Exhaustive search of convex pentagons which tile the plane

Michaël Rao

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Which convex polygon can tile the plane?
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(allowing rotations/translations/mirrors)

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- Only one open case: Pentagons
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Pentagons: history

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What is a “type”?

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Examples:
- Type 1: $\alpha_1 + \alpha_2 = \pi$
- Type 2: $\alpha_1 + \alpha_3 = \pi$ and $\ell_1 = \ell_3$
- Type 4: $\alpha_3 = \alpha_5 = \pi/2$, $\ell_2 = \ell_3$ and $\ell_4 = \ell_5$...
What is a “type”?

A “type” is a set of pentagons (a type is not a tiling, neither a set of tiling...)

A type is all the pentagons that respect

- a set $C_a$ of linear conditions on angles (form: $v \cdot \alpha = 2\pi$ with $v \in \mathbb{N}^5$)
- a set $C_l$ of linear conditions on sides (form: $v \cdot \ell = 0$ with $v \in \mathbb{Z}^5$)
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- ...
We present an exhaustive search of all convex pentagons which tile the plane.
We present an exhaustive search of all convex pentagons which tile the plane.

Let $\mathcal{P}$ be a convex pentagon which tiles the plane.

- **Part 1**: There exist a tiling by $\mathcal{P}$ such that each vertex category has positive density.
- The set of vertex category (i.e. conditions implied by angles) must be “good”
- **Part 2**: There are only 371 good sets to consider
- **Part 3**: For each good set: we do an exhaustive search
- **Result**: we found only the 15 known families (and some special cases).
Let $\mathcal{P}$ be a convex pentagon

- the vertices are $s_1, \ldots, s_5$, in clockwise order
- the angles are respectively $\alpha_1 \times \pi, \ldots, \alpha_5 \times \pi$

\[
\forall 1 \leq i \leq 5, \quad 0 < \alpha_i < 1
\]

\[
\sum_{i=1}^{5} \alpha_i = (1, 1, 1, 1, 1) \cdot \alpha = 3
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\[ \sum_{i=1}^{5} \alpha_i = (1, 1, 1, 1, 1) \cdot \alpha = 3 \]

Let $\mathcal{T}$ be tiling of the plane by $\mathcal{P}$ (we allow rotation/translation/mirror)

(Note: no hypothesis on periodicity / transitivity)
Let $s$ be a vertex of $\mathcal{T}$ (i.e. a vertex of one pentagon in $\mathcal{T}$)

The \textit{vector category} of $s$, denoted $V(s)$, is the vector $v \in \mathbb{N}^5$ s.t. there are $v_i$ angles $s_i$ around $s$. 
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For every vertex $s$, $V(s) \cdot \alpha = 2$
Let $s$ be a vertex of $T$ (i.e. a vertex of one pentagon in $T$)

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For every vertex $s$, $V(s) \cdot \alpha = 2$

$\mathcal{W}$ : set of vectors categories of vertices in $T$. 

Attention ! Two cases of vertices:

"Half" : $s$ is in the border of a tile $P$, but not a vertex of $P$

"Full" : $s$ is a vertex of every tile around $s$

We have to "correct" the vector category of "half" vertices.

Here, for the sake of simplicity, we do not talk about half vertices...
Vector category

Let $s$ be a vertex of $\mathcal{T}$ (i.e. a vertex of one pentagon in $\mathcal{T}$)

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A toy problem

Suppose that the density of each vector category is definite.

\[
density(\nu) = \frac{\text{number of vertices } s \text{ with } V(s) = \nu}{\text{number of tiles}}
\]

What are the densities of \( \nu_a \)',s, \( \nu_b \)',s, \( \nu_c \)',s?

\[
\Rightarrow \quad density(\nu_a) + density(\nu_b) + density(\nu_c) = (1, 1, 1, 1, 1)
\]

\[
\Rightarrow \quad density(\nu_a) = 1, \quad density(\nu_b) = \frac{1}{2}, \quad density(\nu_c) = 0
\]

Can we tile only with \( \nu_a \) and \( \nu_b \)?
A toy problem

Suppose that the density of each vector category is definite.

\[
\text{density}(v) = \frac{\text{number of vertices } s \text{ with } V(s) = v}{\text{number of tiles}}
\]

and \( \mathcal{W} \) is the following:

\[ v_a = (1, 1, 1, 0, 0) \]
\[ v_b = (0, 0, 0, 2, 2) \]
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What are the densities of $\nu$’s?
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What are the densities of \( \nu \)'s?

\[ \Rightarrow d_a \nu_a + d_b \nu_b + d_c \nu_c = (1, 1, 1, 1, 1) \]
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\end{align*}
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Can we tile only with \(\nu_a\) and \(\nu_b\) ?
**Definition (Positive density tiling)**

\( T \) has *positive density* if for every \( v \in \mathcal{W} \), the density of \( V(s) \) is positive.
Definition (Positive density tiling)

$\mathcal{T}$ has *positive density* if for every $v \in \mathcal{W}$, the density of $V(s)$ is positive.

Problem: the density is not always defined for an arbitrary tiling...
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Lemma

*If a tiling by $\mathcal{P}$ exists, then a tiling of positive density by $\mathcal{P}$ exists.*
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- Otherwise, suppose \( v \in \mathcal{W} \) with density 0
Positive density tilings

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- There are sub-tilings of an arbitrarily large disk without a vertex \(v\)
 positive density tiling

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- There are sub-tilings of an arbitrarily large disk without a vertex \( v \)
  (take a grid of girth \( x \): if there is a \( v \) in every cell \( \rightarrow \) contradiction)
- By compactness one can construct a tiling without \( v \)
Back to the toy problem

\[ \nu_a = (1, 1, 1, 0, 0) \text{ of density } d_a \quad \nu_b = (0, 0, 0, 2, 2) \text{ of density } d_b \]

\[ \nu_c = (1, 1, 0, 1, 0) \text{ of density } d_c \quad \mathcal{W} = \{ \nu_a, \nu_b, \nu_c \} \]
Back to the toy problem

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Density of angle 1 = \( d_a + d_c \)

Density of angle 3 = \( d_a \)

Density of angle 5 = 2\( d_b \)
Back to the toy problem

\( \nu_a = (1, 1, 1, 0, 0) \) of density \( d_a \) \hspace{1cm} \( \nu_b = (0, 0, 0, 2, 2) \) of density \( d_b \)
\( \nu_c = (1, 1, 0, 1, 0) \) of density \( d_c \) \hspace{1cm} \( \mathcal{W} = \{ \nu_a, \nu_b, \nu_c \} \)

Density of angle 1 = \( d_a + d_c \) \hspace{1cm} Density of angle 3 = \( d_a \)
Density of angle 5 = 2\( d_b \)
Density of angle 1 - Density of angle 3 = \( d_c \)
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Density of angle 1 - Density of angle 3 = \( d_c \)

Densities of angles must be the same (1 for all)

Density of angle 1 - Density of angle 3 = 0, thus \( d_c = 0 \)
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\( v_b = (0, 0, 0, 2, 2) \) of density \( d_b \)  
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Let \( u \in \mathbb{R}^5 \) such that \( \sum_i u_i = 0 \)  
\( \Rightarrow \sum_i (\text{density of angle } i) \times u_i = 0 \)
Back to the toy problem

\( \nu_a = (1, 1, 1, 0, 0) \) of density \( d_a \)  \( \nu_b = (0, 0, 0, 2, 2) \) of density \( d_b \)  
\( \nu_c = (1, 1, 0, 1, 0) \) of density \( d_c \)  \( \mathcal{W} = \{ \nu_a, \nu_b, \nu_c \} \)

Density of angle 1 = \( d_a + d_c \)  Density of angle 3 = \( d_a \)  
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Densities of angles must be the same (1 for all)  
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Let \( u \in \mathbb{R}^5 \) such that \( \sum_i u_i = 0 \)  
\[ \Rightarrow \sum_i (\text{density of angle } i) \times u_i = 0 \]

Moreover \( \sum_{\nu \in \mathcal{W}} d_\nu \times \nu = (1, 1, 1, 1, 1) \)  
\[ \Rightarrow \sum_{\nu \in \mathcal{W}} d_\nu \times (\nu \cdot u) = 0 \]
Back to the toy problem

\( v_a = (1, 1, 1, 0, 0) \) of density \( d_a \)
\( v_b = (0, 0, 0, 2, 2) \) of density \( d_b \)
\( v_c = (1, 1, 0, 1, 0) \) of density \( d_c \)

\[ \mathcal{W} = \{v_a, v_b, v_c\} \]

Density of angle 1 = \( d_a + d_c \)
Density of angle 3 = \( d_a \)
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Let \( u \in \mathbb{R}^5 \) such that \( \sum_i u_i = 0 \)
\[ \Rightarrow \sum_i (\text{density of angle } i) \times u_i = 0 \]

Moreover \[ \sum_{v \in \mathcal{W}} d_v \times v = (1, 1, 1, 1, 1) \]
\[ \Rightarrow \sum_{v \in \mathcal{W}} d_v \times (v \cdot u) = 0 \]

\[ v_a \cdot u = 0 \quad v_b \cdot u = 0 \quad v_c \cdot u = 1 \quad \text{with } u = (1, 0, -1, 0, 0) \]
Back to the toy problem

\( \nu_a = (1, 1, 1, 0, 0) \) of density \( d_a \) \hspace{1cm} \( \nu_b = (0, 0, 0, 2, 2) \) of density \( d_b \)

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Let \( u \in \mathbb{R}^5 \) such that \( \sum_i u_i = 0 \)

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Moreover \( \sum_{\nu \in \mathcal{W}} d_{\nu} \times \nu = (1, 1, 1, 1, 1) \)

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\( \nu_a \cdot u = 0 \hspace{1cm} \nu_b \cdot u = 0 \hspace{1cm} \nu_c \cdot u = 1 \) with \( u = (1, 0, -1, 0, 0) \)

\[ \Rightarrow \text{density of } \nu_c \text{ must be } 0 \]
**Good set**

<table>
<thead>
<tr>
<th>Definition (Good set)</th>
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\( \mathcal{X} \subseteq \mathbb{N}^5 \) is *good* if \( \forall u \in \mathbb{R}^5 \) with \( \sum u = 0 \), either:

- \( u \cdot v = 0 \) for every \( v \in \mathcal{X} \), or
- there are \( v, v' \in \mathcal{X} \) such that \( u \cdot v < 0 < u \cdot v' \).

In the toy example:

- \( v_1 = (1, 1, 1, 0, 0) \)
- \( v_2 = (0, 0, 0, 2, 2) \)
- \( v_3 = (1, 1, 0, 1, 0) \)

\( \{v_1, v_2, v_3\} \) is *not good*, with \( u = (1, 0, -1, 0, 0) \)

\( \{v_1, v_2\} \) is *good* since \( 2 \times u \cdot v_1 + u \cdot v_2 = 0 \).
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  - \( \{v_1, v_2, v_3\} \) is
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\( v_3 = (1, 1, 0, 1, 0) \)

- \( \{v_1, v_2, v_3\} \) is not good, with \( u = (1, 0, -1, 0, 0) \)
- \( \{v_1, v_2\} \) is
Definition (Good set)

$\mathcal{X} \subseteq \mathbb{N}^5$ is good if $\forall u \in \mathbb{R}^5$ with $\sum u = 0$, either:

- $u \cdot v = 0$ for every $v \in \mathcal{X}$, or
- there are $v, v' \in \mathcal{X}$ such that $u \cdot v < 0 < u \cdot v'$.

In the toy example:

$v_1 = (1, 1, 1, 0, 0)$
$v_2 = (0, 0, 0, 2, 2)$
$v_3 = (1, 1, 0, 1, 0)$

- $\{v_1, v_2, v_3\}$ is not good, with $u = (1, 0, -1, 0, 0)$
- $\{v_1, v_2\}$ is good since $2 \times u \cdot v_1 + u \cdot v_2 = 0$
Lemma

If $\mathcal{T}$ has positive density, then $\mathcal{W}$ is good.
Lemma

If $\mathcal{T}$ has positive density, then $\mathcal{W}$ is good.

Otherwise, suppose $u \in \mathbb{R}^5$ such that $\sum_{i=1}^{5} u_i = 0$ s.t.:

- $\forall v \in \mathcal{W}$ with $u \cdot v \geq 0$.
- there is a $v^+ \in \mathcal{W}$ with $u \cdot v^+ > 0$. 

Part 1/3: Positive density tiling and good sets
Positive density imply $\mathcal{W}$ is good

Lemma

If $T$ has positive density, then $\mathcal{W}$ is good.

Otherwise, suppose $u \in \mathbb{R}^5$ such that $\sum_{i=1}^{5} u_i = 0$ s.t.:

- $\forall \nu \in \mathcal{W}$ with $u \cdot \nu \geq 0$.
- There is a $\nu^+ \in \mathcal{W}$ with $u \cdot \nu^+ > 0$.

We count the densities of angles in the tiling:

$$\sum_{\nu \in \mathcal{W}} \nu \times d_{\nu} = (1, 1, 1, 1, 1)$$
Positive density imply $\mathcal{W}$ is good

**Lemma**

*If $\mathcal{T}$ has positive density, then $\mathcal{W}$ is good.*

Otherwise, suppose $u \in \mathbb{R}^5$ such that $\sum_{i=1}^{5} u_i = 0$ s.t.:

- $\forall v \in \mathcal{W}$ with $u \cdot v \geq 0$.
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We count the densities of angles in the tiling:

$$\sum_{v \in \mathcal{W}} v \times d_v = (1, 1, 1, 1, 1)$$

$$\sum_{v \in \mathcal{W}} (u \cdot v) \times d_v = 0$$
Lemma

*If $\mathcal{T}$ has positive density, then $\mathcal{W}$ is good.*

Otherwise, suppose $u \in \mathbb{R}^5$ such that $\sum_{i=1}^{5} u_i = 0$ s.t.:
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We count the densities of angles in the tiling:

$$\sum_{v \in \mathcal{W}} v \times d_v = (1, 1, 1, 1, 1)$$

$$\sum_{v \in \mathcal{W}} (u \cdot v) \times d_v = 0$$

Contradiction since:

$$\sum_{v \in \mathcal{W}} (u \cdot v) \times d_v \geq (u \cdot v^+) \times d_{v^+} > 0$$
Let $\mathcal{X} \subseteq \mathbb{N}^5$

**Definition ($\mathcal{P}_X$)**

$\mathcal{P}_X$ is the convex polytope of $\alpha = (\alpha_1, \ldots, \alpha_5) \in \mathbb{R}^5$ s.t.

- $\forall i \in \{1, \ldots, 5\}, \ 0 \leq \alpha_i \leq 1$,
- $\sum_{i=1}^{5} \alpha_i = 3$,
- $\forall \nu \in \mathcal{X}, \ \alpha \cdot \nu = 2$. 

What are the good sets $\mathcal{X}$ such that $\mathcal{P}_X \cap [0,1]^5 \neq \emptyset$?

Spoil: only finitely many...
Let $\mathcal{X} \subseteq \mathbb{N}^5$

**Definition (\(\mathcal{P}_\mathcal{X}\))**

\(\mathcal{P}_\mathcal{X}\) is the convex polytope of \(\alpha = (\alpha_1, \ldots, \alpha_5) \in \mathbb{R}^5\) s.t.

- \(\forall i \in \{1, \ldots, 5\}, \ 0 \leq \alpha_i \leq 1\),
- \(\sum_{i=1}^{5} \alpha_i = 3\),
- \(\forall \mathbf{v} \in \mathcal{X}, \ \alpha \cdot \mathbf{v} = 2\).

In a tiling by a convex pentagon: \(\alpha \in \mathcal{P}_\mathcal{W}\), thus \(\mathcal{P}_\mathcal{W} \cap ]0, 1[^5 \neq \emptyset\).
Let $X \subseteq \mathbb{N}^5$

**Definition** ($\mathcal{P}_X$)

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What are the good sets $X$ such that $\mathcal{P}_X \cap ]0, 1[^5 \neq \emptyset$?
Let $\mathcal{X} \subseteq \mathbb{N}^5$

**Definition ($\mathcal{V}_\mathcal{X}$)**

$\mathcal{V}_\mathcal{X}$ is the convex polytope of $\alpha = (\alpha_1, \ldots, \alpha_5) \in \mathbb{R}^5$ s.t.
- $\forall i \in \{1, \ldots, 5\}$, $0 \leq \alpha_i \leq 1$,
- $\sum_{i=1}^{5} \alpha_i = 3$,
- $\forall \nu \in \mathcal{X}$, $\alpha \cdot \nu = 2$.

In a tiling by a convex pentagon: $\alpha \in \mathcal{V}_\mathcal{W}$, thus $\mathcal{V}_\mathcal{W} \cap ]0, 1[^5 \neq \emptyset$

What are the good sets $\mathcal{X}$ such that $\mathcal{V}_\mathcal{X} \cap ]0, 1[^5 \neq \emptyset$ ?

Spoil: only finitely many...
What are the good sets $\mathcal{X}$ such that $\mathcal{P}_\mathcal{X} \cap [0, 1]^5 \neq \emptyset$?
Part 2: Computation of all good sets

What are the good sets $\mathcal{X}$ such that $\mathcal{P}_\mathcal{X} \cap ]0, 1[^5 \neq \emptyset$?

- We present an algorithm which generates all good sets.
What are the good sets \( \mathcal{X} \) such that \( \Psi_{\mathcal{X}} \cap [0, 1[^5 \neq \emptyset \) ?

- We present an algorithm which generates all good sets.
- One execution of this algorithms terminates, and returns 371 good sets.
What are the good sets $\mathcal{X}$ such that $\mathcal{P}_\mathcal{X} \cap [0, 1[^5 \neq \emptyset$?

- We present an algorithm which generates all good sets.
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- Moreover, one can show that this algorithm always terminates.
What are the good sets $\mathcal{X}$ such that $\mathcal{P}_{\mathcal{X}} \cap [0, 1[^5 \neq \emptyset$?

- We present an algorithm which generates all good sets.
- One execution of this algorithms terminates, and returns 371 good sets.
- Moreover, one can show that this algorithm always terminates.

We suppose w.l.o.g. that:

- $1 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4 \geq \alpha_5 \geq 0$ (\(\mathcal{P}_{\mathcal{X}} \supseteq \mathcal{P}_{\mathcal{X}}\) instead of \(\mathcal{P}_{\mathcal{X}}\))
- $\mathcal{X}$ is maximal, i.e. every condition implied by conditions in $\mathcal{X}$ is in $\mathcal{X}$
1: procedure Recurse($\mathcal{X}$)
2: \hspace{1em} $\mathcal{X} \leftarrow \text{Compat}(\mathcal{X})$ (i.e. complete $\mathcal{X}$ to make it maximal)
3: \hspace{1em} if $\mathcal{P} \cap \mathbb{N}^5 = \emptyset$ then return end if
4: \hspace{1em} if $\mathcal{X}$ is good then
5: \hspace{2em} Add $\mathcal{X}$ to the list of good sets
6: \hspace{1em} end if
7: \hspace{1em} Let $u \in \mathbb{R}^5$ such that:
8: \hspace{2em} $u \cdot (1, 1, 1, 1, 1) = 0$
9: \hspace{2em} $\forall v \in \mathcal{X}, \; u \cdot v = 0$

10: $V \leftarrow \{v \in \mathbb{N}^5 : v \cdot u \geq 0 \}$
11: \hspace{1em} for every $w \in V \setminus \mathcal{X}$ do
12: \hspace{2em} Recurse($\mathcal{X} \cup \{w\}$)
13: \hspace{1em} end for
14: end procedure

Recurse($\mathcal{X}$) computes all max good sets $Y \supseteq \mathcal{X}$ with $\mathcal{P} \cap \mathbb{N}^5 \neq \emptyset$.

Line 8: $V$ is finite.

Line 7: such a $u$ always exists.

Recurse always terminates: finitely many good sets with $\mathcal{P} \cap \mathbb{N}^5 \neq \emptyset$. Part 2/3: Computing all good sets
1: **procedure** `Recurse(\mathcal{X})`
2: \( \mathcal{X} \leftarrow \text{Compat}(\mathcal{X}) \)  (i.e. complete \( \mathcal{X} \) to make it maximal)
3: \textbf{if} \( \mathcal{P}_{\mathcal{X}} \cap 0, 1^5 = \emptyset \) \textbf{then} \textbf{return} \textbf{end if}
4: \textbf{if} \( \mathcal{X} \) is good \textbf{then}
5: \hspace{1em} Add \( \mathcal{X} \) to the list of good sets
6: \textbf{end if}
7: \textbf{Let} \( u \in \mathbb{R}^5 \) such that:
   - \( u \cdot (1, 1, 1, 1, 1) = 0 \)
   - \( \forall v \in \mathcal{X}, u \cdot v = 0 \)
8: \( V \leftarrow \{v \in \mathbb{N}^5 : v \cdot u \geq 0 \} \)
9: \textbf{for} every \( w \in V \setminus \mathcal{X} \) \textbf{do}
10: \hspace{1em} \textbf{Recurse}(\mathcal{X} \cup \{w\})
11: \textbf{end for}
12: \textbf{end procedure}

\textbf{Recurse}(\mathcal{X}) \text{ computes all max good sets } \mathcal{Y} \supseteq \mathcal{X} \text{ with } \mathcal{P}_{\mathcal{Y}} \cap 0, 1^5 \neq \emptyset.
Get a finite set $V$

Let $m \chi \in [0, 1]^5$ be such that $(m \chi)_i = \min\{\alpha_i : \alpha \in \mathcal{P}_{\chi}^>\}$

(for short $m = m \chi$)
Get a finite set $V$

Let $m_X \in [0, 1]^5$ be such that $(m_X)_i = \min\{\alpha_i : \alpha \in \mathcal{P}_X^>\}$  
(for short $m = m_X$)

Since $1 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4 \geq \alpha_5 \geq 0$ then:

$$m_1 \geq \frac{3}{5}, \ m_2 \geq \frac{1}{2}, \ m_3 \geq \frac{1}{3}, \ \text{and} \ m_i \geq m_{i+1}$$
Get a finite set $V$

Let $m \in [0, 1]^5$ be such that $(m)_i = \min\{\alpha_i : \alpha \in \mathcal{P}_X^>\}$
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Since $1 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4 \geq \alpha_5 \geq 0$ then:
$m_1 \geq \frac{3}{5}, m_2 \geq \frac{1}{2}, m_3 \geq \frac{1}{3},$ and $m_i \geq m_{i+1}$

For every $v \in \mathbb{N}^5$, if $v_i \times m_i > 2$ then $\mathcal{P}_X^> \cup \{v\} \cap [0, 1]^5 = \emptyset$.

$m_i > 0 \Rightarrow v_i$ must be bounded
Get a finite set $V$

Let $m_{\mathcal{X}} \in [0,1]^5$ be such that $(m_{\mathcal{X}})_i = \min\{\alpha_i : \alpha \in \mathcal{P}_{\mathcal{X}}^\geq\}$
(for short $m = m_{\mathcal{X}}$)

Since $1 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4 \geq \alpha_5 \geq 0$ then:

$m_1 \geq \frac{3}{5}, m_2 \geq \frac{1}{2}, m_3 \geq \frac{1}{3}$, and $m_i \geq m_{i+1}$

For every $v \in \mathbb{N}^5$, if $v_i \times m_i > 2$ then $\mathcal{P}_{\mathcal{X} \cup \{v\}}^\geq \cap [0,1]^5 = \emptyset$.

$m_i > 0 \Rightarrow v_i$ must be bounded

Let $u \in \mathbb{R}^5$ s.t. $\sum u = 0$ and $\forall i$ with $m_i = 0$ then $u_i > 0$  
→ the “negative part” in $v \cdot m$ is bounded  
→ each $v_i$ must be bounded, otherwise $v \cdot \alpha \geq v \cdot m > 2$.  

Part 2/3: Computing all good sets
1: **procedure** `Recurse(X)`
2: \[ X \leftarrow \text{Compat}(X) \quad \text{(i.e. complete } X \text{ to make it maximal)} \]
3: \[ \text{if } \forall X \ni 0, 1^5 = \emptyset \text{ then return end if} \]
4: \[ \text{if } X \text{ is good then} \]
5: \[ \text{Add } X \text{ to the list of good sets} \]
6: \[ \text{end if} \]
7: \[ \text{Let } u \in \mathbb{R}^5 \text{ such that:} \]
\[ \item u \cdot (1, 1, 1, 1, 1) = 0 \]
\[ \item \forall v \in X, u \cdot v = 0 \]
8: \[ V \leftarrow \{ v \in \mathbb{N}^5 : v \cdot u \geq 0 \} \]
9: \[ \text{for every } w \in V \setminus X \text{ do} \]
10: \[ \text{Recurse}(X \cup \{w\}) \]
11: \[ \text{end for} \]
12: \[ \text{end procedure} \]

`Recurse(X)` computes all max good sets \( Y \supseteq X \) with \( \forall Y \ni 0, 1^5 \neq \emptyset \)
1: procedure \texttt{Recurse}(\mathcal{X})
2: \hspace{1em} \mathcal{X} \leftarrow \text{Compat}(\mathcal{X}) \quad (\text{i.e. complete } \mathcal{X} \text{ to make it maximal})
3: \hspace{1em} \text{if } \mathcal{P}^\geq_{\mathcal{X}} \cap ]0, 1[^5 = \emptyset \text{ then return end if}
4: \hspace{1em} \text{if } \mathcal{X} \text{ is good then}
5: \hspace{2em} \text{Add } \mathcal{X} \text{ to the list of good sets}
6: \hspace{1em} \text{end if}
7: \hspace{1em} \text{Let } u \in \mathbb{R}^5 \text{ such that:}
8: \hspace{1em} \bullet \quad u \cdot (1, 1, 1, 1, 1) = 0
9: \hspace{1em} \bullet \quad \forall v \in \mathcal{X}, \ u \cdot v = 0
10: \hspace{1em} \bullet \quad \forall i \in \{4, 5\}, \ (m_{\mathcal{X}})_i = 0 \Rightarrow u_i > 0
11: \hspace{1em} V \leftarrow \{v \in \mathbb{N}^5 : v \cdot u \geq 0 \text{ and } v \cdot m_{\mathcal{X}} \leq 2\}
12: \hspace{1em} \text{for every } w \in V \setminus \mathcal{X} \text{ do}
13: \hspace{2em} \text{Recurse}(\mathcal{X} \cup \{w\})
14: \hspace{1em} \text{end for}
15: \hspace{1em} \text{end procedure}

$\textbf{Recurse}(\mathcal{X})$ computes all max good sets $\mathcal{Y} \supseteq \mathcal{X}$ with $\mathcal{P}^\geq_{\mathcal{Y}} \cap ]0, 1[^5 \neq \emptyset$
1: **procedure** Recurse(\(\mathcal{X}\))

2: \(\mathcal{X} \leftarrow \text{Compat}(\mathcal{X})\) (i.e. complete \(\mathcal{X}\) to make it maximal)

3: **if** \(\mathcal{Y} \supseteq \mathcal{X} \cap ]0, 1[^5 = \emptyset\) **then** return **end if**

4: **if** \(\mathcal{X}\) is good **then**

5: Add \(\mathcal{X}\) to the list of good sets

6: **end if**

7: Let \(u \in \mathbb{R}^5\) such that:
  - \(u \cdot (1, 1, 1, 1, 1) = 0\)
  - \(\forall v \in \mathcal{X}, u \cdot v = 0\)
  - \(\forall i \in \{4, 5\}, (m_X)_i = 0 \Rightarrow u_i > 0\)

8: \(V \leftarrow \{v \in \mathbb{N}^5 : v \cdot u \geq 0\ \text{and} v \cdot m_X \leq 2\}\)

9: for every \(w \in V \setminus \mathcal{X}\) do

10: Recurse(\(\mathcal{X} \cup \{w\}\))

11: **end for**

12: **end procedure**

**Recurse(\(\mathcal{X}\))** computes all max good sets \(\mathcal{Y} \supseteq \mathcal{X}\) with \(\mathcal{Y} \supseteq \mathcal{X} \cap ]0, 1[^5 \neq \emptyset\)

*Line 8: \(V\) is finite*
Choice for $u$

Note: if $\alpha$ and $\alpha' \in \mathcal{P}_{\geq X}$, then with $u = \alpha - \alpha'$:

- $\sum u = 0$
- $u \cdot v = 0$
Choice for $u$

Note: if $\alpha$ and $\alpha' \in \mathcal{P}^\geq X$, then with $u = \alpha - \alpha'$:

- $\sum u = 0$
- $u \cdot v = 0$

For the last condition on $m_X$:

If $m_4 > 0$ and $m_5 = 0$, there is $\alpha' \in \mathcal{P}^\geq X$ s.t. $\alpha'_5 = 0$

If $m_4 = m_5 = 0$, there is $\alpha' \in \mathcal{P}^\geq X$ s.t. $\alpha'_4 = \alpha'_5 = 0$

(and take $\alpha \in \mathcal{P}^\geq X \cap ]0, 1[^5$)
1: \textbf{procedure} \textsc{Recurse}(\mathcal{X})
2: \quad \mathcal{X} \leftarrow \text{Compat}(\mathcal{X}) \quad \text{(i.e. complete } \mathcal{X} \text{ to make it maximal)}
3: \quad \text{if } \mathcal{Y}^\geq \mathcal{X} \cap 0, 1^5 = \emptyset \text{ then return end if}
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6: \quad \text{end if}
7: \quad \text{Let } u \in \mathbb{R}^5 \text{ such that:}
8: \quad \quad u \cdot (1, 1, 1, 1, 1) = 0
9: \quad \quad \forall v \in \mathcal{X}, \; u \cdot v = 0
10: \quad \quad \forall i \in \{4, 5\}, \; (m_\mathcal{X})_i = 0 \Rightarrow u_i > 0
11: \quad V \leftarrow \{v \in \mathbb{N}^5 : v \cdot u \geq 0 \text{ and } v \cdot m_\mathcal{X} \leq 2\}
12: \quad \text{for every } w \in V \setminus \mathcal{X} \text{ do}
13: \quad \quad \text{\textsc{Recurse}(\mathcal{X} \cup \{w\})}
14: \quad \text{end for}
15: \quad \text{end procedure}

\textsc{Recurse}(\mathcal{X}) \text{ computes all max good sets } \mathcal{Y} \supseteq \mathcal{X} \text{ with } \mathcal{Y}^\geq \mathcal{X} \cap 0, 1^5 \neq \emptyset

Line 8: \( V \) is finite
1: **procedure** `Recurse`(\(\mathcal{X}\))

2: \(\mathcal{X} \leftarrow \text{Compat}(\mathcal{X})\) (i.e. complete \(\mathcal{X}\) to make it maximal)

3: **if** \(\mathcal{P}_{\mathcal{X}} \geq \) \([0, 1]^5 = \emptyset\) **then** return **end if**

4: **if** \(\mathcal{X}\) is good **then**

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   - \(u \cdot (1, 1, 1, 1, 1) = 0\)
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9: **for** every \(w \in V \setminus \mathcal{X}\) **do**

10: `Recurse`(\(\mathcal{X} \cup \{w\}\))

11: **end for**

12: **end procedure**

`Recurse`(\(\mathcal{X}\)) computes all max good sets \(\mathcal{Y} \supseteq \mathcal{X}\) with \(\mathcal{P}_{\mathcal{Y}} \geq \) \([0, 1]^5 \neq \emptyset\)

Line 8: \(V\) is finite  
Line 7: such a \(u\) always exists
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2: \(\mathcal{X} \leftarrow \text{Compat}(\mathcal{X})\)  (i.e. complete \(\mathcal{X}\) to make it maximal)
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\textbf{Recurse}(\(\mathcal{X}\)) computes all max good sets \(\mathcal{Y} \supseteq \mathcal{X}\) with \(\mathcal{Y}_{X} \cap [0,1[^5] \neq \emptyset\)

Line 8: \(V\) is finite  Line 7: such a \(u\) always exists
\textbf{Recurse} always terminates: finitely many good sets with \(\mathcal{Y}_{X} \cap [0,1[^5] \neq \emptyset\).
We execute \texttt{Recurse}(\emptyset) and it finds 193 non-empty sets.
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- 193 non-empty maximal good sets $\mathcal{X}$ with $\mathbb{P}^{\geq}_{\mathcal{X}} \cap [0, 1[^5 \neq \emptyset$
Good sets: results

We execute \textsc{Recurse}(\emptyset) and it finds 193 non-empty sets.

- 193 non-empty maximal good sets \( \mathcal{X} \) with \( \mathcal{Y}_{\mathcal{X}} \cap ]0, 1[^5 \neq \emptyset \)
- Take all permutations: 3495 non-empty maximal good sets \( \mathcal{X} \) with \( \mathcal{Y}_{\mathcal{X}} \cap ]0, 1[^5 \neq \emptyset \)
Good sets: results

We execute \texttt{RECURSE}(\emptyset) and it finds 193 non-empty sets.

- 193 non-empty maximal good sets $\mathcal{X}$ with $\mathcal{P}_{\mathcal{X}} \supseteq \mathcal{X} \cap [0, 1[^5 \neq \emptyset$
- Take all permutations: 3495 non-empty maximal good sets $\mathcal{X}$ with $\mathcal{P}_{\mathcal{X}} \cap [0, 1[^5 \neq \emptyset$
- Keep only one represents for each class up to rotation/mirror, one have the 371 sets.
Good sets: results

371 sets to consider:

- 2 s.t. $\mathcal{P}_X$ has dimension 3
- 26 s.t. $\mathcal{P}_X$ has dimension 2
- 92 s.t. $\mathcal{P}_X$ has dimension 1
- 251 s.t. $\mathcal{P}_X$ has dimension 0
Good sets: results

371 sets to consider:

- 2 s.t. $\mathcal{P}_X$ has dimension 3
- 26 s.t. $\mathcal{P}_X$ has dimension 2
- 92 s.t. $\mathcal{P}_X$ has dimension 1
- 251 s.t. $\mathcal{P}_X$ has dimension 0

- 90 of “Type 1” (that is $\alpha_i + \alpha_{i+1} = 1$)
For each family in the 371 families of maximal good sets, we do an exhaustive search of a tiling.
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We chose a (maximal) good set $\mathcal{X}$.
We do an exhaustive search of all tilings, allowing only “vector categories” in $\mathcal{X}$. 
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We chose a (maximal) good set $\mathcal{X}$. We do an exhaustive search of all tilings, allowing only “vector categories” in $\mathcal{X}$.

We backtrack if the conditions (angles and lengths) imply
For each family in the 371 families of maximal good sets, we do an exhaustive search of a tiling.

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We backtrack if the conditions (angles and lengths) imply

- we are in a known case: known family (Types 1 to 15 in Table 1), or a special case of a known family (Types 16 to 19)
For each family in the 371 families of maximal good sets, we do an exhaustive search of a tiling.

We chose a (maximal) good set $\mathcal{X}$.
We do an exhaustive search of all tilings, allowing only “vector categories” in $\mathcal{X}$.

We backtrack if the conditions (angles and lengths) imply

- we are in a known case: known family (Types 1 to 15 in Table 1), or a special case of a known family (Types 16 to 19)
- or no convex pentagon exists with these conditions
Backtracking: general idea

The object on which we work and backtrack is a pair \((G, Q)\):

- \(G\) is a embedded planar graph which represent the partial tiling ("Tiling graph")
- \(Q\) is a set of conditions we know on the lengths of the pentagon: \(i.e.\) a linear program (LP) on \(\ell_1 \ldots \ell_5\)

We add linear conditions on sides "on the fly"
Tiling graph

Tiling graph: embedded planar graph with labels on angles and edges

Two types of faces: usual and special

- usual: corresponds to a pentagon in the tiling. The degree is 5, and the angles are marked from 1 to 5 (in CW or CCW)

- special: corresponds to frontier between tiles, or an unknown area of the plane. Angles are marked with $\emptyset$, $\pi$ or ?

A special face is complete if there no “?”
Example of a tiling graph (Type 15). Unmarked angles are labeled “?”
Length suppositions

A *run* on a special face is a succession of consecutive $\emptyset$ and $\pi$ angles.
Length suppositions

A *run* on a special face is a succession of consecutive $\emptyset$ and $\pi$ angles. Each run corresponds to aligned points in the tiling.
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Length suppositions

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- If $Q$ does not permit to decide among the 3 possibilities: $s < s'$, $s = s'$ and $s > s'$, then we branch on the 3 possibilities: we add the corresponding condition in $Q$ and recurse
Branching on length suppositions: example

\[ Q : \]

\[ \ell_4 - \ell_5 = 0 \]

\((y, z)\) is a complete face. So we (already) have \(\ell_4 = \ell_5\) in the LP. 

\((w, t, w')\) is a run. The length \(wt\) is \(\ell_3\), and the length \(tw'\) is \(\ell_3\). So we merge \(w\) and \(w'\), and mark angle \((t, w, t)\) as \(\emptyset\).
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\((u, t, y, u')\) is also a run.
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Is \(u\) and \(u'\) the same vertex?
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We don’t know. We branch.
Branching on length suppositions: example

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\[(u, t, y, u')\] is also a run.

Is \(u\) and \(u'\) the same vertex? Is \(\ell_3 = \ell_4 + \ell_5\)?

We don’t know. We branch.

first case : add \(\ell_3 > \ell_4 + \ell_5\) to \(Q\) and branch

second case : add \(\ell_3 = \ell_4 + \ell_5\) to \(Q\) and branch
Branching on length suppositions: example

\[(\text{case 2}) \ Q : \ell_4 - \ell_5 = 0, \ell_3 - \ell_4 - \ell_5 = 0\]
Branching on length suppositions: example

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\((u, t, y, u')\) is a run, and we know that \(u\) and \(u'\) have the same position: we merge \(u\) and \(u'\).
Branching on length suppositions: example

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\[(case 2) \quad Q : \ell_4 - \ell_5 = 0, \ell_3 - \ell_4 - \ell_5 = 0\]

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\(u\) is now complete: the angle \(r, u, r'\) is labeled \(\emptyset\)

in the run \((r, u, r')\), \(r\) and \(r'\) have the same position: we merge...
Branching on a new tile

In other cases, we add a new tile (a new “usual face”)
Branching on a new tile

In other cases, we add a new tile (a new “usual face”)

We take a non-complete vertex $w$ in the graph, and we try (branch on) every possibility to add a new face adjacent to $w$
Existence of the pentagon

Given the LP $Q$, we denote by $Q$ the set of solutions $\ell$ of $Q$ with $\sum \ell = 1$

Let $s(\alpha)$ be the vector such that $s(\alpha)_i = (i - 1) - \sum_{j=1}^{i-1} \alpha_i$.

One have:

$$\sum_{i} \ell_i \exp(s(\alpha)_i \times \pi \times \sqrt{-1}) = 0.$$ (1)
Existence of the pentagon

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We backtrack if there is no convex pentagon exists with the properties, that is if the following condition is not fulfilled:

$$\exists \ell \in \mathcal{Q} \cap ]0, 1[^5, \exists \alpha \in \mathcal{P} \cap ]0, 1[^5, \sum \ell_i \exp(s(\alpha)_i \times \pi \times \sqrt{-1}) = 0 \quad (2)$$
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$$\exists \ell \in \Omega \cap [0, 1[^5, \exists \alpha \in \Phi \cap [0, 1[^5, \sum_i \ell_i \exp(s(\alpha)_i \times \pi \times \sqrt{-1}) = 0 \quad (2)$$

If $\dim(\Phi) = 0$ then $\alpha \in Q^5$, and easy to decide: we compute on $Q[\cos(\pi/q)]$ for a $q \in \mathbb{N}$. 

Part 3/3: Testing a family corresponding to a good set
Existence of the pentagon

Given the LP $Q$, we denote by $Q$ the set of solutions $\ell$ of $Q$ with $\sum \ell = 1$

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If $\dim(\mathcal{P}) = 0$ then $\alpha \in Q^5$, and easy to decide: we compute on $Q[\cos(\pi/q)]$ for a $q \in \mathbb{N}$.

If $\dim(\mathcal{P}) > 0$: we backtrack if we have a certificate (computations in $Q$) that there a no solution. Problem: this cannot detect “degenerate case”. So we manually add some degenerate case. (Types 20 to 24 in Table 1).
### Conditions for which we backtrack (Table 1)

<table>
<thead>
<tr>
<th>Type 1</th>
<th>$a + b + c = 2\pi$</th>
<th>Type 2</th>
<th>$a + b + d = 2\pi$</th>
<th>$C = E$</th>
</tr>
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<tbody>
<tr>
<td>Type 3</td>
<td>$3e = 2\pi$</td>
<td>$C + E = D$</td>
<td>$a + b + d = 2\pi$</td>
<td>$D = E$</td>
</tr>
<tr>
<td></td>
<td>$d + 2e = 2\pi$</td>
<td>$A = B$</td>
<td>$2e = \pi$</td>
<td>$B = C$</td>
</tr>
<tr>
<td></td>
<td>$b + 2e = 2\pi$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Type 5</td>
<td>$3e = 2\pi$</td>
<td>$D = E$</td>
<td>$A = B$</td>
<td></td>
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<tr>
<td></td>
<td>$a + b + d = 2\pi$</td>
<td>$B = C$</td>
<td></td>
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<tr>
<td></td>
<td>$2e = \pi$</td>
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</tr>
<tr>
<td>Type 6</td>
<td>$d + 2e = 2\pi$</td>
<td>$A = C = D = E$</td>
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<tr>
<td>Type 7</td>
<td>$a + 2c = 2\pi$</td>
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<td>$A = C = D = E$</td>
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</tr>
<tr>
<td>Type 8</td>
<td>$d + 2e = 2\pi$</td>
<td>$A = B = C = D$</td>
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<td></td>
<td>$b + c = 2\pi$</td>
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<tr>
<td>Type 9</td>
<td>$d + 2e = 2\pi$</td>
<td>$A = B = C = D$</td>
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<td></td>
<td>$2a + c = 2\pi$</td>
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<tr>
<td>Type 11</td>
<td>$c + 2d = 2\pi$</td>
<td>$A = B = C = 2E$</td>
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<td></td>
<td>$b + d + e = 2\pi$</td>
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<td></td>
<td>$a + 2b = 2\pi$</td>
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<tr>
<td>Type 12</td>
<td>$c + 2d = 2\pi$</td>
<td>$A + C = B = 2E$</td>
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<td></td>
<td>$b + d + e = 2\pi$</td>
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<tr>
<td></td>
<td>$a + 2b = 2\pi$</td>
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<tr>
<td>Type 13</td>
<td>$b + 2d = 2\pi$</td>
<td>$A = 2B = 2C$</td>
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<td></td>
<td>$a + b + d = 2\pi$</td>
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<td>$2e = \pi$</td>
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<tr>
<td>Type 14</td>
<td>$c + 2d = 2\pi$</td>
<td>$A = B = 2C = 2E$</td>
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<td></td>
<td>$b + d + e = 2\pi$</td>
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<tr>
<td></td>
<td>$a + 2b = 2\pi$</td>
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</tr>
<tr>
<td>Type 15</td>
<td>$c + 2d = 2\pi$</td>
<td>$B = D = E$</td>
<td>$2A = D = E$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$2b + e = 2\pi$</td>
<td>$C = 2B$</td>
<td>$A = C$</td>
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<tr>
<td></td>
<td>$2a + d = 2\pi$</td>
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<tr>
<td></td>
<td>$2e = \pi$</td>
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<tr>
<td>Type 16</td>
<td>$b + c + e = 2\pi$</td>
<td>$D = E$</td>
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<tr>
<td></td>
<td>$2b + d = 2\pi$</td>
<td>$A = B$</td>
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<tr>
<td></td>
<td>$a + 2c = 2\pi$</td>
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</tr>
<tr>
<td>Type 17</td>
<td>$c + 2e = 2\pi$</td>
<td>$A = B = C = D = E$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$2b + d = 2\pi$</td>
<td></td>
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<tr>
<td></td>
<td>$2e = \pi$</td>
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</tr>
<tr>
<td>Type 18</td>
<td>$d + 2e = 2\pi$</td>
<td>$A = B$</td>
<td></td>
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</tr>
<tr>
<td></td>
<td>$c + 2e = 2\pi$</td>
<td>$B = E$</td>
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<tr>
<td></td>
<td>$b + d + e = 2\pi$</td>
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<tr>
<td>Type 19</td>
<td>$c + 2e = 2\pi$</td>
<td>$A = B = C = D$</td>
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<tr>
<td></td>
<td>$b + 2d = 2\pi$</td>
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<td>$A = B = C = D$</td>
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<td>Type 20</td>
<td>$d + 2e = 2\pi$</td>
<td>$A = C + D$</td>
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<td></td>
<td>$2a + b = 2\pi$</td>
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<td></td>
<td>$2b + d = 2\pi$</td>
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<tr>
<td>Type 21</td>
<td>$d + 2e = 2\pi$</td>
<td>$A = B = C = E$</td>
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<td>$2a + b = 2\pi$</td>
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<td>$A = B = C = E$</td>
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<tr>
<td>Type 22</td>
<td>$2b + d = 2\pi$</td>
<td>$A = 2C = 2D$</td>
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<td></td>
<td>$a + b + d = 2\pi$</td>
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<td></td>
<td>$2e = \pi$</td>
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<tr>
<td>Type 23</td>
<td>$2b + d = 2\pi$</td>
<td>$2c + d = 2\pi$</td>
<td>$2D = A + C$</td>
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<td>$a + b + d = 2\pi$</td>
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<td>$2e = \pi$</td>
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<tr>
<td>Type 24</td>
<td>$2b + c = 2\pi$</td>
<td>$2c + d = 2\pi$</td>
<td>$2E = A + C$</td>
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<td>$a + b + d = 2\pi$</td>
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<td></td>
<td>$a + 2b = 2\pi$</td>
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</tbody>
</table>
Part 3: Results

For every family, the exhaustive search is finite. That is: if a pentagon does not respect condition of Type $i$ for a $i \in \{1, \ldots, 24\}$, then it cannot tile the plane.

- Types 1 to 15 are the already known families.
- Types 16 to 19 are special cases of known families.
- Types 20 to 24 are “degenerate” ($\dim(\mathcal{P}) > 0$): there are no convex pentagons which respects these conditions.
Future

- Reproduce the exhaustive search (∼5000 lines in C++)
Future

- Reproduce the exhaustive search (∼5000 lines in C++): Done
Future

- Reproduce the exhaustive search ($\sim 5000$ lines in C++): Done
- Re-reproduce the exhaustive search (some else ?)
Future

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- Formal proof ? (Coq or similar)
Future

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Simplifications:
- Direct proof for the positive density (?)
- Direct (not algorithmic) proof for the finiteness of good sets ?
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  - Infinitely many types tiling the plane... How to manage ?
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Future:
- Non-convex polygons :
  - Infinitely many types tiling the plane... How to manage ?
- Search for an aperiodic polygon ("Ein-stein" tile) ?
Thanks!