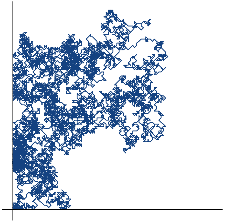
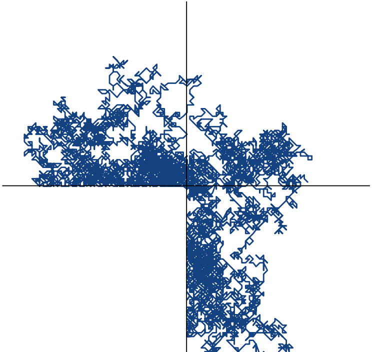
Counting lattice paths confined to cones

Mireille Bousquet-Mélou, CNRS, Bordeaux, France





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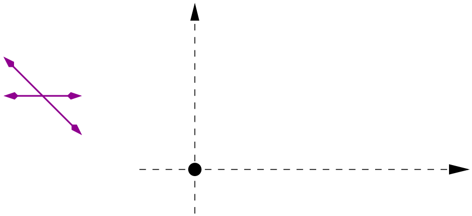
A typical question

Let $\mathcal{S}$ be a finite subset of $\mathbb{Z}^d$ (set of steps) and $p_0 \in \mathbb{Z}^d$ (starting point).

Example. $\mathcal{S} = \{10, \bar{1}0, 1\bar{1}, \bar{1}1\}$, $p_0 = (0,0)$



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A path (walk) of length n starting at p_0 is a sequence $(p_0, p_1, \ldots, p_n)$ such that $p_{i+1} - p_i \in \mathcal{S}$ for all i.

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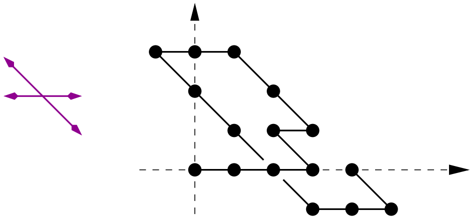

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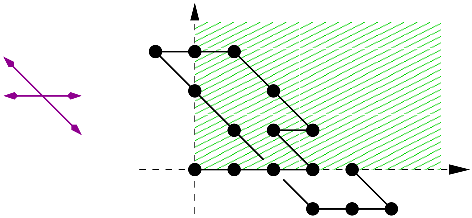
Let C be a cone of $\mathbb{R}^d$.

Example. $\mathcal{S} = \{10, \bar{1}0, 1\bar{1}, \bar{1}1\}$, $p_0 = (0,0)$ and $C = \mathbb{R}_+^2$.



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Questions

- What is the number $a(n)$ of n-step walks starting at p_0 and contained in C?
- For $\boldsymbol{i} = (i_1, \dots, i_d) \in C$, what is the number $a(\boldsymbol{i}; n)$ of such walks that end at $\boldsymbol{i}$?

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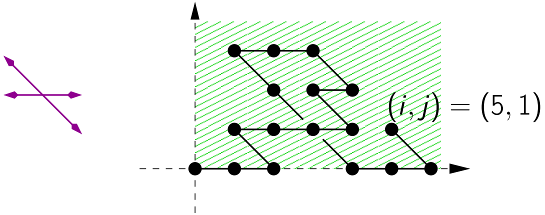

Example [Gouyou-Beauchamps 86], [mbm-Mishna 10]

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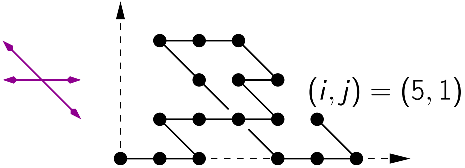

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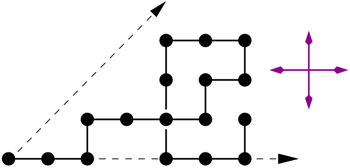

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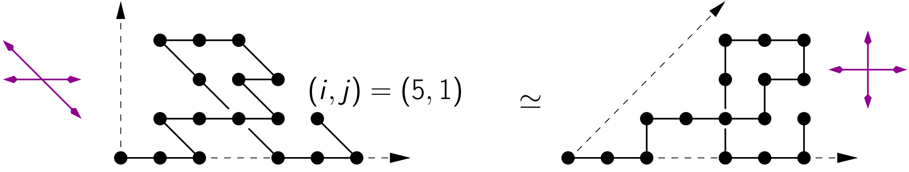

Nice numbers

If $n = 2m + \delta$, with $\delta \in \{0, 1\}$,

$$a(n) = \frac{n!(n+1)!}{m!(m+1)!(m+\delta)!(m+\delta+1)!}.$$

Moreover, if $n = 2m + i$,

$$a(i, j; n) = \frac{(i+1)(j+1)(i+j+2)(i+2j+3)n!(n+2)!}{(m-j)!(m+1)!(m+i+2)!(m+i+j+3)!}.$$

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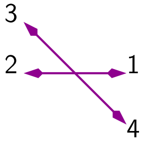

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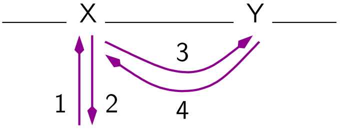

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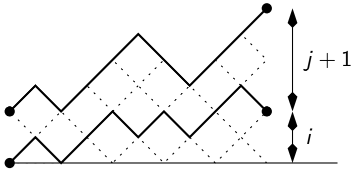

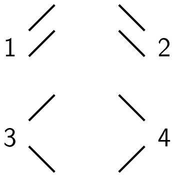

$+$ Young tableaux of height 4 [Gouyou-Beauchamps 89]

Why count walks in cones?

- Many discrete objects can be encoded in that way:
  - in combinatorics, statistical physics...
  - in (discrete) probability theory: random walks, queuing theory...

- To reach a better understanding of functional equations with divided differences

$$Q(x,y) = 1 + txyQ(x,y) + t\frac{Q(x,y) - Q(0,y)}{x} + t\frac{Q(x,y) - Q(x,0)}{y}$$

Many contributions

Adan, Banderier, Bernardi, Bostan, Budd, Cori, Denisov, Duchon, Dulucq, Fayolle, Gessel, Fisher, Flajolet, Gouyou-Beauchamps, Guttmann, Guy, Janse van Rensburg, Johnson, Kauers, Koutschan, Krattenthaler, Kurkova, Kreweras, van Leeuwarden, MacMahon, Melczer, Mishna, Niederhausen, Petkovšek, Prellberg, Raschel, Rechnitzer, Sagan, Salvy, Viennot, Wachtel, Wilf, Yeats, Zeilberger...

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~~Specific question~~
~~Ad hoc solution~~ ⟶ Systematic approach

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- Generating functions:

$$A(t) = \sum_{n \geq 0} a(n) t^n, \qquad A(x_1, \ldots, x_d; t) = \sum_{\boldsymbol{i}, n} a(\boldsymbol{i}; n) \boldsymbol{x}^{\boldsymbol{i}} t^n$$

$$= \sum_{w \text{ walk}} \boldsymbol{x}^{\boldsymbol{i}(w)} t^{|w|}$$

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Remarks

- $A(1, \ldots, 1; t) = A(t)$
- if $C \subset \mathbb{R}_+^d$, then $A(0, \ldots, 0; t)$ counts walks ending at $(0, \ldots, 0)$
- $A(0, x_2, \ldots, x_d; t)$ counts walks ending on the hyperplane $i_1 = 0$

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Can one express these series? What is their *nature*?

A hierarchy of formal power series

- Rational series

$$A(t) = \frac{P(t)}{Q(t)}$$

- Algebraic series

$$P(t, A(t)) = 0$$

- Differentially finite series (D-finite)

$$\sum_{i=0}^{d} P_i(t) A^{(i)}(t) = 0$$

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Multi-variate series: one DE per variable

A (very) basic cone: the full space

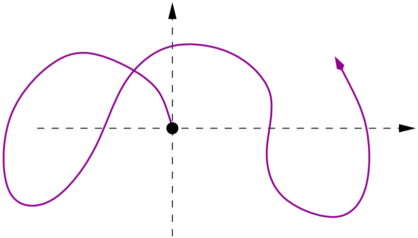
Rational series

If $\mathcal{S} \subset \mathbb{Z}^d$ is finite and $C = \mathbb{R}^d$, then $A(\boldsymbol{x}; t)$ is rational:

$$a(n) = |\mathcal{S}|^n \quad \Leftrightarrow \quad A(t) = \sum_{n \geq 0} a(n) t^n = \frac{1}{1 - |\mathcal{S}|\, t}$$

More generally:

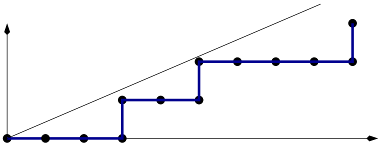
$$A(\boldsymbol{x}; t) = \frac{1}{1 - t \sum_{s \in \mathcal{S}} \boldsymbol{x}^{\boldsymbol{s}}}.$$



Caveat: rational cones only!

The bounding hyperplanes are given by linear equations with integer/rational coefficients.

Example: the generating function of walks with N and E steps under a line of irrational slope is not known.

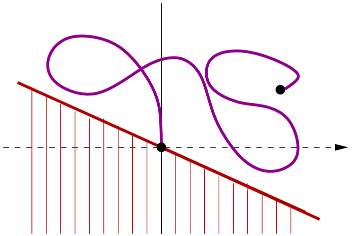


Also well-known: a (rational) half-space

Algebraic series

If $\mathcal{S} \subset \mathbb{Z}^d$ is finite and C is a rational half-space, then $A(\boldsymbol{x}; t)$ is algebraic, given by an explicit system of polynomial equations.

[Gessel 80]; [mbm-Petkovšek 00], [Duchon 00], [Banderier & Flajolet 02]...



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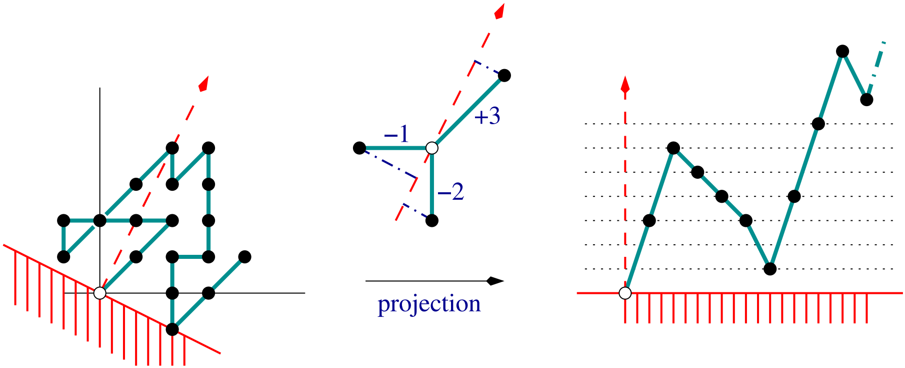
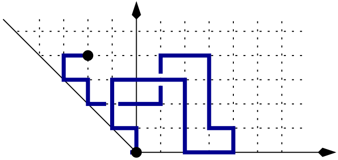
By projection: Equivalent to walks in 1D confined to a half-line

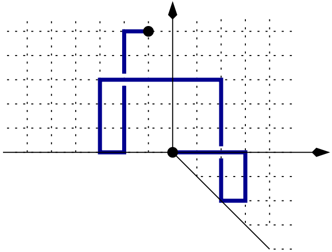


The "next" case: two bounding hyperplanes

- Convex cone: walks in a quadrant



- Non-convex cone: walks avoiding a quadrant



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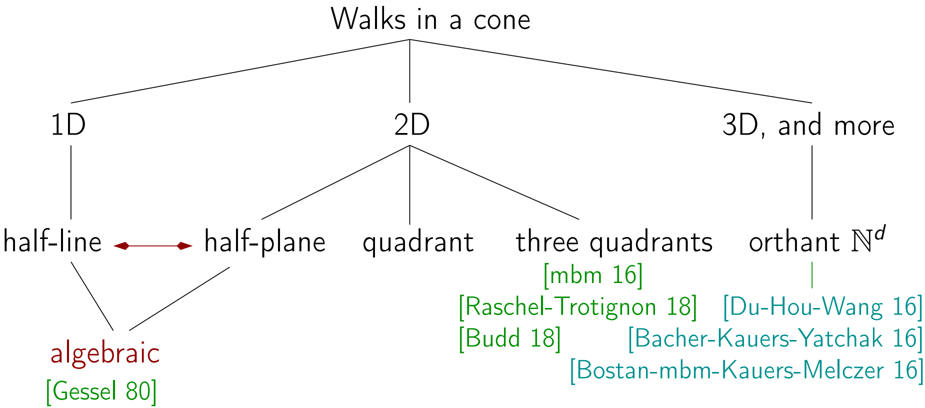

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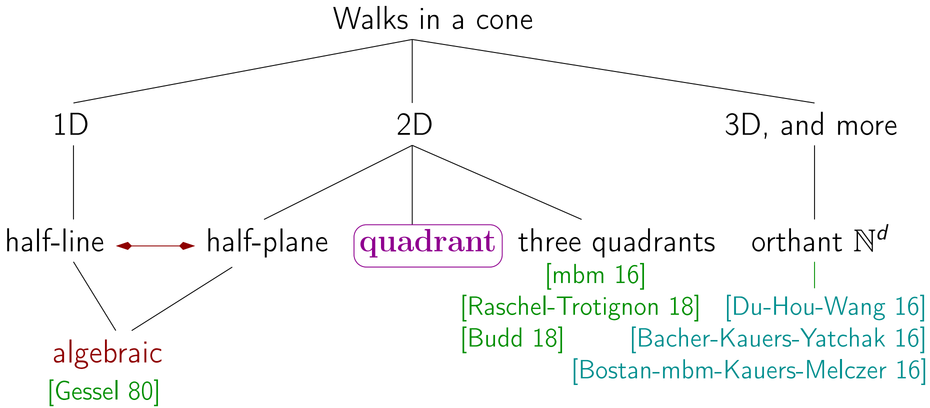


A typical question

Let $\mathcal{S}$ be a finite subset of $\mathbb{Z}^2$ (set of steps) and $p_0 \in \mathbb{Z}^d$ (starting point).

Questions

- What is the number $q(n)$ of n-step walks starting at p_0 and contained in the quadrant $\mathbb{N}^2$?
- For $(i, j) \in \mathbb{N}^2$, what is the number $q(i, j; n)$ of such walks that end at (i, j)?

The associated generating function:

$$Q(x, y; t) = \sum_{i,j,n \geq 0} q(i, j; n) x^i y^j t^n = ?$$

II. Walks in a quadrant: asymptotic enumeration

1. Excursions (prescribed endpoint (i, j))
2. All quadrant walks

Expected:

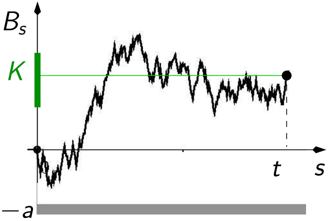
$$q(i, j; n) \sim \kappa \mu_e^n n^{\gamma_e} \qquad\qquad q(n) \sim \kappa \mu_w^n n^{\gamma_w}$$

possibly with periodicity conditions on n.

The 1D case: excursions on a half-line

- The brownian exponent: as $t \to \infty$,

$$\mathbb{P}(B_s > -a \text{ for } s \in [0,t] \text{ and } B_t \in K) \sim \kappa t^{-3/2}$$



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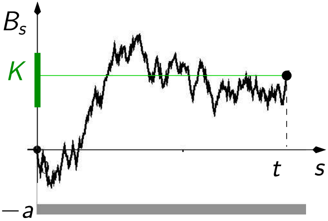
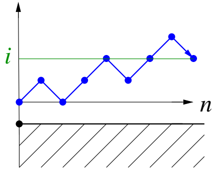


- Lattice walks: let $\mathcal{S}$ be a (finite) step set, and let $S(x) = \sum_{s \in \mathcal{S}} x^s$. For walks on the non-negative half-line ending at position i,

$$h(i;n) \sim \kappa \mu^n n^{-3/2}$$

where

$$\mu = \min_{x>0} S(x)$$
$$= S(x_c) \quad \text{with} \quad S'(x_c) = 0$$



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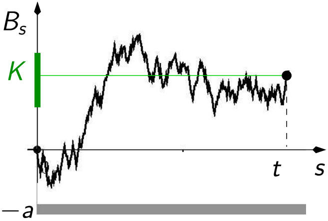

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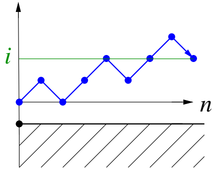

Remark: $\mu \leq S(1)$, with equality iff $S'(1) = \sum_{s \in \mathcal{S}} s = 0 \quad \Leftrightarrow$ NO DRIFT

The 2D case: excursions in a quadrant

- The (2D) brownian exponent: as $t \to \infty$,

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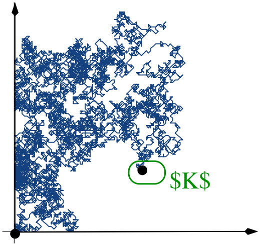

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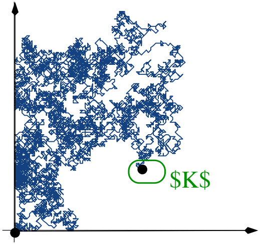

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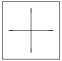
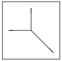
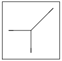


$q(0,0;n) \sim \quad \kappa\, 4^n n^{-3}$





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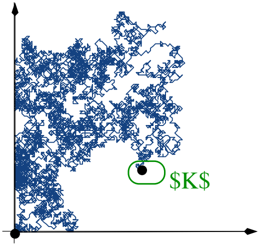

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	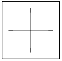	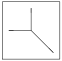	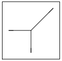
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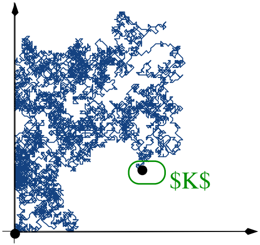

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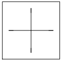
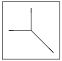
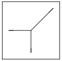




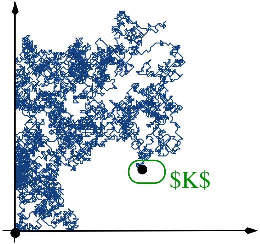
$$q(0, 0; n) \sim \quad \kappa\, 4^n n^{-3} \quad \Big| \quad \kappa\, 3^n n^{-4} \quad \Big| \quad \kappa\, 3^n n^{-5/2}$$

What's happening?

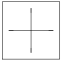
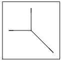
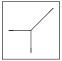
The 2D case: excursions in a quadrant

- The (2D) brownian exponent: as $t \to \infty$,

  $$\mathbb{P}(B_s > -\boldsymbol{a} \text{ for } s \in [0,t] \text{ and } B_t \in K) \sim \kappa t^{-3}$$



- Lattice walks:

$$q(0,0;n) \sim \quad \kappa\, 4^n n^{-3} \quad \Big| \quad \kappa\, 3^n n^{-4} \quad \Big| \quad \kappa\, 3^n n^{-5/2}$$

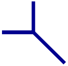
What's happening?

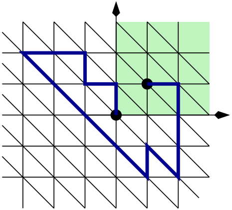
The x- and y-moves are correlated.

Excursions in a quadrant: an example

Take a uniform random walk with steps $(0,1),(-1,0),(1,-1)$. We have

$$\mathbb{E}(X)=\mathbb{E}(Y)=0, \quad \mathbb{E}(X^2)=\mathbb{E}(Y^2)=2/3 \quad \text{and} \quad \mathbb{E}(XY)=\mathbf{-1/3}.$$





Excursions in a quadrant: an example

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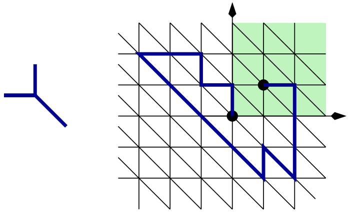
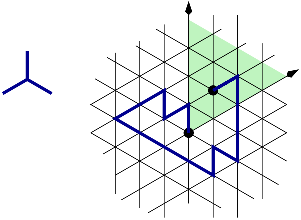




Excursions in a quadrant: an example

Take a uniform random walk with steps $(0,1),(-1,0),(1,-1)$. We have

$$\mathbb{E}(X) = \mathbb{E}(Y) = 0, \quad \mathbb{E}(X^2) = \mathbb{E}(Y^2) = 2/3 \quad \text{and} \quad \mathbb{E}(XY) = \mathbf{-1/3}.$$

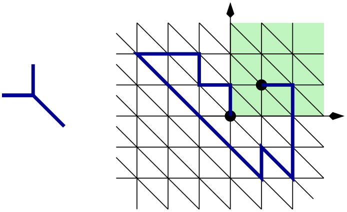
Define

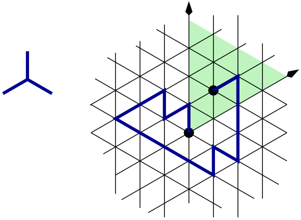
$$Y' = \frac{1}{\sqrt{3}}(X + 2Y),$$

so that

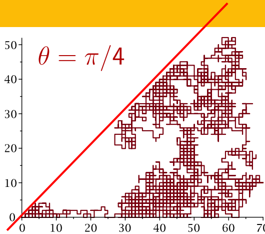
$$\mathbb{E}(X) = \mathbb{E}(Y') = 0, \quad \mathbb{E}(X^2) = \mathbb{E}(Y'^2) = 2/3 \quad \text{and} \quad \mathbb{E}(XY') = \mathbf{0}.$$

But the confining quadrant has become a wedge of $\pi/3$!





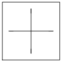
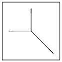
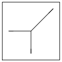
The 2D case: excursions in a wedge



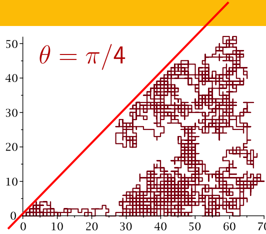
- Brownian exponents: as $t \to \infty$,

$$\mathbb{P}(B_s \in W_\theta \text{ for } s \in [0,t] \text{ and } B_t \in K) \sim \kappa t^{-1-\pi/\theta}$$

- Lattice walks:

			
θ	$\pi/2$	$\pi/3$	$2\pi/3$
$q(0,0;n) \sim$	$\kappa\, 4^n n^{-3}$	$\kappa\, 3^n n^{-4}$	$\kappa\, 3^n n^{-5/2}$

The 2D case: excursions in a wedge



- Brownian exponents: as $t \to \infty$,

$$\mathbb{P}(B_s \in W_\theta \text{ for } s \in [0, t] \text{ and } B_t \in K) \sim \kappa t^{-1-\pi/\theta}$$

- Lattice walks: let $\mathcal{S} \subset \mathbb{Z}^2$ be a (finite) step set, and let $S(x, y) = \sum_{(k,\ell)\in\mathcal{S}} x^k y^\ell$. For walks in the quadrant $\mathbb{N}^2$ ending at (i, j),

$$q(i, j; n) \sim \kappa \mu^n n^{-1-\pi/\theta}$$

where

$$\mu = \min_{x>0,y>0} S(x, y)$$
$$= S(x_c, y_c) \quad \text{with} \quad S'_1(x_c, y_c) = S'_2(x_c, y_c) = 0$$

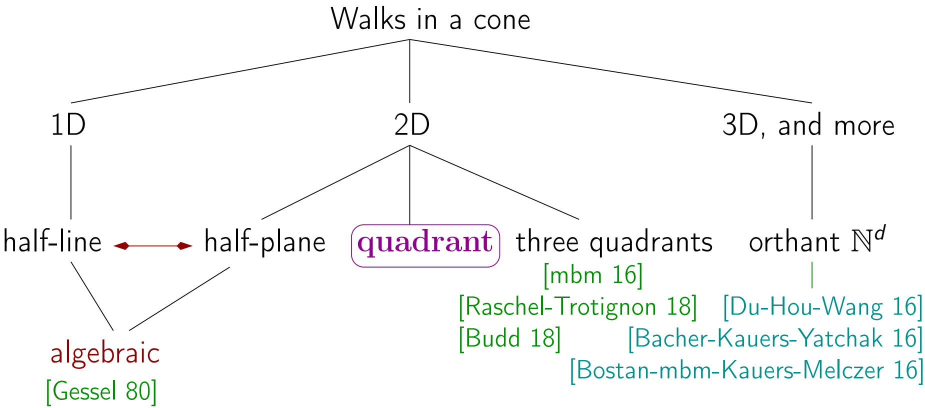
and

$$\theta = \arccos\left(-\frac{S''_{1,2}(x_c, y_c)}{\sqrt{S''_{1,1}(x_c, y_c)S''_{2,2}(x_c, y_c)}}\right)$$

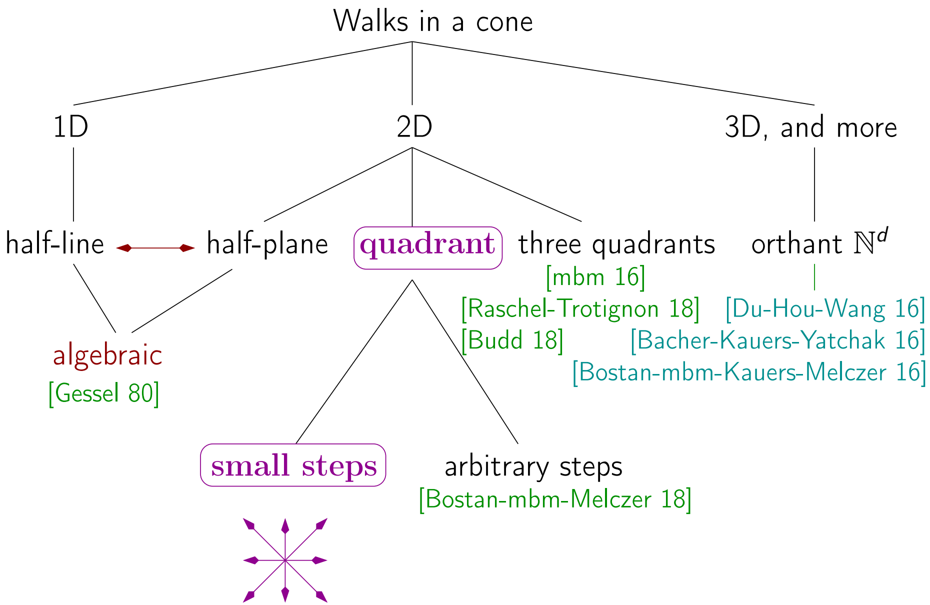
[Denisov & Wachtel 15]

III. Walks in a quadrant: exact enumeration

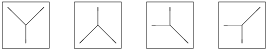
Counting walks confined to cones



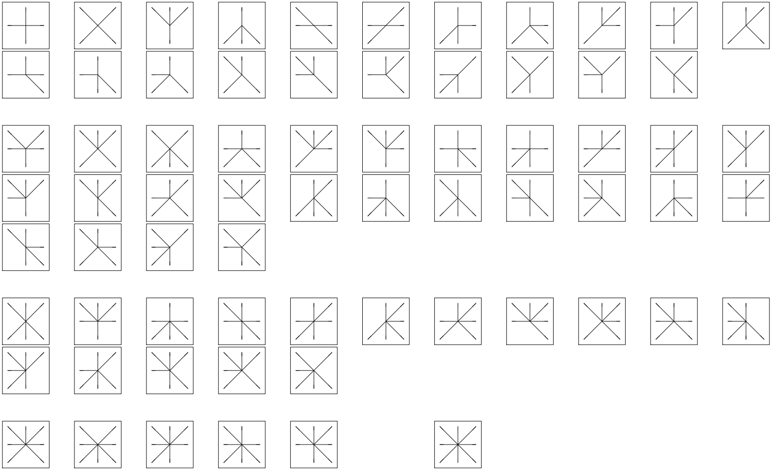
Counting walks confined to cones

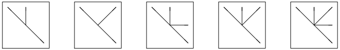


The 79 interesting distinct quadrant models with small steps



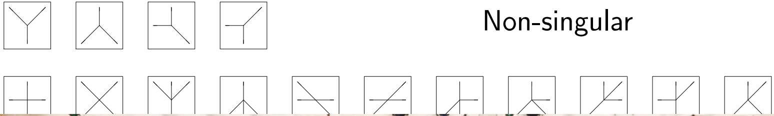
Non-singular



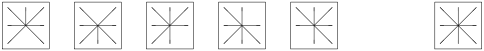


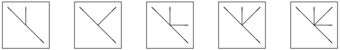
Singular

The 79 interesting distinct quadrant models with small steps



Non-singular





Singular

The starting point: a recurrence relation...

The numbers $q(i,j;n)$ satisfy

$$q(i,j;n) = \begin{cases} 0 & \text{if } i<0 \text{ or } j<0 \text{ or } n<0, \\ \mathbb{1}_{i=j=0} & \text{if } n=0, \\ \sum_{(i,'j')\in\mathcal{S}} q(i-i',j-j';n-1) & \text{otherwise.} \end{cases}$$

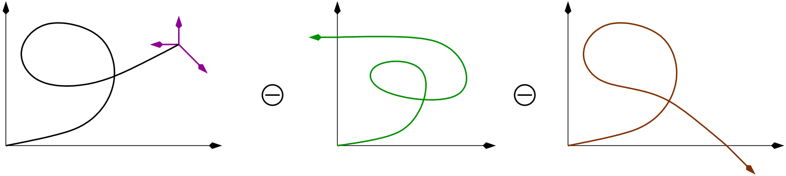
$\Rightarrow$ an equation for

$$Q(x,y;t) = \sum_{i,j,n\geq 0} q(i,j;n)x^i y^j t^n$$

... and the corresponding functional equation

Example: $\mathcal{S} = \{01, \bar{1}0, 1\bar{1}\}$, with $\bar{x} := 1/x$ and $\bar{y} := 1/y$

$$Q(x, y; t) \equiv Q(x, y) = 1 + t(y + \bar{x} + x\bar{y})Q(x, y) - t\bar{x}Q(0, y) - tx\bar{y}Q(x, 0)$$



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or

$$\big(1 - t(y + \bar{x} + x\bar{y})\big)Q(x, y) = 1 - t\bar{x}Q(0, y) - tx\bar{y}Q(x, 0),$$

... and the corresponding functional equation

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$$\big(1 - t(y + \bar{x} + x\bar{y})\big)Q(x, y) = 1 - t\bar{x}Q(0, y) - tx\bar{y}Q(x, 0),$$

or

$$\boxed{\big(1 - t(y + \bar{x} + x\bar{y})\big)xyQ(x, y) = xy - tyQ(0, y) - tx^2Q(x, 0)}$$

- The polynomial $1 - t(y + \bar{x} + x\bar{y})$ is the kernel of this equation

- The equation is linear, with two catalytic variables x and y (tautological at $x = 0$ or $y = 0$)

Exact results: examples

- expressions for $q(n)$, for $q(i,j;n)$
- expressions for the generating functions

$$Q(t) := \sum_n q(n) t^n \qquad \text{and} \qquad Q(x,y;t) := \sum_{i,j,n} x^i y^j t^n q(i,j;n)$$

- nature of these generating functions

A hierarchy of formal power series

- Rational series

$$A(t) = \frac{P(t)}{Q(t)}$$

- Algebraic series

$$P(t, A(t)) = 0$$

- Differentially finite series (D-finite)

$$\sum_{i=0}^{d} P_i(t) A^{(i)}(t) = 0$$

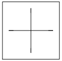
- D-algebraic series

$$P(t, A(t), A'(t), \ldots, A^{(d)}(t)) = 0$$

1. Expressions for numbers

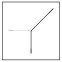
- Square lattice walks

$$q(0,0;2n) = C_n C_{n+1} \quad \text{with} \quad C_n = \frac{1}{n+1}\binom{2n}{n}$$



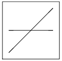
- Kreweras' walks [Kreweras 65]

$$q(0,0;3n) = \frac{4^n}{(n+1)(2n+1)}\binom{3n}{n}$$



- Gessel's walks [Kauers-Zeilberger 09]

$$q(0,0;2n) = 16^n \frac{(5/6)_n(1/2)_n}{(5/3)_n(2)_n}$$

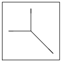


where $(a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$ is the ascending factorial.

2. Expressions for series

- As positive parts of rational series in t (with $\bar{x} := 1/x$, $\bar{y} := 1/y$):

$$Q(x,y;t) = [x^{\geq 0} y^{\geq 0}] \frac{1 - \bar{x}^2 y + \bar{x}^3 - \bar{x}^2 \bar{y}^2 + \bar{y}^3 - x\bar{y}^2}{1 - t(y + \bar{x} + x\bar{y})}$$

 [mbm-Mishna 09]

2. Expressions for series

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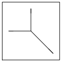
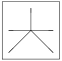


[mbm-Mishna 09]

- As explicit integrals involving hypergeometric series

$$Q(0,0;t) = \frac{2}{t^2} \int_0^t \int_0^u \frac{1}{(1-4\,v^2)^{3/2}}\ {}_2F_1\left(\frac{3}{4}, \frac{5}{4}; 2; 64\,\frac{(v+1)\,v^3}{(1-4\,v^2)^2}\right) \mathrm{d}v\,\mathrm{d}u$$



[Bostan-Chyzak-van Hoeij-Kauers-Pech 17]

2. Expressions for series (cont'd)

- In terms of Weierstrass' function and elliptic integrals

$$t(1+y)Q(0,y;t)+\frac{1}{y}=\frac{I'(0)}{I(y)-I(0)}-\frac{I'(0)}{I(-1)-I(0)}-1$$

with

$$I(y)\equiv I(y;t)=\wp\left(\mathcal{R}(y;t),\omega_1(t),\omega_3(t)\right)$$

where

- $\wp$ is Weierstrass' elliptic function
- its periods ω_1 and ω_3 are explicit elliptic integrals
- its argument $\mathcal{R}$ is also an explicit elliptic integral

[Bernardi-mbm-Raschel 17(a)]

2. Expressions for series (cont'd)

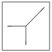
- Climax: an integral expression involving the same ingredients

$$\tilde{K}(0,y;t)Q(0,y;t) - \tilde{K}(0,0;t)Q(0,0;t) = yX_0(y;t) +$$
$$\frac{1}{2i\pi}\int_{y_1(t)}^{y_2(t)} u\,[X_0(u;t) - X_1(u;t)]\left[\frac{\partial_u I(u;t)}{I(u;t)-I(y;t)} - \frac{\partial_u I(u;t)}{I(u;t)-I(0;t)}\right] \mathrm{d}u$$

where $\tilde{K}(x,y;t) = xy(1-tS(x,y))$, X_0, X_1, y_1 and y_2 are explicit algebraic series and $I(y;t)$ is as given on the previous slide.

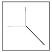
Valid for all (non-singular) small step models
[Raschel 12]

3. Nature of the series



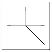
Algebraic [Kreweras 65, Gessel 86]

$$(1 - t(\bar{x} + \bar{y} + xy))xyQ(x,y) = xy - tyQ(0,y) - txQ(x,0)$$



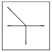
D-finite, but transcendental [Gessel 90]

$$(1 - t(y + \bar{x} + x\bar{y}))xyQ(x,y) = xy - tyQ(0,y) - tx^2Q(x,0)$$



Not D-finite, but D-algebraic [Bernardi, mbm & Raschel 17]

$$(1 - t(x + \bar{x} + y + x\bar{y}))xyQ(x,y) = xy - tyQ(0,y) - tx^2Q(x,0)$$



Not D-algebraic (in y) [Dreyfus, Hardouin, Roques & Singer 17]

$$(1 - t(x\bar{y} + \bar{x} + \bar{y} + y))xyQ(x,y) = xy - tyQ(0,y) - tx(1+x)Q(x,0)$$

Classification of quadrant walks: a variety of tools

quadrant models: 79

- $|G|<\infty$: 23 → **D-finite**
  - OS=0: 4 → **algebraic**
  - OS≠0: 19 → **DF transc.**
- $|G|=\infty$: 56 → **Not D-finite**
  - decoupled: 9 → **D-alg.**
  - not decoupled: 47 → **not D-alg.**

Formal power series algebra

Complex analysis

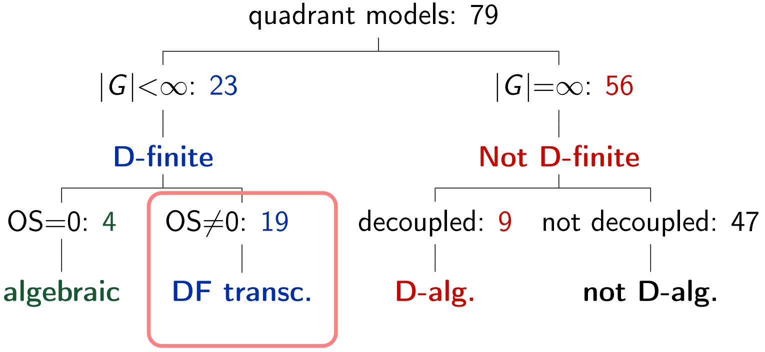
Differential Galois theory

Computer algebra

Random walks in probability

D-finite series
- effective closure properties
- arithmetic properties
- asymptotics
- G-functions

III.1. The group of the walk, and D-finite cases



Formal power series algebra

[mbm & Mishna 09]

The group of the model

Example. Take $\mathcal{S} = \{\bar{1}0, 01, 1\bar{1}\}$, with step polynomial

$$S(x, y) = \frac{1}{x} + y + \frac{x}{y} = \bar{x} + y + x\bar{y}$$

The group of the model

Example. Take $\mathcal{S} = \{\bar{1}0, 01, 1\bar{1}\}$, with step polynomial

$$S(x,y) = \frac{1}{x} + y + \frac{x}{y} = \bar{x} + y + x\bar{y}$$

Observation: $S(x,y)$ is left unchanged by the rational transformations

$$\Phi : (x,y) \mapsto (\bar{x}y, y) \quad \text{and} \quad \Psi : (x,y) \mapsto (x, x\bar{y})\,.$$

The group of the model

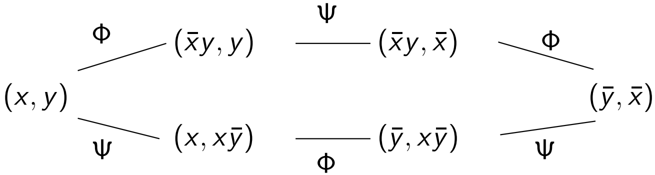
Example. Take $\mathcal{S} = \{\bar{1}0, 01, 1\bar{1}\}$, with step polynomial

$$S(x, y) = \frac{1}{x} + y + \frac{x}{y} = \bar{x} + y + x\bar{y}$$

Observation: $S(x, y)$ is left unchanged by the rational transformations

$$\Phi : (x, y) \mapsto (\bar{x}y, y) \quad \text{and} \quad \Psi : (x, y) \mapsto (x, x\bar{y})\,.$$

They are involutions, and generate a finite dihedral group G:



The group of the model

Example. Take $\mathcal{S} = \{\bar{1}0, 01, 1\bar{1}\}$, with step polynomial

$$S(x,y) = \frac{1}{x} + y + \frac{x}{y} = \bar{x} + y + x\bar{y}$$

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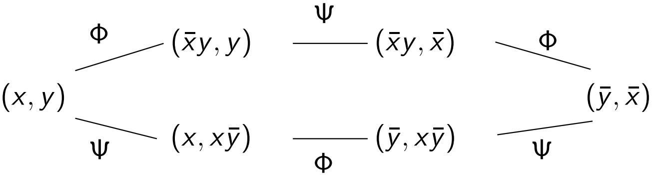


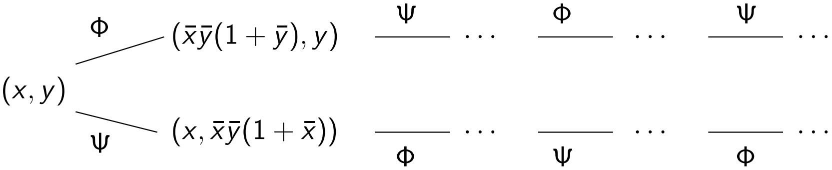
Remark. G can be defined for any quadrant model with small steps

The group is not always finite

- If $\mathcal{S} = \{0\bar{1}, \bar{1}\bar{1}, \bar{1}0, 11\}$, then $S(x, y) = \bar{x}(1 + \bar{y}) + \bar{y} + xy$ and

$$\Phi : (x, y) \mapsto (\bar{x}\bar{y}(1 + \bar{y}), y) \quad \text{and} \quad \Psi : (x, y) \mapsto (x, \bar{x}\bar{y}(1 + \bar{x}))$$

generate an infinite group:



The algebraic kernel method

- The equation reads (with $K(x,y) = 1 - t(y + \bar{x} + x\bar{y})$):

$$K(x,y)xyQ(x,y) = xy - tx^2Q(x,0) - tyQ(0,y)$$

- The orbit of (x,y) under G is

$$(x,y) \overset{\Phi}{\longleftrightarrow} (\bar{x}y, y) \overset{\Psi}{\longleftrightarrow} (\bar{x}y, \bar{x}) \overset{\Phi}{\longleftrightarrow} (\bar{y}, \bar{x}) \overset{\Psi}{\longleftrightarrow} (\bar{y}, x\bar{y}) \overset{\Phi}{\longleftrightarrow} (x, x\bar{y}) \overset{\Psi}{\longleftrightarrow} (x,y).$$

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- All transformations of G leave $K(x,y)$ invariant. Hence

$$\begin{array}{rclclcl}
K(x,y)\; xyQ(x,y) & = & xy & - & tx^2Q(x,0) & - & tyQ(0,y) \\
K(x,y)\; \bar{x}y^2Q(\bar{x}y,y) & = & \bar{x}y^2 & - & t\bar{x}^2y^2Q(\bar{x}y,0) & - & tyQ(0,y) \\
K(x,y)\; \bar{x}^2yQ(\bar{x}y,\bar{x}) & = & \bar{x}^2y & - & t\bar{x}^2y^2Q(\bar{x}y,0) & - & t\bar{x}Q(0,\bar{x}) \\
\cdots & = & & & \cdots & & \\
K(x,y)\; x^2\bar{y}Q(x,x\bar{y}) & = & x^2\bar{y} & - & tx^2Q(x,0) & - & tx\bar{y}Q(0,x\bar{y}).
\end{array}$$

⇒ Form the alternating sum of the equation over all elements of the orbit:

$$K(x,y)\Big(xyQ(x,y) - \bar{x}y^2Q(\bar{x}y,y) + \bar{x}^2yQ(\bar{x}y,\bar{x}) - \bar{x}\bar{y}Q(\bar{y},\bar{x}) + x\bar{y}^2Q(\bar{y},x\bar{y}) - x^2\bar{y}Q(x,x\bar{y})\Big) = xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}$$

(the orbit sum).

Why is this interesting?

$$xyQ(x,y) - \bar{x}y^2Q(\bar{x}y,y) + \bar{x}^2yQ(\bar{x}y,\bar{x}) - \bar{x}\bar{y}Q(\bar{y},\bar{x}) + x\bar{y}^2Q(\bar{y},x\bar{y}) - x^2\bar{y}Q(x,x\bar{y}) = \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}$$

- Both sides are power series in t, with coefficients in $\mathbb{Q}[x,\bar{x},y,\bar{y}]$.

Why is this interesting?

$$xyQ(x,y) - \bar{x}y^2Q(\bar{x}y,y) + \bar{x}^2yQ(\bar{x}y,\bar{x}) - \bar{x}\bar{y}Q(\bar{y},\bar{x}) + x\bar{y}^2Q(\bar{y},x\bar{y}) - x^2\bar{y}Q(x,x\bar{y}) = \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}$$

- Both sides are power series in t, with coefficients in $\mathbb{Q}[x, \bar{x}, y, \bar{y}]$.
- Extract the part with positive powers of x and y:

$$xyQ(x,y) = [x^{>0}y^{>0}]\,\frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}$$

is a D-finite series.
[Lipshitz 88]

The kernel method in general (finite groups)

- For all models with a finite group,

$$\sum_{g\in G} \mathrm{sign}(g) g(xyQ(x,y;t)) = \frac{1}{K(x,y;t)} \sum_{g\in G} \mathrm{sign}(g) g(xy) = \frac{OS}{K(x,y;t)},$$

where $g(Q(x,y)) := Q(g(x,y))$.

- The right-hand side is an explicit rational series.

[mbm-Mishna 10]

The kernel method in general (finite groups)

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where $g(Q(x,y)) := Q(g(x,y))$.

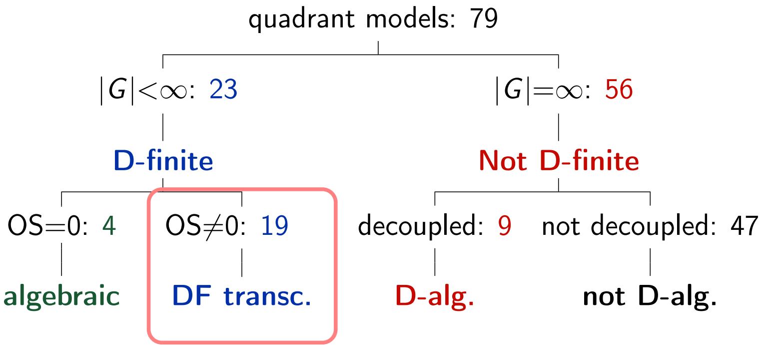
- The right-hand side is an explicit rational series.

- For the 19 models where the orbit sum is non-zero,

$$xyQ(x,y;t) = [x^{>0}y^{>0}] \frac{OS}{K(x,y;t)}$$

is a D-finite series.
[mbm-Mishna 10]

The group of the walk, and D-finite cases



Formal power series algebra

[mbm & Mishna 09]

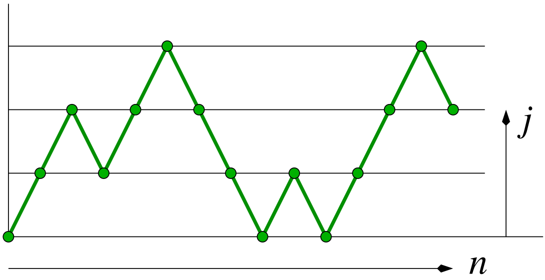
III.2. Computer algebra: a guess & check approach

In 1D: Walks on a half-line with steps $+1, -1$

- Generating function:

$$H(y;t) \equiv H(y) = \sum_{n\geq 0} \sum_{j\geq 0} h(j;n) y^j t^n$$

$h(j;n)$: number of n-step walks on the half-line ending at ordinate j



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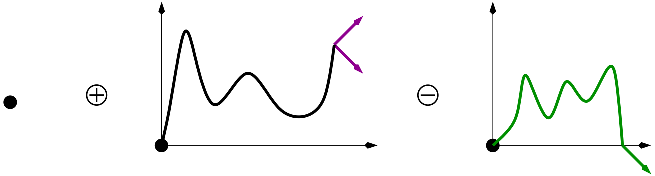
- Generating function:

$$H(y;t) \equiv H(y) = \sum_{n \geq 0} \sum_{j \geq 0} h(j;n) y^j t^n$$

- Step-by-step construction:

$$H(y) = 1 + t(y + \bar{y})H(y) - t\bar{y}H(0)$$

with $\bar{y} = 1/y$.



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$$H(y;t) \equiv H(y) = \sum_{n\geq 0}\sum_{j\geq 0} h(j;n) y^j t^n$$

- Step-by-step construction:

$$H(y) = 1 + t(y+\bar{y})H(y) - t\bar{y}H(0)$$

or

$$\big(1 - t(y+\bar{y})\big)H(y) = 1 - t\bar{y}H(0).$$

Guess & check for walks on a half-line

- The equation:

$$\big(1 - t(y + \bar{y})\big) H(y; t) = 1 - t\bar{y} H(0; t). \tag{1}$$

Guess & check for walks on a half-line

- The equation:

$$\big(1 - t(y + \bar{y})\big) H(y;t) = 1 - t\bar{y} H(0;t). \tag{1}$$

- A uniqueness result: there exists a unique series $H(y;t) \equiv H(y)$ in t, with coefficients in $\mathbb{Q}[y]$, satisfying the above equation.

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- The equation:

$$\big(1 - t(y + \bar{y})\big) H(y;t) = 1 - t\bar{y} H(0;t). \qquad (1)$$

- A uniqueness result: there exists a unique series $H(y;t) \equiv H(y)$ in t, with coefficients in $\mathbb{Q}[y]$, satisfying the above equation.

- Generate the first coefficients:

$$H(0) \equiv H(0;t) = 1 + t^2 + 2\,t^4 + 5\,t^6 + 14\,t^8 + 42\,t^{10} + O(t^{12}).$$

Guess & check for walks on a half-line

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- Guess that [Gfun]

$$H(0) = 1 + t^2 H(0)^2 \quad \text{and} \quad H(y) = 1 + t(ty^2 + t - y)H(y)^2 + 2tyH(y).$$

Guess & check for walks on a half-line

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Guess & check for walks on a half-line

- The equation:

$$\big(1 - t(y + \bar{y})\big) H(y; t) = 1 - t\bar{y} H(0; t). \tag{1}$$

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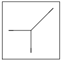
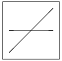
- This equation has a unique solution $\tilde{H}(y)$ that is a power series in t, and its coefficients are polynomials in y.

- Check (using resultants, and the first coeffs.) that $\tilde{H}(y)$ satisfies (1).

Some guess & check quadrant results

- Algebraicity results for $Q(x, y; t)$
  - Kreweras' walks [Bostan, Kauers 10]
  - Gessel's walks [Bostan, Kauers 10]





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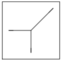
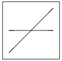
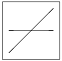


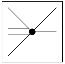
  - Gessel's walks [Bostan, Kauers 10]



- D-finiteness results for $Q(x, y; t)$
  - Gessel's walks (for $Q(0, 0; t)$ only) [Kauers, Zeilberger 09]



  - For weighted D-finite models that resist the algebraic kernel method



    [Bostan, mbm, Kauers, Melczer 16]

Some guess & check quadrant results

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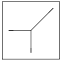
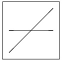
  - Kreweras' walks [Bostan, Kauers 10]





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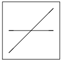

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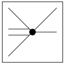
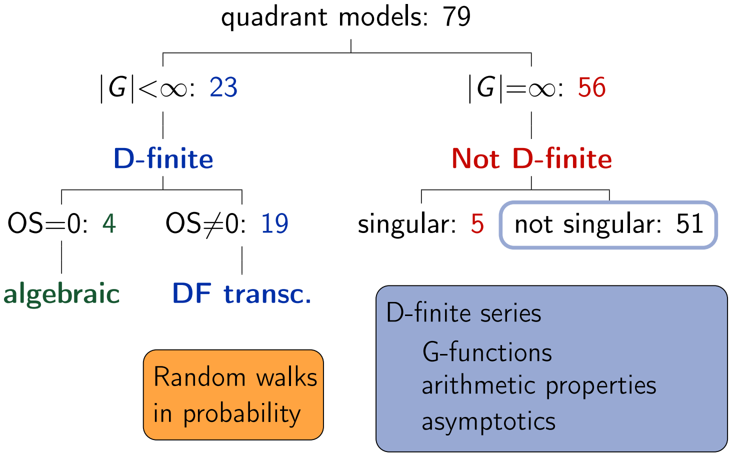


    [Bostan, mbm, Kauers, Melczer 16]

For Gessel's walks, the polynomial annihilating $Q(x, y; t)$ has size about 30Gb.

Even with big computers, one needs to be clever!

III.3. Non-D-finiteness via asymptotics



Non-D-finiteness via asymptotics

- The excursion exponent: let $\mathcal{S} \subset \mathbb{Z}^2$ be a (finite) step set, and let $S(x,y) = \sum_{(k,\ell)\in\mathcal{S}} x^k y^\ell$. For walks in the quadrant $\mathbb{N}^2$ ending at $(0,0)$:

$$q(0,0;n) \sim \kappa \mu^n n^{-1-\pi/\theta}$$

where

$$\theta = \arccos(-c), \qquad c = \frac{S''_{1,2}(x_c,y_c)}{\sqrt{S''_{1,1}(x_c,y_c)S''_{2,2}(x_c,y_c)}}$$

with $S'_1(x_c,y_c) = S'_2(x_c,y_c) = 0$.

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with $S'_1(x_c,y_c) = S'_2(x_c,y_c) = 0$.

Asymptotics of D-finite generating functions (G-functions)
[André 89], [Chudnovsky[2] 85], [Katz 70]

If the exponent $-1-\pi/\theta$ is irrational, then $Q(0,0;t)$ cannot be D-finite.

Non-D-finiteness via asymptotics

• The excursion exponent: let $\mathcal{S} \subset \mathbb{Z}^2$ be a (finite) step set, and let $S(x,y) = \sum_{(k,\ell)\in\mathcal{S}} x^k y^\ell$. For walks in the quadrant $\mathbb{N}^2$ ending at $(0,0)$:

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Asymptotics of D-finite generating functions (G-functions) [André 89], [Chudnovsky^2 85], [Katz 70]

If c is not the root of a cyclotomic polynomial, then $Q(0,0;t)$ cannot be D-finite.

Non-D-finiteness via asymptotics

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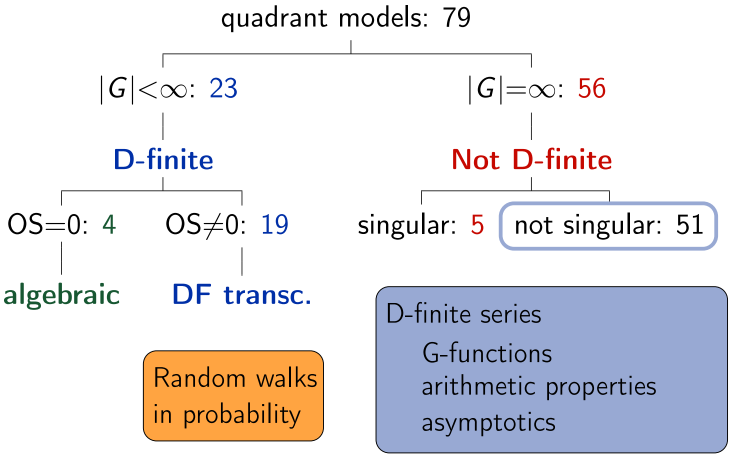
Asymptotics of D-finite generating functions (G-functions) [André 89], [Chudnovsky^2 85], [Katz 70]

If c is not the root of a cyclotomic polynomial, then $Q(0,0;t)$ cannot be D-finite.

Non-D-finiteness [Bostan, Raschel, Salvy 14]

The series $Q(0,0;t)$ is not D-finite for 51 of the 79 small step models.

Non-D-finiteness via asymptotics



Classification of quadrant walks: a variety of tools

quadrant models: 79

- $|G|<\infty$: 23
  - **D-finite**
    - OS$=$0: 4
      - **algebraic**
    - OS$\neq$0: 19
      - **DF transc.**
- $|G|=\infty$: 56
  - **Not D-finite**
    - decoupled: 9
      - **D-alg.**
    - not decoupled: 47
      - **not D-alg.**

Formal power series algebra

Complex analysis

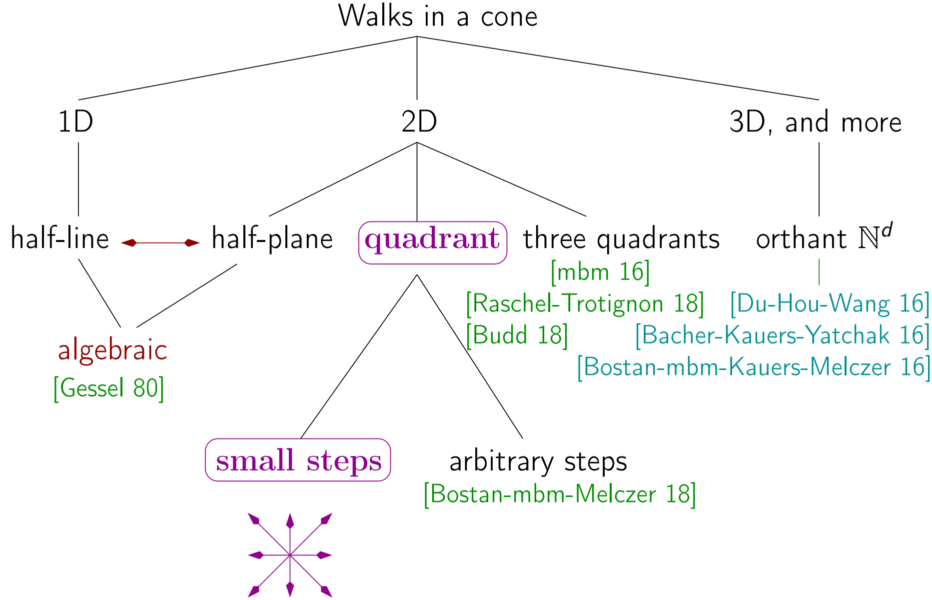
Differential Galois theory

Computer algebra

Random walks in probability

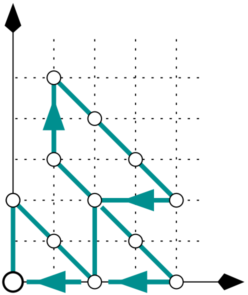
D-finite series
- effective closure properties
- arithmetic properties
- asymptotics
- G-functions

Beyond small steps in the quadrant



Beyond small steps in the quadrant

• Arbitrary steps in the quadrant ⇒ equivalence between D-finiteness and finite "group"?



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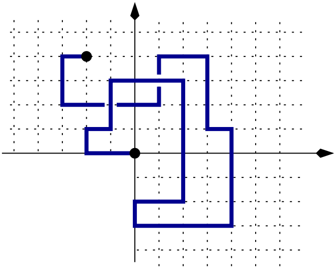
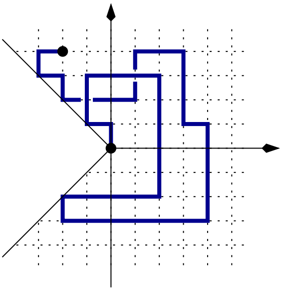
• Walks avoiding a quadrant: same dichotomy between D-finite and non-D-finite models?





Beyond small steps in the quadrant

• Arbitrary steps in the quadrant ⇒ equivalence between D-finiteness and finite "group"?

• Walks avoiding a quadrant: same dichotomy between D-finite and non-D-finite models?

• Walks with small steps in $\mathbb{N}^3$: some non-D-finite models with a finite group?

Example. The model $\{111, \bar{1}00, 0\bar{1}0, 00\bar{1}\}$ has a finite group of order 24. The orbit sum vanishes. Is it D-finite?