Exhaustive search of convex pentagons which tile the plane

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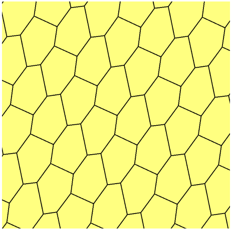
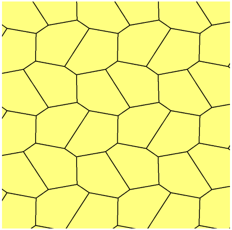
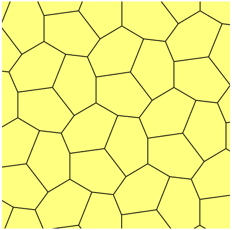

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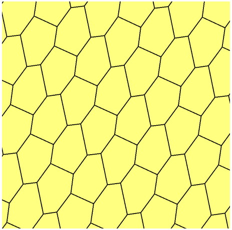
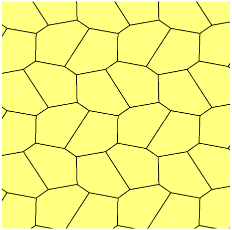
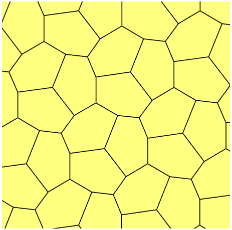

- Only one open case: Pentagons

Pentagons: history

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- Mann, McLoud & Von Derau (2015): Type 15.

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- a set C_a of linear conditions on angles (form: $v \cdot \alpha = 2\pi$ with $v \in \mathbb{N}^5$)
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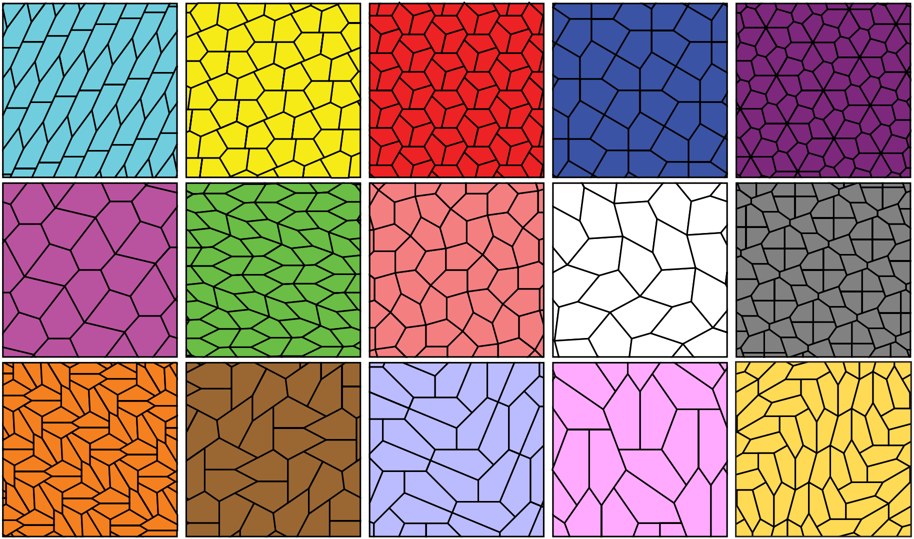
Examples:

- Type 1: $\alpha_1 + \alpha_2 = \pi$
- Type 2: $\alpha_1 + \alpha_3 = \pi$ and $\ell_1 = \ell_3$
- Type 4: $\alpha_3 = \alpha_5 = \pi/2$, $\ell_2 = \ell_3$ and $\ell_4 = \ell_5$
- ...



(Wikipedia)

Sketch

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Let $\mathcal{P}$ be a convex pentagon which tiles the plane.

- Part 1: There exist a tiling by $\mathcal{P}$ such that each vertex category has positive density
- The set of vertex category (i.e. conditions implied by angles) must be "good"
- Part 2: There are only 371 good sets to consider
- Part 3: For each good set : we do an exhaustive search
- Result: we found only the 15 known families (and some special cases).

Part 1: positive density tiling and good sets

Let $\mathcal{P}$ be a convex pentagon

- the vertices are $s_1, \dots s_5$, in clockwise order
- the angles are respectively $\alpha_1 \times \pi, \dots, \alpha_5 \times \pi$

$$\forall 1 \leq i \leq 5, \quad 0 < \alpha_i < 1$$

$$\sum_{i=1}^{5} \alpha_i = (1,1,1,1,1) \cdot \alpha = 3$$

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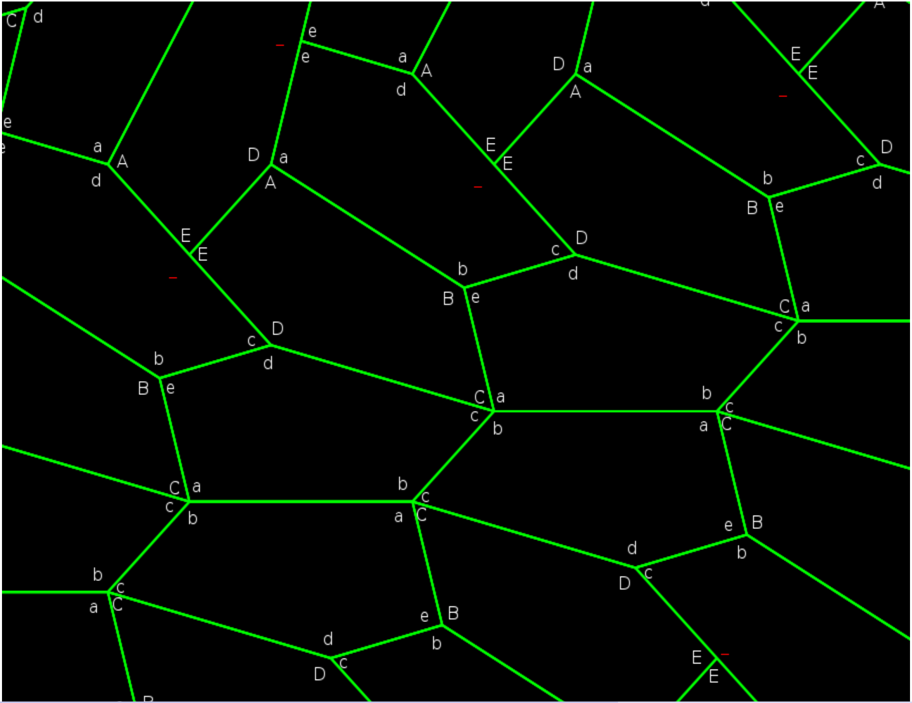
Let $\mathcal{T}$ be tiling of the plane by $\mathcal{P}$ (we allow rotation/transtation/mirror)

(Note: no hypothesis on periodicity / transitivity)



Vector category

Let s be a vertex of $\mathcal{T}$ (*i.e.* a vertex of one pentagon in $\mathcal{T}$)

The *vector category* of s, denoted $V(s)$, is the vector $v \in \mathbb{N}^5$ s.t. there are v_i angles s_i around s.

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Attention ! Two cases of vertices:

- "Half" : s is in the border of a tile P, but not a vertex of P
- "Full" : s is a vertex of every tile around s

We have to "correct" the vector category of "half" vertices.
Here, for the sake of simplicity, we do not talk about half vertices...

A toy problem

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Can we tile only with v_a and v_b ?

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- By compactness one can construct a tiling without v

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$v_a \cdot u = 0$ $v_b \cdot u = 0$ $v_c \cdot u = 1$ with $u = (1,0,-1,0,0)$

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$v_a \cdot u = 0 \quad v_b \cdot u = 0 \quad v_c \cdot u = 1 \quad$ with $u = (1, 0, -1, 0, 0)$
$\Rightarrow$ density of v_c must be 0

Good set

Definition (Good set)

$\mathcal{X} \subseteq \mathbb{N}^5$ is *good* if $\forall u \in \mathbb{R}^5$ with $\sum u = 0$, either:

- $u \cdot v = 0$ for every $v \in \mathcal{X}$, or
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$v_1 = (1, 1, 1, 0, 0)$
$v_2 = (0, 0, 0, 2, 2)$
$v_3 = (1, 1, 0, 1, 0)$

- $\{v_1, v_2, v_3\}$ is not good, with $u = (1, 0, -1, 0, 0)$
- $\{v_1, v_2\}$ is good since $2 \times u \cdot v_1 + u \cdot v_2 = 0$

Positive density imply $\mathcal{W}$ is good

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Contradiction since:

$$\sum_{v \in \mathcal{W}} (u \cdot v) \times d_v \geq (u \cdot v^+) \times d_{v^+} > 0$$

$\mathfrak{P}_{\mathcal{X}}$

Let $\mathcal{X} \subseteq \mathbb{N}^5$

Definition ($\mathfrak{P}_{\mathcal{X}}$)

$\mathfrak{P}_{\mathcal{X}}$ is the convex polytope of $\alpha = (\alpha_1, \dots \alpha_5) \in \mathbb{R}^5$ s.t.

- $\forall i \in \{1, \dots, 5\}$, $0 \leq \alpha_i \leq 1$,
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Spoil: only finitely many...

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We suppose w.l.o.g. that:

- $1 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4 \geq \alpha_5 \geq 0 \quad (\mathfrak{P}^{\geq}_{\mathcal{X}}$ instead of $\mathfrak{P}_{\mathcal{X}})$
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procedure $\textsc{Recurse}(\mathcal{X})$

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Let $m_{\mathcal{X}} \in [0,1]^5$ be such that $(m_{\mathcal{X}})_i = \min\{\alpha_i : \alpha \in \mathfrak{P}^{\geq}_{\mathcal{X}}\}$
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Let $u \in \mathbb{R}^5$ s.t. $\sum u = 0$ and $\forall i$ with $m_i = 0$ then $u_i > 0$
$\rightarrow$ the "negative part" in $v \cdot m$ is bounded
$\rightarrow$ each v_i must be bounded, otherwise $v \cdot \alpha \geq v \cdot m > 2$.

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Note : if α and $\alpha' \in \mathfrak{P}_{\mathcal{X}}^{\geq}$, then with $u = \alpha - \alpha'$:

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For the last condition on $m_{\mathcal{X}}$:

If $m_4 > 0$ and $m_5 = 0$, there is $\alpha' \in \mathfrak{P}^{\geq}_{\mathcal{X}}$ s.t. $\alpha'_5 = 0$

If $m_4 = m_5 = 0$, there is $\alpha' \in \mathfrak{P}^{\geq}_{\mathcal{X}}$ s.t. $\alpha'_4 = \alpha'_5 = 0$

(and take $\alpha \in \mathfrak{P}^{\geq}_{\mathcal{X}} \cap]0,1[^5$)

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$\textsc{Recurse}$ always terminates: finitely many good sets with $\mathfrak{P}^{\geq}_{\mathcal{X}} \cap]0,1[^5 \neq \emptyset$.

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- Take all permutations: 3495 non-empty maximal good sets $\mathcal{X}$ with $\mathfrak{P}_{\mathcal{X}} \cap]0,1[^5 \neq \emptyset$
- Keep only one represents for each class up to rotation/mirror, one have the 371 sets.

Good sets: results

371 sets to consider:

- 2 s.t. $\mathfrak{P}_{\mathcal{X}}$ has dimension 3
- 26 s.t. $\mathfrak{P}_{\mathcal{X}}$ has dimension 2
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- 90 of “Type 1” (that is $\alpha_i + \alpha_{i+1} = 1$)

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- or no convex pentagon exists with these conditions

Backtracking: general idea

The object on which we work and backtrack is a pair (G, Q):

- G is a embedded planar graph which represent the partial tiling ("Tiling graph")
- Q is a set of conditions we know on the lengths of the pentagon: *i.e.* a linear program (LP) on $\ell_1 \ldots \ell_5$

We add linear conditions on sides "on the fly"

Tiling graph

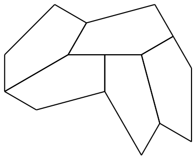
Tiling graph : embedded planar graph with labels on angles and edges

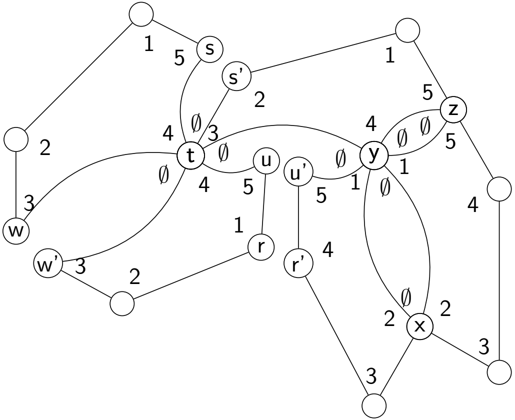
Two types of faces: usual and special

- usual: corresponds to a pentagon in the tiling. The degree is 5, and the angles are marked from 1 to 5 (in CW or CCW)
- special: corresponds to frontier between tiles, or an unknown area of the plane. Angles are marked with $\emptyset$, π or ?

A special face is *complete* if there no "?"

Tiling graph: example





Example of a tiling graph (Type 15). Unmarked angles are labeled "?"

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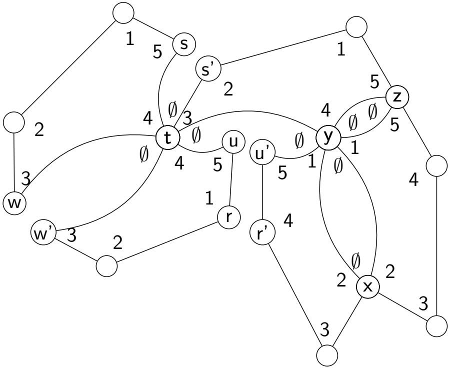
- If Q does not permit to decide among the 3 possibilities: $s < s'$, $s = s'$ and $s > s'$, then we branch on the 3 possibilities: we add the corresponding condition in Q and recurse

Branching on length suppositions: example



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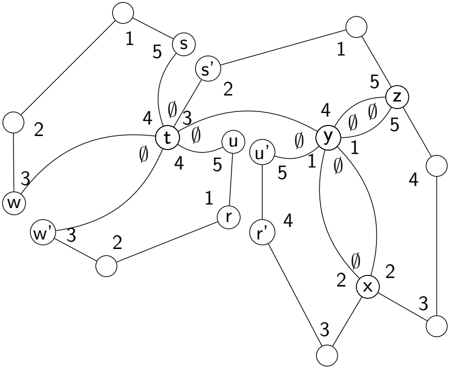
Branching on length suppositions: example



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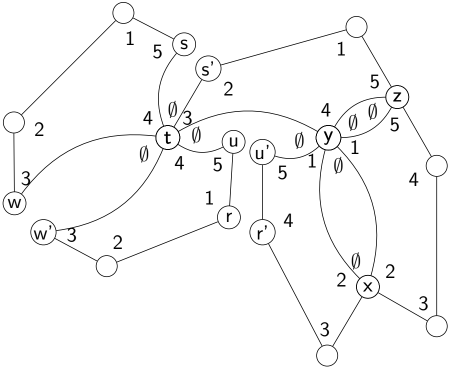

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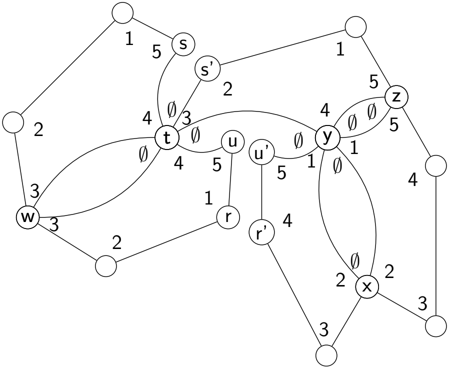

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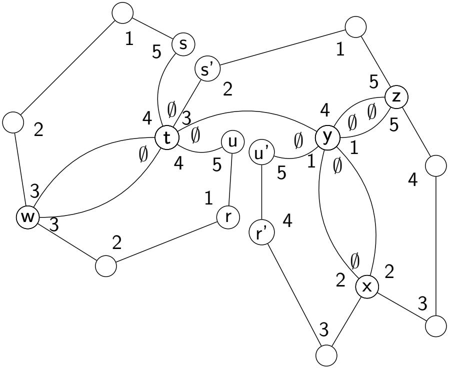

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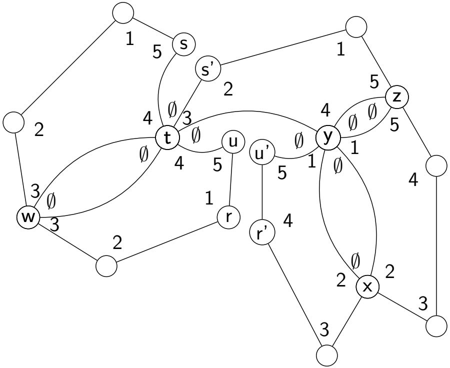

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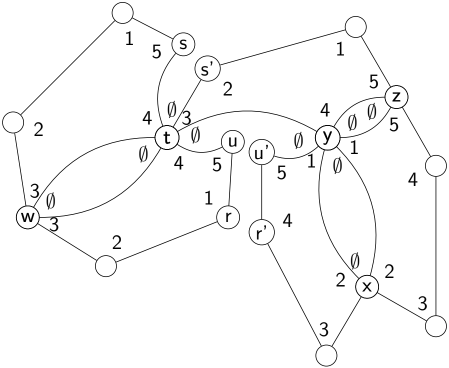

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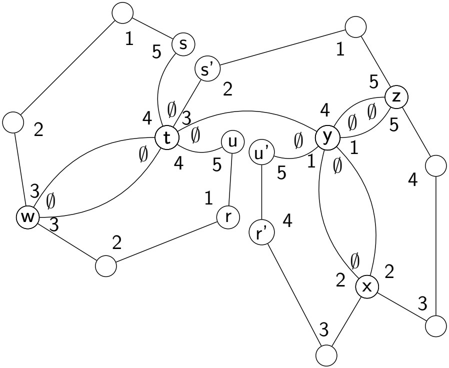

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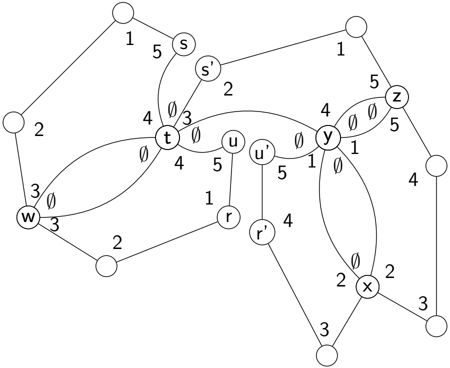

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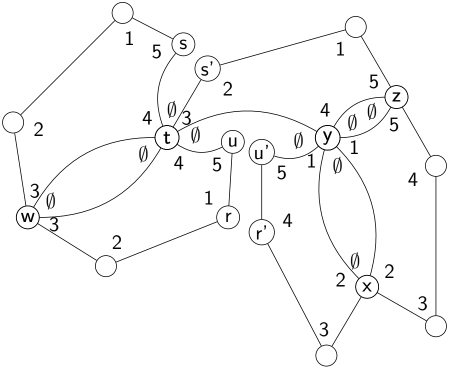

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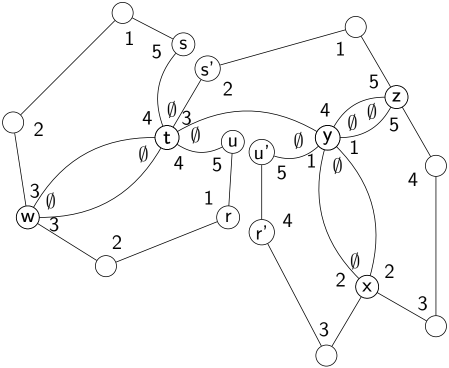

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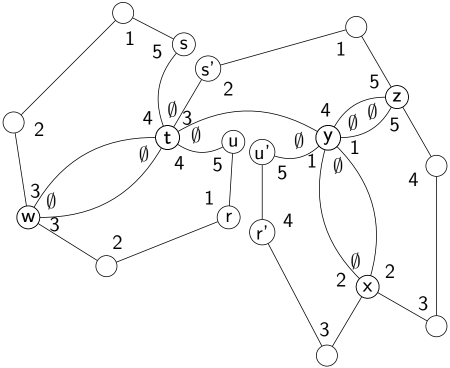

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We don't know. We branch.
first case : add $\ell_3 > \ell_4 + \ell_5$ to Q and branch
second case : add $\ell_3 = \ell_4 + \ell_5$ to Q and branch

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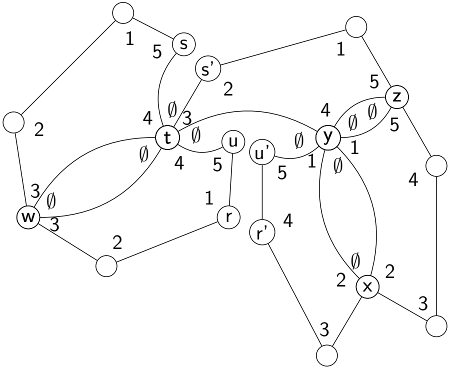

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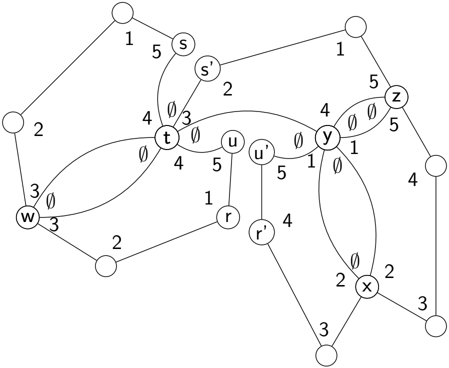

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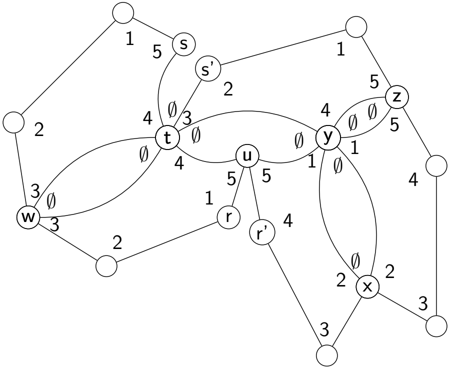

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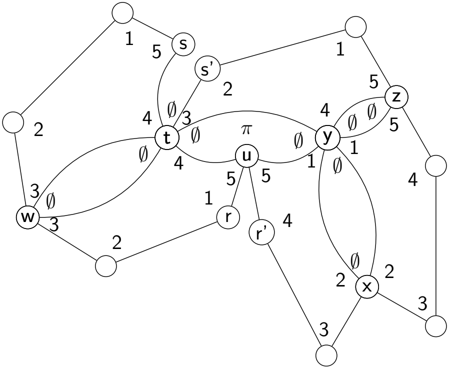

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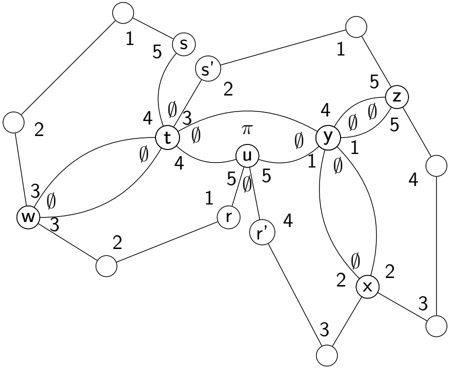

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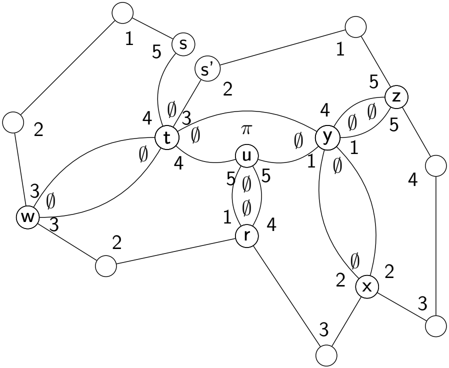

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in the run (r, u, r'), r and r' have the same position: we merge...

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We take a non-complete vertex w in the graph, and we try (branch on) every possibility to add a new face adjacent to w

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Given the LP Q, we denote by $\mathfrak{Q}$ the set of solutions ℓ of Q with $\sum \ell = 1$
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$$\sum_i \ell_i \exp(s(\alpha)_i \times \pi \times \sqrt{-1}) = 0. \tag{1}$$

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If $\dim(\mathfrak{P}) > 0$: we backtrack if we have a certificate (computations in $\mathbb{Q}$) that there a no solution. Problem: this cannot detect "degenerate case". So we manually add some degenerate case. (Types 20 to 24 in Table 1).

Conditions for which we backtrack (Table 1)

Type 1 (i=1)	$a+b+c=2\pi$		Type 2 (i=2)	$a+b+d=2\pi$	$C=E$
Type 3 (i=31)	$3e=2\pi$ $d+2e=2\pi$ $b+2e=2\pi$	$C+E=D$ $A=B$	Type 4 (i=6)	$a+b+d=2\pi$ $2e=\pi$	$D=E$ $B=C$
Type 5 (i=4)	$3e=2\pi$ $a+b+d=2\pi$	$D=E$ $B=C$	Type 6 (i=13)	$d+2e=2\pi$ $a+c+d=2\pi$	$C=D=E$ $A=B$
Type 7 (i=17)	$d+2e=2\pi$ $a+2c=2\pi$	$A=C=D=E$	Type 8 (i=14)	$d+2e=2\pi$ $2b+c=2\pi$	$A=B=C=D$
Type 9 (i=15)	$d+2e=2\pi$ $2a+c=2\pi$	$A=B=C=D$	Type 10 (i=69)	$2c+d=2\pi$ $b+c+e=2\pi$ $a+2b=2\pi$	$A+C=D=E$
Type 11 (i=67)	$c+2d=2\pi$ $b+d+e=2\pi$ $a+2b=2\pi$	$A=B=C+2E$	Type 12 (i=67)	$c+2d=2\pi$ $b+d+e=2\pi$ $a+2b=2\pi$	$A+C=B=2E$
Type 13 (i=63)	$b+2d=2\pi$ $a+b+d=2\pi$ $2e=\pi$	$A=2B=2C$	Type 14 (i=67)	$c+2d=2\pi$ $b+d+e=2\pi$ $a+2b=2\pi$	$A=B=2C=2E$
Type 15 (i=303)	$c+2d=2\pi$ $2b+e=2\pi$ $2a+d=2\pi$ $2e=\pi$	$B=D=E$ $C=2B$	Type 16 (i=72) $\subset$ T10	$b+c+e=2\pi$ $2b+d=2\pi$ $a+2c=2\pi$	$2A=D=E$ $A=C$
Type 17 (i=25) $\subset$ T2	$c+2e=2\pi$ $2b+d=2\pi$	$A=B=C=D=E$	Type 18 (i=73) $\subset$ T2	$d+2e=2\pi$ $c+2e=2\pi$ $b+d+e=2\pi$	$D=E$ $A=B$
Type 19 (i=23) $\subset$ T1	$c+2e=2\pi$ $b+2d=2\pi$	$A=B=C=D$	Type 20 (i=2) degen.	$a+b+d=2\pi$	$A=C+D$ $B=E$
Type 21 (i=12) degen.	$d+2e=2\pi$ $2a+b=2\pi$	$A=B$ $C=D$	Type 22 (i=27) degen.	$c+2e=2\pi$ $a+2d=2\pi$	$A=B=C=E$
Type 23 (i=64) degen.	$2b+d=2\pi$ $a+b+d=2\pi$ $2e=\pi$	$A=2C=2D$	Type 24 (i=69) degen.	$2c+d=2\pi$ $b+c+e=2\pi$ $a+2b=2\pi$	$2D=A+C$ $2E=A+C$

Part 3: Results

For every family, the exhaustive search is finite

That is: if a pentagon does not respect condition of Type i for a $i \in \{1, \ldots 24\}$, then it cannot tile the plane.

- Types 1 to 15 are the already known families
- Types 16 to 19 are special cases of known families
- Types 20 to 24 are "degenerate" ($\dim(\mathfrak{P}) > 0$): there are no convex pentagons which respects these conditions

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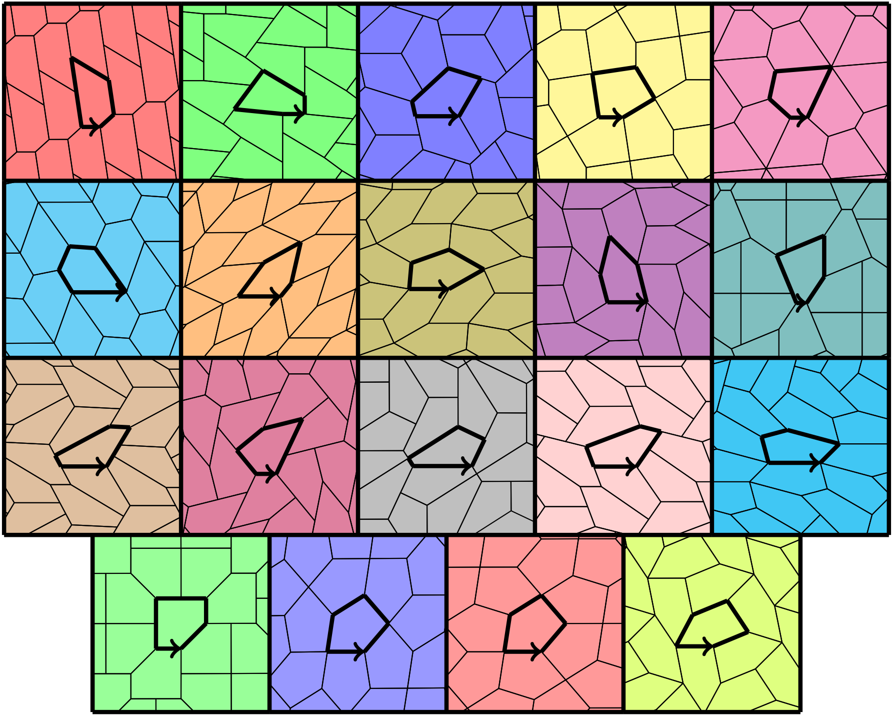
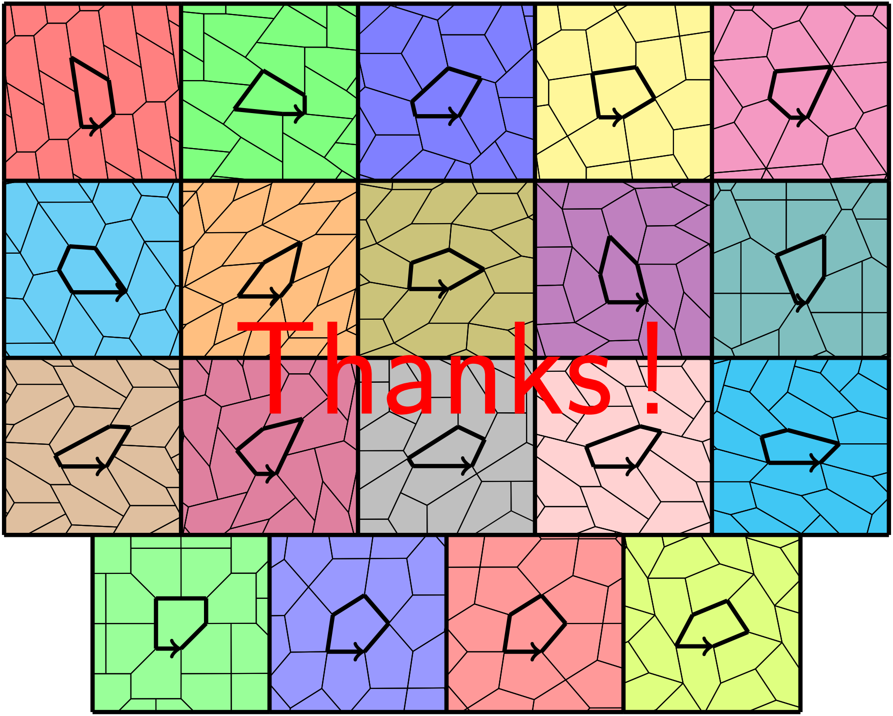
- Search for an aperiodic polygon ("Ein-stein" tile) ?




Thanks !