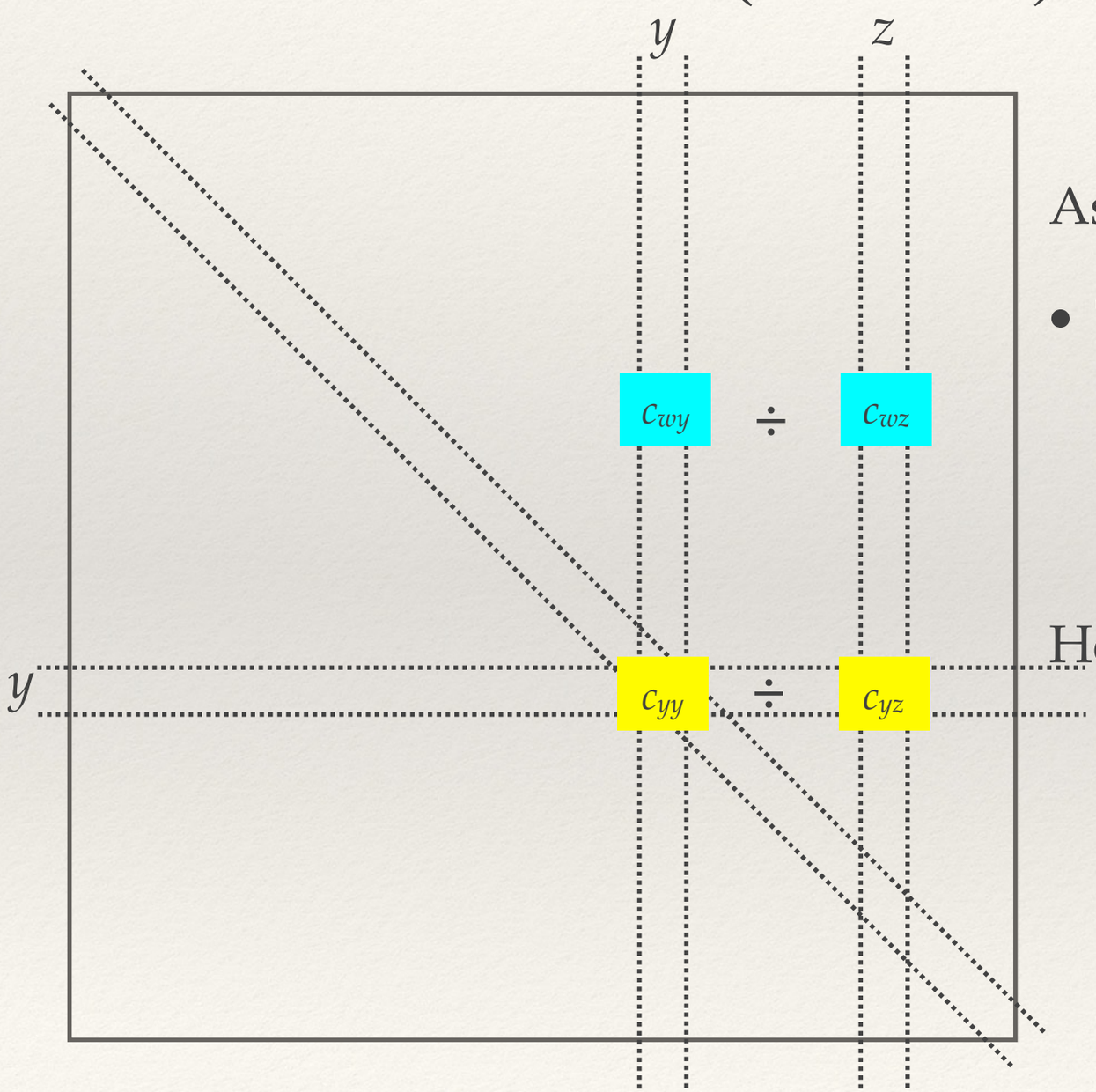


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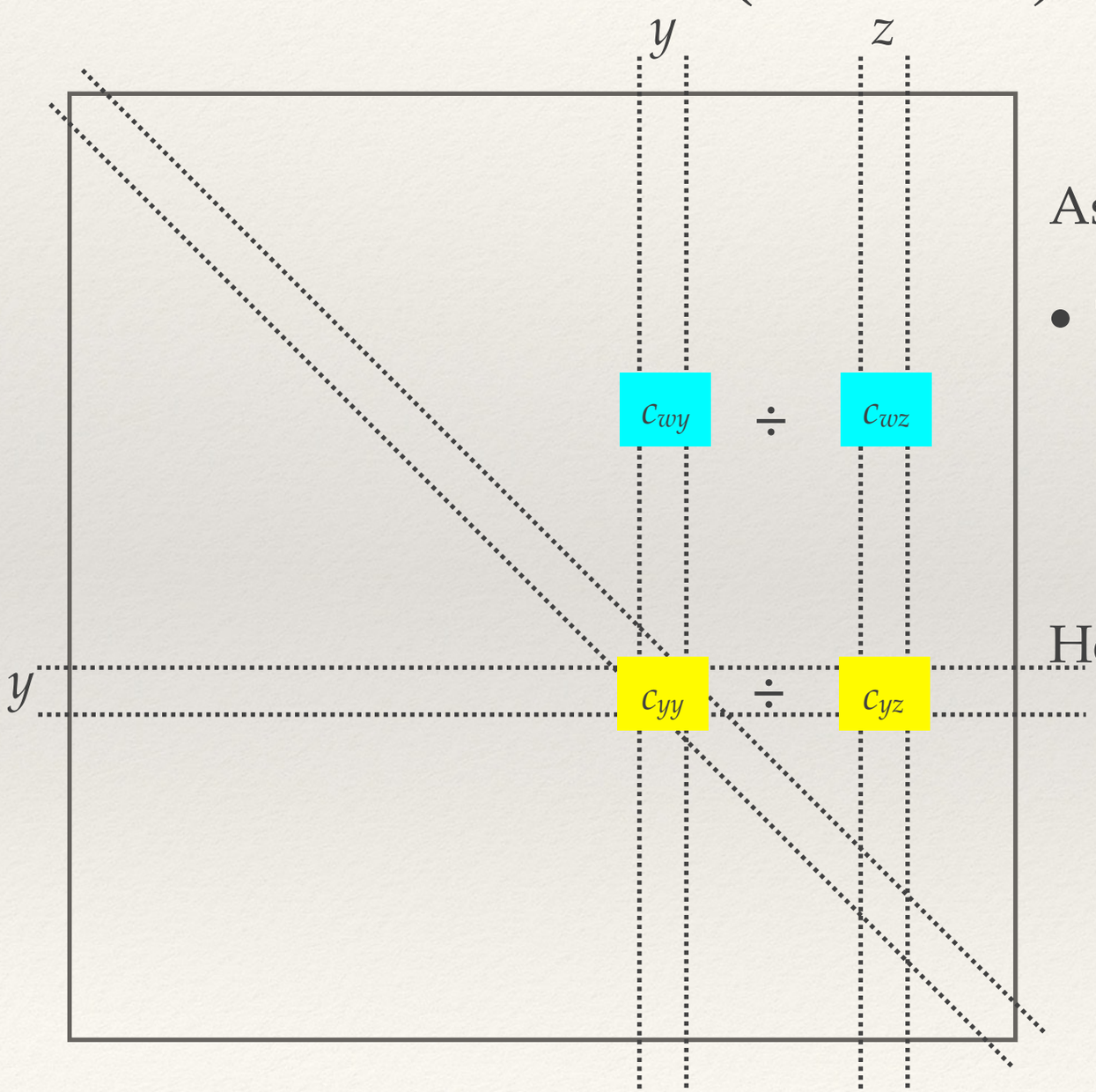


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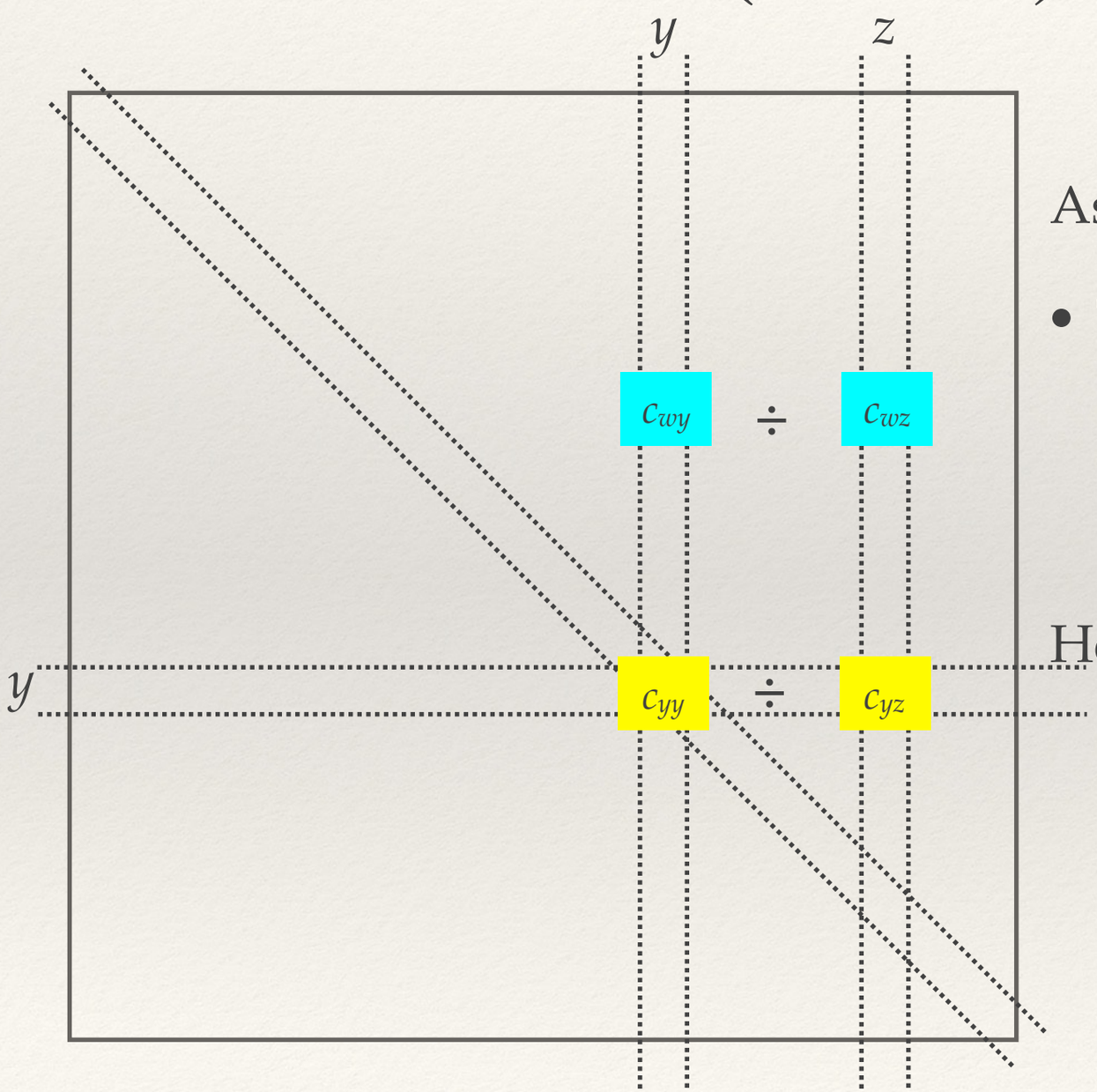


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Thanks for your attention!