Augmented systems in fluid mechanics

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Based on joint works with:
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Main objectives of my talk

- Enlarge the number of eqs are already present in some manipulations.
- Enlarge the number of eqs may help for modeling, mathematics, numerics.

1st Lecture:

An example around Compressible Euler-Korteweg

2nd Lecture:

Some recent references in fluid mechanics: Shallow-water, Green-Nagdhi etc.....
The case of the Compressible Navier-Stokes equations

In all the talk, we consider a periodic domain Ω:
– Get rid of the difficulties due to the boundary
– Play with the structure of the equations only.
– Comments will be done on bounded domains.

Compressible Euler system:

$$\partial_t \rho + \operatorname{div}(\rho u) = 0$$
$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) = 0$$

with $p(\rho) = a\rho^\gamma$ with $a > 0$ and $\gamma > 1$.

Energy:

$$\frac{d}{dt}\int_\Omega \frac{1}{2}\rho|u|^2 + \frac{a}{\gamma - 1}\rho^\gamma = 0.$$

$$
\begin{aligned}
\Big(\int_\Omega \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u)\Big) \cdot u &= \int_\Omega \rho(\partial_t u + u \cdot \nabla u) \cdot u \\
&= \frac{1}{2}\int_\Omega \rho(\partial_t |u|^2 + u \cdot \nabla |u|^2) \\
&= \frac{1}{2}\frac{d}{dt}\int_\Omega \rho |u|^2
\end{aligned}
$$

$$
\Longrightarrow \frac{1}{2}\frac{d}{dt}\int_\Omega \rho|u|^2 + \int_\Omega \nabla p(\rho) \cdot u = 0 \qquad (1)
$$

Energy equation:

$$
\partial_t(\rho e(\rho)) + \mathrm{div}(\rho e(\rho) u) + p(\rho)\mathrm{div} u = 0 \qquad (2)
$$

with $e(\rho) = \int_0^\rho p(\tau)/\tau^2 d\tau = a\rho^{\gamma-1}/(\gamma - 1)$.

Add (1) and $\int_\Omega$ (2) $\Longrightarrow$

$$
\begin{aligned}
E(\rho, u) &= \int_\Omega (\frac{1}{2}\rho|u|^2 + \frac{a}{\gamma - 1}\rho^\gamma) && (3)\\
&= \int_\Omega (\frac{1}{2}\rho|u|^2 + \frac{a}{\gamma - 1}\rho^\gamma) \\
&= \int_\Omega (\frac{1}{2}\rho_0|u_0|^2 + \frac{a}{\gamma - 1}\rho_0^\gamma) && (4)
\end{aligned}
$$

Modulated energy with target (r, U)

$$
E(\rho, u|r, U) = \int_\Omega \frac{1}{2}\rho|u - U|^2 + \Big(H(\rho) - H(r) - H'(r)(\rho - r)\Big)
$$

with $H(\rho) = \rho e(\rho)$ convex with $p(\rho) = a\rho^\gamma$..

Note that we write $\rho|u|^2 = |m|^2/\rho$ where $m = \rho u$
$\Longrightarrow$ modulation with convex properties :

$$
\frac{|\rho u|^2}{\rho} - \frac{|rU|^2}{r} - \frac{2rU \cdot (\rho u - rU)}{r} + \frac{|rU|^2}{r^2}(\rho - r) = \rho|u - U|^2
$$

Note that we have used an augmented system namely:

Mass, Momentum, Internal energy.

An application of the relative entropy:

Definition. The pair $(\overline{\varrho}, \overline{u})$ is a dissipative solution of the compressible Euler equations if and only if $(\overline{\varrho}, \overline{u})$ satisfies the relative energy inequality

$$E(\varrho, u, \big| r, U)(t) \leq E(\varrho, u \big| r, U)(0) \exp\Big[c_0(r) \int_0^t \|\mathrm{div}\, U(\tau)\|_{L^\infty(\Omega)} d\tau\Big]$$

$$+ \int_0^t \exp\Big[c_0(r) \int_s^t \|\mathrm{div}\, U(\tau)\|_{L^\infty(\Omega)}\Big] \int_\Omega \varrho E(r, U) \cdot (U - \overline{u})\, dxds$$

for all smooth test functions U defined on $[0, T] \times \overline{\Omega})$ with $(r, E(r, U))$ given through

$$\partial_t r + \mathrm{div}(rU) = 0,$$
$$E(r, U) = \partial_t U + U \cdot \nabla U + \nabla H'(r)$$

with $0 < c < r < c^{-1} < +\infty$.

See:
– P.-L. Lions: Book 1998 Oxford for incompressible Euler.
– C. Bardos, T. Nguyen: 2016
– F. Sueur : 2014

This may be helpful for:

– Weak-Strong uniqueness
– Asymptotic analysis
– Definition of weakest solution when difficulties to deal with nonlinearities.

Compressible Euler-Korteweg:

$$\partial_t \rho + \mathrm{div}(\rho u) = 0$$

$$\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p(\rho) = \varepsilon^2 \mathrm{div}\, \mathbb{K}$$

where

$$\mathbb{K} = \Big(\rho\, \mathrm{div}(K(\rho)\nabla\rho) + \frac{1}{2}(K(\rho) - \rho K'(\rho))|\nabla\rho|^2\Big)\mathbb{I}_{\mathbb{R}^d} - K(\rho)\nabla\rho \otimes \nabla\rho$$

Energy:

$$\frac{d}{dt}\int_\Omega \Big(\frac{1}{2}\rho|u|^2 + \frac{a}{\gamma-1}\rho^\gamma + \varepsilon^2\frac{K(\rho)}{2}|\nabla\rho|^2\Big).$$

$$E(\rho, u, \nabla\rho) = \int_\Omega \Big(\frac{1}{2}\rho|u|^2 + \frac{a}{\gamma - 1}\rho^\gamma + \varepsilon^2 \frac{K(\rho)}{2}|\nabla\rho|^2 \Big).$$

Modulated energy with target $(r, U, \nabla r)$

$$E(\rho, u, \nabla\rho|, r, U\nabla r) = \frac{1}{2}\int_\Omega \rho|u - U|^2 + \frac{1}{2}\varepsilon^2 \int_\Omega I_T + \int_\Omega H(\rho|r)$$

where

$$H(\rho|r) = H(\rho) - H(r) - H'(r)(\rho - r)$$

and

$$I_T = K(\rho)|\nabla\rho|^2 - K(r)|\nabla r|^2 - K'(r)|\nabla r|^2(\rho - r) - 2K(r)\nabla r \cdot (\nabla\rho - \nabla r).$$

If $K(\rho) = \rho^s$, convexity of the functional I_T requires $-1 \leq s \leq 0$.
To get Gronwall Lemma: control of terms coming from K ask for $s + 2 \leq \gamma$.
J. Giesselmann, C. Lattanzio, A. Tzavaras: (2017).

$$
\begin{aligned}
E(\rho,u,\nabla\rho) &= \int_\Omega \Big(\frac{1}{2}\rho|u|^2 + \frac{a}{\gamma-1}\rho^\gamma + \varepsilon^2\frac{1}{2}|\nabla\int_0^\rho \sqrt{K(\tau)}d\tau|^2\Big).\\
&= \int_\Omega \Big(\frac{1}{2}\rho|u|^2 + \frac{a}{\gamma-1}\rho^\gamma + \varepsilon^2\frac{\rho}{2}|\nabla\int_0^\rho \sqrt{\frac{K(\tau)}{\tau}}d\tau|^2\Big).
\end{aligned}
$$

Euler Lagrange associated to $\displaystyle\int_\Omega \frac{1}{2}|\nabla\Psi(\rho)|^2$ where $\Psi(\rho) = \frac{1}{2}|\nabla\int_0^\rho \sqrt{K(\tau)}d\tau|^2$:

$$
\begin{aligned}
\int_\Omega \nabla\Psi(\rho)\cdot\nabla(\partial_t\Psi(\rho)) &= -\int_\Omega \Delta\Psi(\rho)\,\partial_t\Psi(\rho)\\
&= -\int_\Omega \Delta\Psi(\rho)\,\Psi'(\rho)\,\partial_t\rho\\
&= \int_\Omega \Delta\Psi(\rho)\,\Psi'(\rho)\,\mathrm{div}(\rho u)\\
&= -\int_\Omega u\cdot\Big[\rho\nabla(\Psi'(\rho)\Delta\Psi(\rho))\Big]
\end{aligned}
$$

$$\partial_t \rho + \mathrm{div}(\rho u) = 0$$

$$\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p(\rho) = \varepsilon^2 \rho \nabla \Big(\Psi'(\rho) \Delta \Psi(\rho) \Big)$$

where

$$\Psi(\rho) = \int_0^\rho \sqrt{K(\tau)}\, d\tau.$$

$K(\rho) = 1/\rho \implies$ Quantum-Euler system $\implies 2\rho\nabla(\frac{1}{\sqrt{\rho}}\Delta\sqrt{\rho})$.
$K(\rho) = 1 \implies$ Euler with surface tension $\implies \rho\nabla\Delta\rho$.

In compressible Euler system:

The term $e(\rho)$ was really important to deals with the pressure term.

Importance of the term $\nabla \int_0^\rho \sqrt{K(\tau)/\tau}\, d\tau$?

Recall that $E(\rho, u, \nabla\rho)$ may be written as

$$\tilde{E}(\rho, u, v) = \int_\Omega \Big(\frac{1}{2}\rho|u|^2 + \rho\, e(\rho) + \varepsilon^2 \frac{\rho}{2}|v|^2\Big)$$

with $v = \nabla \int_0^\rho \sqrt{K(\tau)/\tau}\, d\tau$

An Augmented system helping at continuous and discrete level ?

Let us choose $K(\rho) = 1/\rho$ for simplicity then $v = \nabla \log \rho$. Remark that

$$\mathbb{K} = \Big(\rho \,\mathrm{div}(K(\rho)\nabla\rho) + \frac{1}{2}(K(\rho) - \rho K'(\rho))|\nabla\rho|^2\Big)\mathbb{I}_{\mathbb{R}^d} - K(\rho)\nabla\rho \otimes \nabla\rho$$

which may be written

$$\begin{aligned} \mathbb{K} &= \Delta\rho - \rho\nabla \log \rho \otimes \nabla \log \rho \\ &= \mathrm{div}(\rho\nabla\nabla \log \rho) = \mathrm{div}(\rho\nabla v). \end{aligned} \qquad (5)$$

Moreover we have, differentiating the mass equations,

$$\partial_t \nabla\rho + \nabla \mathrm{div}(\rho u) = 0$$

and therefore

$$\partial_t(\rho v) + \mathrm{div}(\rho v \otimes u) = -\nabla \mathrm{div}(\rho u) + \mathrm{div}(\rho v \otimes u) = -\mathrm{div}(\rho\, {}^t\nabla u)$$

In the variable $(\rho, u, \bar{v})$ with $\bar{v} = \varepsilon v$, the Euler-Korteweg system reads

$$\partial_t \rho + \mathrm{div}(\rho u) = 0$$

$$\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p(\rho) = \varepsilon \mathrm{div}(\rho \nabla v).$$

$$\partial_t(\rho \bar{v}) + \mathrm{div}(\rho \bar{v} \otimes u) = -\varepsilon \mathrm{div}(\rho^t \nabla u)$$

with $\bar{v} = \varepsilon \nabla \log \rho$.

Note that for the energy, it is easy to get it:
Scalar product of Eq u with u and scalar product of Eq v with v. Integration in space and use of the mass equation.

Modulated energy with target $(r, U, \bar{V})$.

$$E(\rho, u, \bar{v}|r, U, \bar{V}) = \frac{1}{2} \int_\Omega \rho|u - U|^2 + H(\rho) - H(r) - H'(r)(\rho - r) + \rho|\bar{v} - \bar{V}|^2.$$

Using augmented version for $K(\rho) = \rho^s$:
OK Gronwall if $s + 2 < \gamma$ with $-1 \leq s$ ok.

D.B., M. Gisclon, I. Lacroix-Violet. (2018).

Important remark: A global weak solution of the Euler-Korteweg is a global weak solution of the augmented system. Play with augmented system is appropriate for theoretical and numerical purposes !!

– Weak-Strong uniqueness, Dissipative solutions, singular limits.

CRAS 2015 D.B., F. Couderc, P. Noble, J.–P. Vila

$$[E-K] \quad \begin{aligned} \partial_t \varrho + \mathrm{div}(\varrho \mathbf{u}) = 0, \\ \partial_t (\varrho \mathbf{u}) + \mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla P(\varrho) = \operatorname{div} \mathbf{K} \end{aligned}$$

with

$$\mathbf{K} = \Big(\rho \operatorname{div}(K(\rho)\nabla\rho) + \frac{1}{2}(K(\rho) - \rho K'(\rho))|\nabla\rho|^2\Big)\mathrm{Id} - K(\rho)\nabla\rho\otimes\nabla\rho$$

where $K(\rho)$ is the capillary coefficient. Note that

$$\operatorname{div}\mathbf{K} = \rho\nabla(\sqrt{K(\rho)}\Delta(\int_0^\rho \sqrt{K(s)}\, ds)$$

$$= \operatorname{div}\Big(F(\rho)\nabla\nabla\varphi(\rho)\Big) + \nabla\Big((F'(\rho)\rho - F(\rho))\Delta\varphi(\rho)\Big)$$

with $\sqrt{\rho}\varphi'(\rho) = \sqrt{K(\rho))}$, $F'(\rho) = \sqrt{F(\rho)\rho}$.

$\Longrightarrow$ extended formulation of the Euler-Korteweg system with $w = \nabla\varphi(\rho)$.

$\Longrightarrow$ Stable schemes under hyperbolic CFL condition.

Numerical simulations

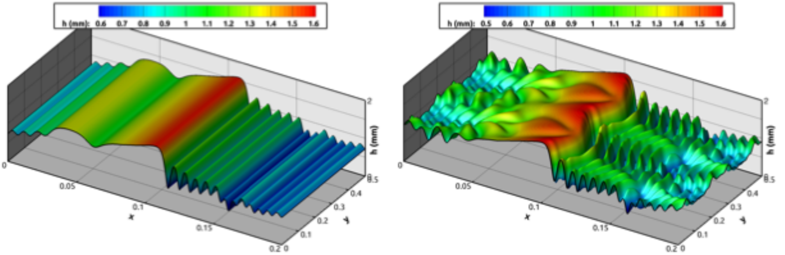


Figure 1. Numerical simulation of a roll-wave in presence of surface tension. On the left: one dimensional roll-wave without transverse perturbations. On the right: a two-dimensional roll-wave

See works for other studies with augmented systems by :

– Dhaouadi, Favrie, Gavrilyuk: NLS ou Euler-Korteweg.

– Favrie, Gavrilyuk : Serre-Green-Naghdi.

– Schochet, Weinstein: NLS (Eq Zakharov)

– Obrecht: Benney-Roskes

– Kazerani : Green-Naghdi.

– Benzoni, Danchin, Descombes: Euler-Korteweg in non conservative form.

– CEA with Jamet et al.: Euler-Korteweg

– Noble, Richard, Ruyer-Quil, Vila: Shallow-water and multifluid systems.

– Besse, Noble : Transparent boundary conditions for dispersive systems.

....

Relative κ entropy (CRAS 2015 D.B, P. Noble, J.–P. Vila)

A useful tool to measure distance between quantities!!

Idea on a simple energy: Assume $e(u,\rho) = e_1(u) + e_2(\rho)$, calculate

$$E(u,\rho) := e(u,\rho) - e(U,r) - \nabla e_1(u)\cdot(u-U) - e_2'(r)(\rho - r)$$

If global strict-convexity then control of

$$|u-U|^2 + |\rho - r|^2$$

See for instance:
C. Dafermos, R. Di Perna, H.T. Yau, Y. Brenier, C. Bardos, F. Otto, A. Tzavaras, L. St Raymond........

See R. Herbin, T. Galloüet, D. Maltese, A. Novotny, E. Feireisl for the compressible Navier-Stokes with constant viscosities, relative entropy and comparision continuous/discrete solutions..

May be complicated depending on the study !!

Let us consider the incompressible Euler equations:

$$\partial_t u + u \cdot \nabla u + \nabla p = 0$$

with energy

$$\int_\Omega |u|^2 \leq \int_\Omega |u_0|^2$$

Let $\bar{u}$ a divergence free smooth function, let us denote

$$PL(\bar{u}, q) = \partial_t \bar{u} + P(\bar{u} \cdot \nabla \bar{u})$$

With P the Leray projector.

Formally

$$\partial_t (u - \bar{u}) + u \cdot \nabla (u - \bar{u}) + (u - \bar{u}) \cdot \nabla \bar{u} + \nabla q = L(\bar{u}, q).$$

Formally

$$\frac{d}{dt}\int_\Omega |u-\bar{u}|^2 + \int_\Omega ((u-\bar{u})\cdot\nabla\bar{u})\cdot(u-\bar{u}) = \int_\Omega L(\bar{u})\cdot(u-\bar{u}).$$

Gronwall with $E(u|\bar{u}) = \int_\Omega |u-\bar{u}|^2$.

A dissipative solution u of the incompressible Euler equations is a solution satisfying

$$E(u|\bar{u}) \le E(u_0|\bar{u}_0)\exp[\int_0^t \|\nabla\bar{u}\|_{L^\infty(\Omega)} d\tau] + \int_0^t \exp[\int_s^t \|\nabla\bar{u}\|_{L^\infty(\Omega)} d\tau] \int_\Omega L(\bar{u})\cdot(u-\bar{u})\, dxds$$

for all $\bar{u}$ smooth divergence free vector field and

$$L(\bar{u}) = P(\bar{u}\cdot\nabla\bar{u})$$

How to prove that a global weak solutions if a dissipative solution ?

Take car of the regularity !!

Recall that

$$\int_\Omega |u - \bar{u}|^2 = \int_\Omega (|u|^2 - |\bar{u}|^2 - 2\bar{u} \cdot (u - \bar{u}).$$

$\Longrightarrow$

1) Show the relative entropy inequality starting with

$$E(u|\bar{u})(t) - E(u|\bar{u})(0).$$

1) Use energy inequality for the first term.
2) Use the energy equality for $\bar{u}$ with $PL(\bar{u})$.
3) Test the equation of $\bar{u}$ by $u - \bar{u}$.

How to construct a dissipative solution ?

Start from the incompressible Navier-Stokes equations
$\Longrightarrow$ global weak solutions
$\Longrightarrow$ dissipative solutions as previously but with

$$PL_\nu(\bar{u}, q) = \partial_t \bar{u} + P(\bar{u} \cdot \nabla \bar{u}) - \nu \Delta \bar{u}.$$

Now let $\nu \to 0$ to get a global dissipative solution of the incompressible Euler equations in the sense given before.

Let us now focus on:

A mixture model/low-mach system with large heat release.

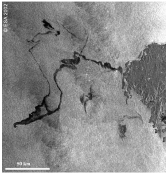
A system which encodes incompressible/compressible features.

What I want to show you:

- The importance of viscosity even if it is small,
- The presence of viscosity effect in dispersive term.
- The existence of two velocities even if the model seems to have only one.

Examples related to the mixture model

Powder-snow avalanche



Spreading of pollutant in water

Mixture system

Consider the following system in periodic box:

$$
[M-NS] \quad \begin{aligned} &\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0, \\ &\partial_t (\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - 2\operatorname{div}(\mu(\varrho) D(\mathbf{u})) - \nabla(\lambda(\varrho)\operatorname{div}\mathbf{u}) + \nabla\Pi = \mathbf{0}, \\ &\operatorname{div}\mathbf{u} = -2\kappa\triangle\varphi(\varrho). \end{aligned}
$$

where $D(\mathbf{u}) = (\nabla u + \nabla^t u)/2$ or equivalently

$$
\begin{aligned} &\partial_t \varrho + \nabla\varrho \cdot (\mathbf{u} + 2\kappa\nabla\varphi(\varrho)) - 2\kappa\operatorname{div}(\varrho\nabla\varphi(\varrho)) = 0, \\ &\partial_t (\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - 2\operatorname{div}(\mu(\varrho) D(\mathbf{u})) - \nabla(\lambda(\varrho)\operatorname{div}\mathbf{u}) + \nabla\Pi = \mathbf{0}, \\ &\operatorname{div}\mathbf{u} = -2\kappa\triangle\varphi(\varrho). \end{aligned}
$$

Note here κ const

Physical literature

Such system:

- 1) Low mach number limit from
  Heat-conducting compressible Navier-Stokes eq. with large heat release.
  See the book by P.–L. Lions.
- 2) Formally obtained as mixture equations with Fick law to close the system. See the book by Rajagopal and Tao.

Some special cases and possible extension:

- 1) For $\varphi(\varrho) = -1/\varrho$ we recover combustion model.
  See works by Embid, Majda, Lions, Lafitte, Dellacherie, Penel...
- 2) For $\varphi(\varrho) = \log \varrho$ we recover pollutant model.
  See works by Graffi, Straughan, Antonsev, Kazhikhov, Monakov...
- 3) See extension to other nonlinearities and to two-fluid system by Dellacherie, Faccanoni, Grec, Penel, Lafitte etc... for other cases.

$\kappa = 0 \implies$ Non-homogeneous incompressible Navier–Stokes equations.

$$
[NH-INS] \quad \begin{aligned} &\partial_t \varrho + \mathrm{div}(\varrho \mathbf{u}) = 0, \\ &\partial_t (\varrho \mathbf{u}) + \mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - 2\,\mathrm{div}(\mu(\varrho) D(\mathbf{u})) + \nabla \Pi = \mathbf{0}, \\ &\mathrm{div}\, \mathbf{u} = 0. \end{aligned}
$$

Global well posedness: A. Kazhikhov '77, J. Simon '87, P.–L. Lions '98.

Mathematical literature on the mixture system

- Local strong solutions

  Beirão Da Veiga '82, Secchi '82,
  Danchin & Liao '12 (in critical Besov spaces).

- Global in time solutions

  Kazhikov & Smagulov '77: Modified conv. term, constraint on c_0
  existence of generalized solution which is unique in 2d,
  Lions '98: 2d weak solutions ($\varphi = -1/\varrho$), small perturb. const. ρ_0,
  Secchi '88: 2d unique solution for small c_0
  Danchin & Liao '12: Small perturb. const. ρ + small initial velocity.

- No smallness assumption

  B., Essoufi & Sy '07, for special relation

  $$\varphi'(s) = \mu'(s)/s, \quad \kappa = 1 \quad \Longrightarrow \quad \text{Kazhikhov-Smagulov type system}$$

  Cai, Liao & Sun '12: Uniqueness in 2d,
  Liao '14: Global strong solution in 2d, critical Besov spaces.

Numerical literature

- J. Etienne, E. Hopfinger, P. Saramito.
  Numerical simulations of high density ratio lock-exchange flows.
  No change of variable.
  Finite element + characteristic method with mesh refinements.
- C. Acary-Robert, D. Bresch, D. Dutykh.
  Numerical simulation of powder-snow avalanche interaction with obstacle.
  Numerical test using Open-Foam,
  change of variable + relation between μ and φ
  Discussion around a new entropy encountered in a theoretical paper.
- C. Calgaro, E. Creusé, T. Goudon.
  Simulation of Mixture Flows: Pollution Spreading and Avalanches.
  Change of variable + get ride of high-order terms
  (Kazhikhov-Smagulov type system).
  Numerical schemes: hybrid Finite Volume/Finite Element method.
  Test and comparison.

Goal of this part on this powder snow avalanches system:
A two-velociy hydrodynamics in this model

The case $\mu'(s) = s\varphi'(s)$:

$\Longrightarrow$ A non-linear hypocoercivity property!
$\Longrightarrow$ A two-velocity hydrodynamic in the spirit of H. Brenner but.......
.... with two different velocities:
not volume and mass velocities as in H. Brenner's work
That means not $\mathbf{u}$ and $\mathbf{u} + 2\kappa\nabla\varphi(\rho)$ but two others specified later on.
$\Longrightarrow$ Global existence of weak solutions for a wide range of coefficient.
$\Longrightarrow$ An answer to an open question in P.–L. Lions's book.
$\Longrightarrow$ An interesting numerical scheme
(work in progress with P. Noble, J.–P. Vila).

The case $\mu'(s) \neq s\varphi'(s)$:

A conclusion under some inequalities constraints.
$\Longrightarrow$ An answer to an other open question in P.–L. Lions's book.

Special case where φ and μ are related: Two velocity hydrodynamics

Let us remark

$$\int \nabla \Pi_1 \cdot \mathbf{u} = 2\,\kappa \int \Pi_1 \Delta\varphi(\varrho)$$

and

$$\int \nabla \Pi_1 \cdot (\mathbf{u} + 2\nabla\varphi(\varrho)) = -2\,(1-\kappa) \int \Pi_1 \Delta\varphi(\varrho)$$

Thus

$$\int \nabla \Pi_1 \cdot ((1-\kappa)\mathbf{u}) + \int \nabla \Pi_1 \cdot (\kappa(\mathbf{u} + 2\nabla\varphi(\varrho))) = 0$$

Momentum equation on $\mathbf{u}$:

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}(\varrho\mathbf{u}\otimes\mathbf{u}) - 2\,\mathrm{div}(\mu(\varrho)D(\mathbf{u})) = -\nabla\Pi_1$$

Momentum equation on $\mathbf{v} = \mathbf{u} + 2\nabla\varphi(\varrho)$:

$$\partial_t(\varrho\mathbf{v}) + \mathrm{div}(\varrho\mathbf{v}\otimes\mathbf{u}) - 2\,\mathrm{div}(\mu(\varrho)A(\mathbf{v})) - 2\nabla\Big((\mu'(\varrho)\varrho - \mu(\varrho))\,\mathrm{div}\,\mathbf{u}\Big) = -\nabla\Pi_1$$

where $A(\mathbf{v}) = (\nabla\mathbf{v} - \nabla^t\mathbf{v})/2$.

Additional entropy equality

Testing Eq on $\mathbf{u}$ by $(1-\kappa)\mathbf{u}$ and Eq on $\mathbf{v}$ by $\kappa\mathbf{v}$ and adding we get

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega \varrho\left((1-\kappa)\frac{|\mathbf{u}|^2}{2}+\kappa\frac{|\mathbf{v}|^2}{2}\right)\,\mathrm{d}x$$
$$+2(1-\kappa)\int_\Omega \mu(\varrho)|D(\mathbf{u})|^2\,\mathrm{d}x+2\kappa\int_\Omega \mu(\varrho)|A(\mathbf{u})|^2\,\mathrm{d}x$$
$$+2(1-\kappa)\kappa^2\int_\Omega (\mu'(\varrho)\varrho-\mu(\varrho))|2\triangle\varphi|^2\,\mathrm{d}x=0,$$

which generalizes the "B-D" entropy to the M-NS system.

Two-velocity hydrodynamic: joint velocity and drift velocity

Remark that:

$$(1-\kappa)|\mathbf{u}|^2 + \kappa|\mathbf{u} + 2\nabla\varphi|^2 = |\mathbf{w}|^2 + (1-\kappa)\kappa|2\nabla\varphi|^2.$$

$\Longrightarrow$ See two-velocity hydrodynamics papers by S.C. Shugrin and S. Gavrilyuk
Defining a new velocity vector field (joint velocity)

$$\mathbf{w} = \mathbf{u} + \kappa\nabla\varphi(\varrho), \quad \text{we see that} \quad \operatorname{div}\mathbf{w} = 0$$

Note that $\mathbf{v_1} = 2\nabla\varphi(\varrho)$ is called the drift velocity.

Appropriate unknowns: $\mathbf{w}$ and $\sqrt{(1-\kappa)\kappa}\mathbf{v_1}$.

If $\kappa = 1$, then we get the following system on $(\varrho, \mathbf{v})$:

$$\partial_t \varrho + \mathrm{div}(\varrho \mathbf{v}) - 2\Delta\mu(\varrho) = 0,$$
$$[KS] \quad \partial_t (\varrho \mathbf{v}) + \mathrm{div}(\varrho \mathbf{v} \otimes \mathbf{u}) - 2\,\mathrm{div}(\mu(\varrho)A(\mathbf{u})) + \nabla \Pi_1 = \mathbf{0},$$
$$\mathrm{div}\,\mathbf{v} = 0$$

with $\mathbf{u} = \mathbf{v} - 2\nabla\varphi(\varrho)$ (Note that $\mathbf{v}$ is divergence free).

$\Longrightarrow$ Kazhikhov-Smagulov type system

Global well posedness without asking any size constraint on the initial density !!

Proved by D.B., E. Hassan Essoufi, M. Sy '07

With the κ-entropy type estimate: More general results!!

If $\kappa \in (0,1)$, then $\mathbf{v} = u + 2\kappa\nabla\varphi(\rho)$ satisfies If $\kappa = 1$, then we get the following system on $(\varrho, \mathbf{v})$:

[*KSkappa*]
$$\begin{aligned} \partial_t \varrho + \mathrm{div}(\varrho \mathbf{v}) - 2\Delta\mu(\varrho) = 0, \\ \partial_t(\varrho \mathbf{v}) + \mathrm{div}(\varrho \mathbf{v} \otimes \mathbf{u}) - 2(1-\kappa)\,\mathrm{div}(\mu(\varrho)D(\mathbf{u})) \\ -2\kappa\,\mathrm{div}(\mu(\varrho)A(\mathbf{u})) + \nabla\Pi_1 = \mathbf{0}, \\ \mathrm{div}\,\mathbf{v} = 0 \end{aligned}$$

Is there an energy in such system ?

Take scalar product by $\mathbf{v}$ of the $\mathbf{v}$ equation, gives

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho|\mathbf{v}|^2 + 2(1-\kappa)\int_\Omega \mu(\rho)|D(u)|^2 - 4(1-\kappa)\kappa\int_\Omega \mu(\rho)D(u) : \nabla\nabla\varphi(\rho) = 0.$$

How to treat the last term ?

Write an equation on $\bar{v} = 2\sqrt{(1-\kappa)\kappa}\nabla\varphi(\rho)$!!!

Augmented system again !!!

Take the scalar product of the equation by $\bar{v}$ and add to the previous equality for provide nice energy.

Special case where φ and μ are related

For $0 < T < \infty$, $\Omega = \mathbb{T}^3$ and the low Mach number system

$$[M-NS] \quad \begin{aligned} &\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0, \\ &\partial_t (\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - 2\operatorname{div}(\mu(\varrho) D(\mathbf{u})) - \nabla(\lambda(\varrho) \operatorname{div} \mathbf{u}) + \nabla \Pi = \mathbf{0}, \\ &\operatorname{div} \mathbf{u} = -2\kappa \triangle \varphi(\varrho), \end{aligned}$$

with $\varphi'(s) = \mu'(s)/s$, $0 < \kappa < 1$ we have:

T1: For the initial conditions satisfying

$$\sqrt{(1-\kappa)\kappa \varrho^0} \in H^1(\Omega), \quad 0 < r \le \varrho^0 \le R < \infty, \quad \mathbf{u}^0 + 2\kappa \nabla \varphi(\rho^0) \in H,$$

$\mu(\varrho)$ such that $\mu(\varrho) \in C^1([r, R])$, $\mu'(\varrho) > 0$, $\mu \ge c > 0$ on $[r, R]$, and

$$\left(\frac{1-d}{d} \mu(\varrho) + \mu'(\varrho) \varrho \right) \ge c > 0.$$

There exists a global in time weak solution* to [M-NS].

T2: For $\kappa \to 0$ this solution converges to the weak solution of the non-homogenous incompressible N-S equations; for $\kappa \to 1$ (and $\kappa \varrho_\kappa^0 \in H^1(\Omega)$) it converges to the weak solutions of the Kazhikhov-Smagulov system.

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General case

For $T < \infty$, $\Omega = \mathbb{T}^3$ and the General low Mach number system

$$[M - NS - G] \quad \begin{aligned} \partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) &= 0, \\ \partial_t (\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - 2 \operatorname{div}(\mu(\varrho) D(\mathbf{u})) - \nabla(\lambda(\varrho) \operatorname{div} \mathbf{u}) + \nabla \Pi &= \mathbf{0}, \\ \operatorname{div} \mathbf{u} &= -2 \triangle \tilde{\varphi}(\varrho), \end{aligned}$$

with $\tilde{\varphi}'(s) = \tilde{\mu}'(s)/s$, we have:

T3: Under previous assumptions on the data and, for $\mu(\cdot) \in C^1([r,R])$, $\mu'(\cdot) > 0$, $\mu \geq c > 0$ on $[r,R]$ and $\tilde{\varphi}(\cdot) \in C^1([r,R])$ and $\mu(\varrho)$, $\tilde{\mu}(\varrho)$ related by

$$c \leq \min_{\varrho \in [r,R]} \left(\mu(\varrho) - \tilde{\mu}(\varrho) \right),$$

$$\max_{\varrho \in [r,R]} \frac{(\mu(\varrho) - \tilde{\mu}(\varrho) - \xi \tilde{\mu}(\varrho))^2}{2\,(\mu(\varrho) - \tilde{\mu}(\varrho))} \leq \xi \min_{\varrho \in [r,R]} \left(\tilde{\mu}'(\varrho)\varrho + \frac{1-d}{d} \tilde{\mu}(\varrho) \right).$$

for some positive constants c, ξ. There exists global weak solution to [M-NS-G].

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Remarks

- For $\mu(\varrho) = \varrho^{\alpha}$ the exponent α from T1 is

$$\alpha > 1 - \frac{1}{d}.$$

In particular, it does not depend on κ.

- Assume that all assumptions of T3 are satisfied and $\mu(\varrho)$, $\tilde{\mu}(\varrho)$ are replaced by

$$\mu(\varrho) = \varrho, \quad \tilde{\mu}(\varrho) = \log \varrho \quad (\textit{i.e.} \;\; \tilde{\varphi}(\varrho) = -1/\varrho).$$

Then there exist a non-empty interval $[\tilde{r}, \tilde{R}]$ such that if

$$0 < \tilde{r} \leq \varrho^{0} \leq \tilde{R} < 0,$$

then the weak solution to [M-NS-G] exists globally in time, which corresponds to the dense gas approximation:

S. Chapman and T.G. Cowling:
The mathematical theory of non-uniform gases, 1970.

$\Longrightarrow$ Generalization of P.–L. Lions's result to the 3d case!

Construction of solution

We consider the augmented regularized system with three unknowns $(\varrho, \mathbf{w}, \mathbf{v_1})$:

$$\partial_t \varrho + \mathrm{div}(\varrho \mathbf{w}) - 2\kappa \triangle \mu(\varrho) = 0,$$

$$\partial_t (\varrho \mathbf{w}) + \mathrm{div}((\varrho \mathbf{w} - 2\kappa \nabla \mu(\varrho)) \otimes \mathbf{w}) - 2(1-\kappa)\,\mathrm{div}(\mu(\varrho) D(\mathbf{w}))$$
$$- 2\kappa\,\mathrm{div}(\mu(\varrho) A(\mathbf{w})) + \nabla \Pi_1 + \varepsilon \triangle^2 \mathbf{w} = -2\kappa(1-\kappa)\,\mathrm{div}(\mu(\varrho)\nabla \mathbf{v_1}),$$

$$\partial_t(\varrho \mathbf{v_1}) + \mathrm{div}((\varrho \mathbf{w} - 2\kappa \nabla \mu(\varrho)) \otimes \mathbf{v_1}) - 2\kappa\,\mathrm{div}(\mu(\varrho)\nabla \mathbf{v_1})$$
$$- 2\kappa \nabla((\mu'(\varrho)\varrho - \mu(\varrho))\,\mathrm{div}\,\mathbf{v_1}) = -2\,\mathrm{div}(\mu(\varrho)\nabla^t \mathbf{w}),$$
$$\mathrm{div}\,\mathbf{w} = 0.$$

Of course, we have to prove that $\mathbf{v_1} = 2\nabla\varphi(\varrho)$ to solve the initial system.
To do that Important property: $\mathrm{div}\,\mathbf{w} = 0$.

Augmented system in other topics: See references given in the first part of the talks

For compressible NS equations (see later-on): an extra integrability needed
This is the term $-\varepsilon\,\mathrm{div}\left(|\nabla \mathbf{w}|^2 \nabla \mathbf{w}\right)$:
An hyper diffusive term introduced by O.A. Ladhyzenskaya.

Extensions to density dependent viscosities compressible NS equations

$$[CNS] \quad \begin{array}{c} \partial_t \varrho + \mathrm{div}(\varrho \mathbf{u}) = 0, \\ \partial_t (\varrho \mathbf{u}) + \mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - 2\,\mathrm{div}(\mu(\varrho) D(\mathbf{u})) - \nabla(\lambda(\varrho)\,\mathrm{div}\,\mathbf{u}) + \nabla P(\varrho) = \mathbf{0} \end{array}$$

with $\lambda(\varrho) = 2(\mu'(\varrho)\varrho - \mu(\varrho))$ (algebraic relation found by D.B., B. Desjardins).

$$P(\rho) = \rho^2/2, \qquad \mu(\varrho) = \mu\varrho \implies \text{Viscous shallow-water type system}$$

Let us introduce an arbitrary coefficient κ such that $0 < \kappa < 1$.

Extensions to density dependent viscosities compressible NS equations

For this compressible barotropic system we have generalized κ- entropy:

$$\int_\Omega \varrho\left(\frac{|\mathbf{w}|^2}{2}+(1-\kappa)\kappa\frac{|2\nabla\varphi(\varrho)|^2}{2}\right)(T)\,\mathrm{d}x+\int_\Omega \varrho e(\varrho)\,\mathrm{d}x$$
$$+2(1-\kappa)\int_0^T\int_\Omega \mu(\varrho)|D(\mathbf{u})|^2\,\mathrm{d}x\,\mathrm{d}t+2(1-\kappa)\int_0^T\int_\Omega(\mu'(\varrho)\varrho-\mu(\varrho))(\mathrm{div}\,\mathbf{u})^2\,\mathrm{d}x\,\mathrm{d}t$$
$$+2\kappa\int_0^T\int_\Omega \mu(\varrho)|A(\mathbf{w})|^2\,\mathrm{d}x\,\mathrm{d}t+2\kappa\int_0^T\int_\Omega\frac{\mu'(\varrho)p'(\varrho)}{\varrho}|\nabla\varrho|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$F\le\int_\Omega \varrho_0\left(\frac{|\mathbf{w}_0|^2}{2}+(1-\kappa)\kappa\frac{|2\nabla\varphi(\varrho_0)|^2}{2}\right)\mathrm{d}x+\int_\Omega \varrho_0 e(\varrho_0)\,\mathrm{d}x,$$

where we have introduced $e(\varrho)$ defined as

$$\frac{\varrho^2\mathrm{d}e(\varrho)}{\mathrm{d}\varrho}=p(\varrho).$$

κ-entropy may be used to construction of κ-entropy solutions with a simple construction scheme for the compressible system with extra terms (singular pressure or drag terms).

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Extensions to density dependent viscosities compressible NS equations

- Global κ-entropy weak solution of the CNS through the κ-entropy:
  Work D.B., B. Desjardins, E. Zatorska (JMPA 2015)
  Non-linear extension of hypocoercivity property known for linearized CNS.
- Interesting framework for asymptotics and numerics (κ relative entropy):
  CRAS 2015 with P. Noble and J.–P. Vila
  (various asymptotics through relative entropy)
  a) Inviscid shallow-water equation
  from the degenerate viscous shallow-water
  b) Low mach number and high Reynolds limit
  Appropriate schemes based on the continuous study: In progress.

Better rate in the asymptotic limits than with constant viscosities.

Thank you very much !!!

Please come back in Auvergne !!!