LABORATORY FOR HYDRAULICS SAINT-VENANT

Maxwell meets Saint-Venant

about viscoelastic flow models

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Outline

1 Motivation and setting of the problem

2 Saint-Venant models for shallow free-surface flows

3 Illustrative simple (1D) flows

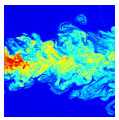
4 Finite-Volume discretization

5 2D numerical results

6 Directions

Flows of complex fluids (non-Newtonian/turbulent)

Consider flowing heterogeneous materials, with complex structure.
Do we have a good *fluid model* for a *continuum* ? [Maxwell 1867]



For turbulent water flows (eddies fluctuations) e.g.,

suspensions of *rigid* particles (volume fraction), or

$\Rightarrow$ suspensions of high-molecular-weight *deformable* particles ?

Find mass density $\rho(t, \boldsymbol{x})$, velocity $\boldsymbol{u}(t, \boldsymbol{x})$, internal energy $e(t, \boldsymbol{x})$, stress $\boldsymbol{\Sigma}(t, \boldsymbol{x})$ solution (at $T = cte$) to:

$$\partial_t \rho + \mathrm{div}(\rho \boldsymbol{u}) = 0$$

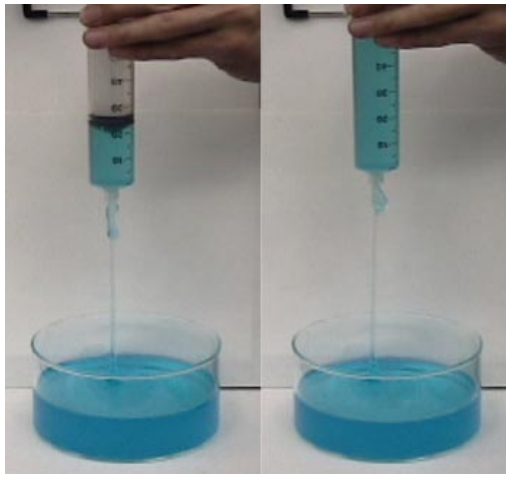
$$\partial_t(\rho \left(\frac{1}{2}|\boldsymbol{u}|^2 + e\right)) + \mathrm{div}(\rho \left(\frac{1}{2}|\boldsymbol{u}|^2 + e\right) \boldsymbol{u} - \boldsymbol{\Sigma} \cdot \boldsymbol{u}) = \boldsymbol{F} \cdot \boldsymbol{u} \cancel{+ \mathrm{div}\, q}$$

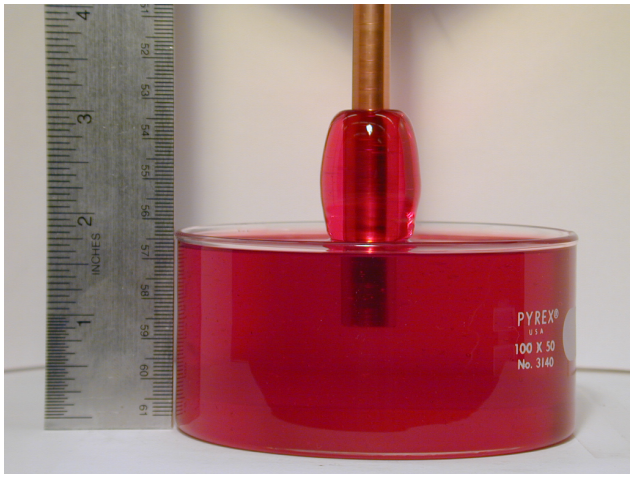
$$\rho\,(\partial_t e + (\boldsymbol{u} \cdot \boldsymbol{\nabla})e) = \boldsymbol{\Sigma} : \boldsymbol{\nabla u}$$

$$\partial_t(\rho \boldsymbol{u}) + \mathrm{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u} - \boldsymbol{\Sigma}) = \boldsymbol{F}$$

Polymer solutions: $\boldsymbol{\Sigma} \neq \nu \boldsymbol{D}(\boldsymbol{u})$

Extensional flows: hardening
(left image: tubeless open-syphon effect)





Shear flows: softening plus normal stresses
(right image: rod-climbing effect [Weissenberg 1947])

(Pictures from Gareth McKinley's group at MIT http://web.mit.edu/nnf/)

Problem: out-of-equilibrium transients, with shear and extension, in realistic times

$$\partial_t \rho + \mathrm{div}(\rho \boldsymbol{u}) = 0$$
$$\partial_t (\rho \boldsymbol{u}) + \mathrm{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u} - \boldsymbol{\Sigma}) = 0$$

- Euler gas dynamics $\boldsymbol{\Sigma} = -p(\rho, T)\boldsymbol{I}$ (*symmetric hyperbolic*); shear & extension hardly modelled
- Navier-Stokes $\boldsymbol{\Sigma} = -p\boldsymbol{I} - \lambda\, \mathrm{div}\, \boldsymbol{u}\, \boldsymbol{I} + 2\mu \boldsymbol{D}(\boldsymbol{u})$ dissipative stress *equilibrium* with strain $\boldsymbol{E} \approx \boldsymbol{D}(\boldsymbol{u})$

$\Rightarrow$ Relax $\boldsymbol{\Sigma}$ or $\boldsymbol{E}$ to capture more physics (like Maxwell, Jeffrey, Wiechert…) at *large deformations* with a *hyperbolic model*

$$\rho\,(\partial_t e + (\boldsymbol{u}\cdot\boldsymbol{\nabla})e) = \boldsymbol{\Sigma} : \boldsymbol{\nabla}\boldsymbol{u}$$
$$\partial_t(\rho\left(\frac{1}{2}|\boldsymbol{u}|^2 + e\right)) + \mathrm{div}(\rho\left(\frac{1}{2}|\boldsymbol{u}|^2 + e\right)\boldsymbol{u} - \boldsymbol{\Sigma}\cdot\boldsymbol{u}) = 0$$

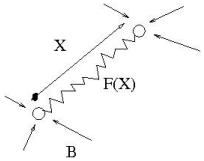
Viscoelastic models for Maxwell fluids

A standard model for polymers is Oldroyd-B ([Maxwell 1867] is 1D):

$\operatorname{div} \boldsymbol{u} = 0,\ \rho \equiv \rho_\infty,\ \boldsymbol{\Sigma} = -p\boldsymbol{I} + \boldsymbol{\tau},\quad \boldsymbol{\tau} = \boldsymbol{\Sigma}_s + \boldsymbol{\Sigma}_p,\quad \boldsymbol{\Sigma}_s = 2\rho_\infty \nu_s \boldsymbol{D}(\boldsymbol{u})$ and

$$\lambda \overset{\nabla}{\boldsymbol{\Sigma}}_p + \boldsymbol{\Sigma}_p = 2\rho_\infty \nu_p\, \boldsymbol{D}(\boldsymbol{u}) \Rightarrow \lambda \overset{\nabla}{\boldsymbol{\tau}} + \boldsymbol{\tau} = \nu_s \lambda\, \overset{\nabla}{\boldsymbol{D}}(\boldsymbol{u}) + (\nu_s + \nu_p)\boldsymbol{D}(\boldsymbol{u})\,. \qquad (1)$$

Elastic (extra-)stresses $\boldsymbol{\Sigma}_p = G(\boldsymbol{C} - \boldsymbol{I})$ interpreted e.g. with the *configuration* $\boldsymbol{C} = \mathbb{E}(X_t \otimes X_t)$ of a random material vector $X_t(x)$



$$dX_t + ((u \cdot \nabla)X_t - (\nabla u)X_t + (2G/\zeta)X_t)\, dt = \sqrt{k_B T/\zeta}\, dB_t$$

$\equiv$ Brownian "hookean dumbbells" with *elasticity modulus* G, and *relaxation time* λ.

When $\frac{1}{\lambda} = \frac{4G}{\zeta} = \frac{2k_B T}{\zeta}$, $G\lambda = \rho_\infty \nu_p$ (1) rewrites $\overset{\nabla}{\boldsymbol{C}} = (\boldsymbol{I} - \boldsymbol{C})/\lambda$

with *upper-convected* time-rate $\overset{\nabla}{\boldsymbol{C}} := \partial_t \boldsymbol{C} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{C} - \nabla \boldsymbol{u}\, \boldsymbol{C} - \boldsymbol{C}\, \nabla \boldsymbol{u}^T$

– like *left* Cauchy-Green deformation tensor $\boldsymbol{F}\boldsymbol{F}^T$ s.t. $D_t \boldsymbol{F} = \nabla \boldsymbol{u}\, \boldsymbol{F}$ –.

High Weissenberg Number Problem (HWNP)

- Not all physics is in (1), but (1) contains the essence of most up-to-date Maxwell models like FENE-P, or Giesekus

$$\lambda \overset{\nabla}{\boldsymbol{\Sigma}}_p + \boldsymbol{\Sigma}_p + \alpha \boldsymbol{\Sigma}_p : \boldsymbol{\Sigma}_p = 2\nu_p \boldsymbol{D}(\boldsymbol{u}) \qquad (2)$$

- FENE-P or Giesekus (2) coupled with *incompressible Navier-Stokes* $D_t \boldsymbol{u} = \operatorname{div} \boldsymbol{\Sigma}$ has global weak *dissipative* solutions [Masmoudi 2011],
  but numerical computations remain difficult when $\lambda \geq L/U$

Results for the drag coefficient K compared to results from literature. The values from literature are always the finest mesh results. In the case of [11], the MIX0 results are utilized.

Wi	K							
	M1	M2	M3	M4	M5	[9]	[10]	[11]
0.1	130.3706	130.3613	130.3620	130.3625	130.3626	130.363	130.364	130.36
0.2	126.6609	126.6288	126.6254	126.6252	126.6252	126.626	126.626	126.62
0.3	123.2622	123.2008	123.1922	123.1913	123.1912	123.193	123.192	123.19
0.4	120.6953	120.6080	120.5931	120.5914	120.5912	120.596	120.593	120.59
0.5	118.9615	118.8505	118.8291	118.8263	118.8260	118.836	118.826	118.83
0.6	117.9542	117.8048	117.7798	117.7756	117.7752	117.775	117.776	117.78
0.7	117.5430	117.3416	117.3193	117.3155	117.3157	117.315	117.316	117.32
0.75	117.5108	117.2940	117.2747	117.2733	117.2752			
0.8	117.5639	117.3539	117.3365	117.3395	117.3454	117.373	117.368	117.36
0.85	117.6809	117.5116	117.4925	117.5016	117.5138			
0.88	117.7743	117.6495	117.6265	117.6402	117.6567			
0.89	117.8085	117.7022	117.6774	117.6927	117.7107			
0.9	117.8442	117.7584	117.7312	117.7483	117.7678	117.787	117.812	117.80

Numerical results from Knechtges et al. JNNFM 2014, 214

The flow past a cylinder benchmark

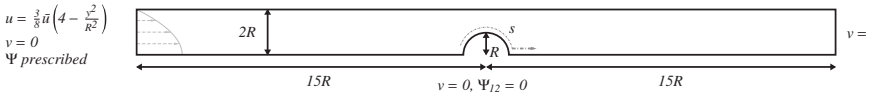


Fig. 2. Sketch of the used geometry including the boundary conditions.

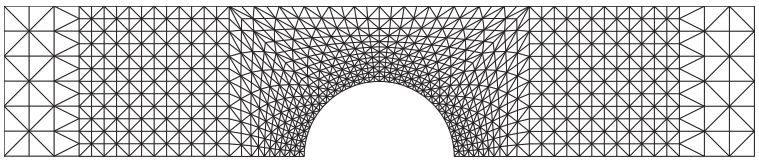
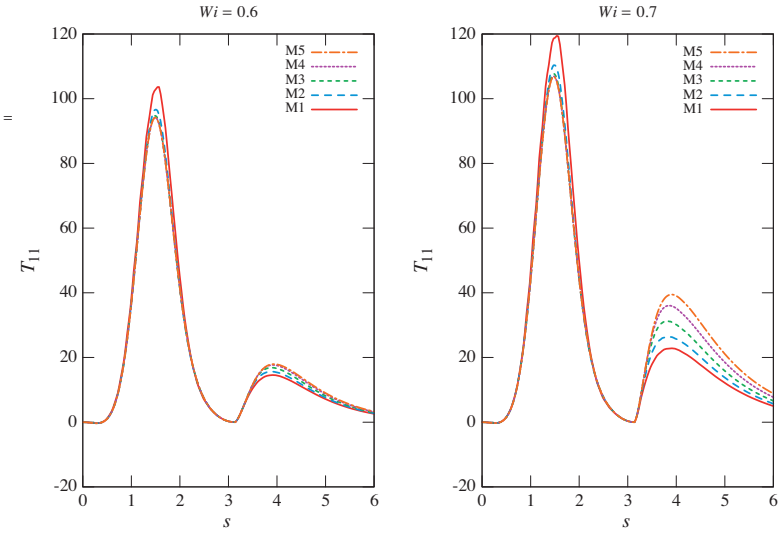


Fig. 3. Excerpt of the Mesh M1.



. The T_{11} component of the polymeric stress along and in the wake of the cylinder for different Weissenberg numbers.

Numerical results from Knechtges et al. JNNFM 2014, 214, 78-87

This work: zero retardation-time in Saint-Venant framework

- Numerical models typically use $\nu_s > 0$ and div $\boldsymbol{u} = 0$

$\Rightarrow$ We discretize O-B (FENE-P, Giesekus etc.) for $\nu_s \geq 0$ explicitly i.e. without iterative algorithm for the approximation of nonlinearities as fixed points

$\Rightarrow$ We consider possibly *non-solenoidal velocities* solutions to *quasilinear* first-order systems with mass and energy conservation: *Saint-Venant* 2D framework for hydrostatic (shallow) free-surface flows

1. When $\nu_s = 0$, Oldroyd-B = "Upper-convected Maxwell"
2. [Phelan et al. 1989, Beris et al. 1990] consider *hyperbolic* quasilinear systems for *compressible* Maxwell fluids, but without exact mass conservation nor a second-principle formulation

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2D thin homogeneous layer $H(t,x,y) > 0$

Assume a 2D stratified flow with a free surface under kinematic condition, then a shallow-water model holds [Saint-Venant 1871]

$$\partial_t H + \operatorname{div}(H\boldsymbol{U}) = 0 \tag{3}$$

$$\partial_t(H\boldsymbol{U}) + \operatorname{div}(H\boldsymbol{U}\otimes\boldsymbol{U} + H(P+\Sigma_{zz})\boldsymbol{I} - H\boldsymbol{\Sigma}_h) = -kH\boldsymbol{U} \tag{4}$$

where the (specific) Cauchy stress terms read $P = gH/2$ and $\boldsymbol{\Sigma}_h = 2\nu D(\boldsymbol{U})$, $\Sigma_{zz} = -(\Sigma_{xx}+\Sigma_{yy})$ for a Newtonian fluid with viscosity $\nu \geq 0$ on a flat plane with rugosity $k \geq 0$.

$$\partial_t(HE) + \operatorname{div}(HE\boldsymbol{U} + H(P+\Sigma_{zz})\boldsymbol{U} - H\boldsymbol{\Sigma}_h\cdot\boldsymbol{U}) = -kH|\boldsymbol{U}|^2 - HD$$

moreover holds with $D = 2\nu\left(|D(\boldsymbol{U})|^2 + 2|\operatorname{div}\boldsymbol{u}|^2\right) \geq 0$ for $E = \frac{1}{2}\left(|\boldsymbol{U}|^2 + gH\right)$: (3–4) symmetric *hyperbolic* when $\nu = 0$, but sustains shear only when $\nu > 0$ with strain at *infinite speed*.

Viscoelastic Saint-Venant: SV-UCM

(3–4) approximates depth-averaged free-surface *viscous* Navier-Stokes equations [Gerbeau-Perthame 2001].
Depth-averaged *viscoelastic* NS equations are approximated e.g. with $\boldsymbol{\Sigma} = G(\boldsymbol{C} - \boldsymbol{I})$ by [Bouchut-Boyaval 2016]:

$$D_t \boldsymbol{C}_h - \nabla \boldsymbol{U} \boldsymbol{C}_h - \boldsymbol{C}_h \nabla \boldsymbol{U}^T = (\boldsymbol{I} - \boldsymbol{C}_h)/\lambda \tag{5}$$

$$D_t C_{zz} + 2 C_{zz} \operatorname{div} \boldsymbol{U} = (1 - C_{zz})/\lambda \tag{6}$$

$E = \left(|\boldsymbol{U}|^2 + gH + G \operatorname{tr}(\boldsymbol{C} - \ln \boldsymbol{C} - \boldsymbol{I}) \right)/2$ satisifies

$$\partial_t (HE) + \operatorname{div}(HE\boldsymbol{U} + H(P + \Sigma_{zz})\boldsymbol{U} - H\boldsymbol{\Sigma}_h \cdot \boldsymbol{U}) \leq -kH|\boldsymbol{U}|^2 - HD$$

i.e. a 2nd-principle with $D := G(\operatorname{tr} \boldsymbol{C} + \operatorname{tr} \boldsymbol{C}^{-1} - 2 \operatorname{tr} \boldsymbol{I})/(2\lambda) > 0$, on $\{H, C_{zz}, \boldsymbol{C}_h = \boldsymbol{C}_h^T > 0\}$ where *hyperbolicity holds* !

Viscoelastic Saint-Venant: SV-UCM

We will next consider "entropy" solutions to
the SV-UCM quasilinear viscoelastic-flow system

$$\partial_t H + \operatorname{div}(H\boldsymbol{U}) = 0$$
$$\partial_t (H\boldsymbol{U}) + \operatorname{div}(H\boldsymbol{U}\otimes\boldsymbol{U} + gH^2/2 + GHC_{zz} - GH\boldsymbol{C}_h) = -kH\boldsymbol{U}$$
$$\partial_t \boldsymbol{C}_h + \boldsymbol{U}\cdot\nabla\boldsymbol{C}_h - \nabla\boldsymbol{U}\boldsymbol{C}_h - \boldsymbol{C}_h\nabla\boldsymbol{U}^T = (\boldsymbol{I} - \boldsymbol{C}_h)/\lambda$$
$$\partial_t C_{zz} + \boldsymbol{U}\cdot\nabla C_{zz} + 2C_{zz}\operatorname{div}\boldsymbol{U} = (1 - C_{zz})/\lambda$$

hyperbolic on $\{H, C_{zz}, \boldsymbol{C}_h = \boldsymbol{C}_h^T > 0\}$. 1D-x eigenvalues read:

$$u - \sqrt{gh + 3G(C_{zz} + C_{xx})} \le u - \sqrt{GC_{xx}} \le u\,(\times 3) \le u + \sqrt{GC_{xx}} \le u + \sqrt{gh + 3G(C_{zz} + C_{xx})}\,.$$

It does not have a conservative form, but the dissipation of
$E = \left(|\boldsymbol{U}|^2 + gH + G\operatorname{tr}(\boldsymbol{C} - \ln\boldsymbol{C} - \boldsymbol{I})\right)/2$ helps hyperbolicity.

Viscoelastic Saint-Venant: SV-TM

Next, we also consider "entropy" solutions to the SV-TM quasilinear viscoelastic-flow system

$$
\begin{aligned}
&\partial_t H + \operatorname{div}(H\boldsymbol{U}) = 0\\
&\partial_t(H\boldsymbol{U}) + \operatorname{div}(H\boldsymbol{U}\otimes\boldsymbol{U} + gH^2/2 - GHC_{zz} + GH\boldsymbol{C}_h) = -kH\boldsymbol{U}\\
&\partial_t \boldsymbol{C}_h + \boldsymbol{U}\cdot\nabla\boldsymbol{C}_h + \nabla\boldsymbol{U}\boldsymbol{C}_h + \boldsymbol{C}_h\nabla\boldsymbol{U}^T = (\boldsymbol{I}-\boldsymbol{C}_h)/\lambda\\
&\partial_t C_{zz} + \boldsymbol{U}\cdot\nabla C_{zz} - 2C_{zz}\operatorname{div}\boldsymbol{U} = (1-C_{zz})/\lambda
\end{aligned}
$$

hyperbolic on $\{H, C_{zz}, \boldsymbol{C}_h = \boldsymbol{C}_h^T > 0\}$ with same eigenvalues.

The same (dissipative) companion law for the free-energy $E = \left(|\boldsymbol{U}|^2 + gH + G\operatorname{tr}(\boldsymbol{C} - \ln\boldsymbol{C} - \boldsymbol{I})\right)/2$ holds that can be required as an *inequality*.

Viscoelastic Saint-Venant: 2 models

- both models contain the 1D model of [Bouchut B. 2013]
- SV-UCM preserves mass unlike [Phelan et al. 1989,Berris et al. 1990]
- SV-TM is a RANS model [Teshukov 2007] for Reynolds stresses $\boldsymbol{\Sigma}_h = G(\boldsymbol{I} - \boldsymbol{C}_h)$, $\Sigma_{zz} = G(1 - C_{zz})$ built with the covariances $\boldsymbol{C}_h$, C_{zz} of velocity fluctuations $\boldsymbol{u}' = \sqrt{G}\boldsymbol{F}$

$$d\boldsymbol{F} = \left(-(\boldsymbol{U} \cdot \boldsymbol{\nabla})\boldsymbol{F} - \boldsymbol{L}\boldsymbol{F} - \frac{1}{\lambda}\boldsymbol{F} \right) dt + \frac{1}{2\sqrt{\lambda}} d\boldsymbol{B}_t \qquad (7)$$

Our free-energy is different than [Ivanova-Gavrilyuk 2017]

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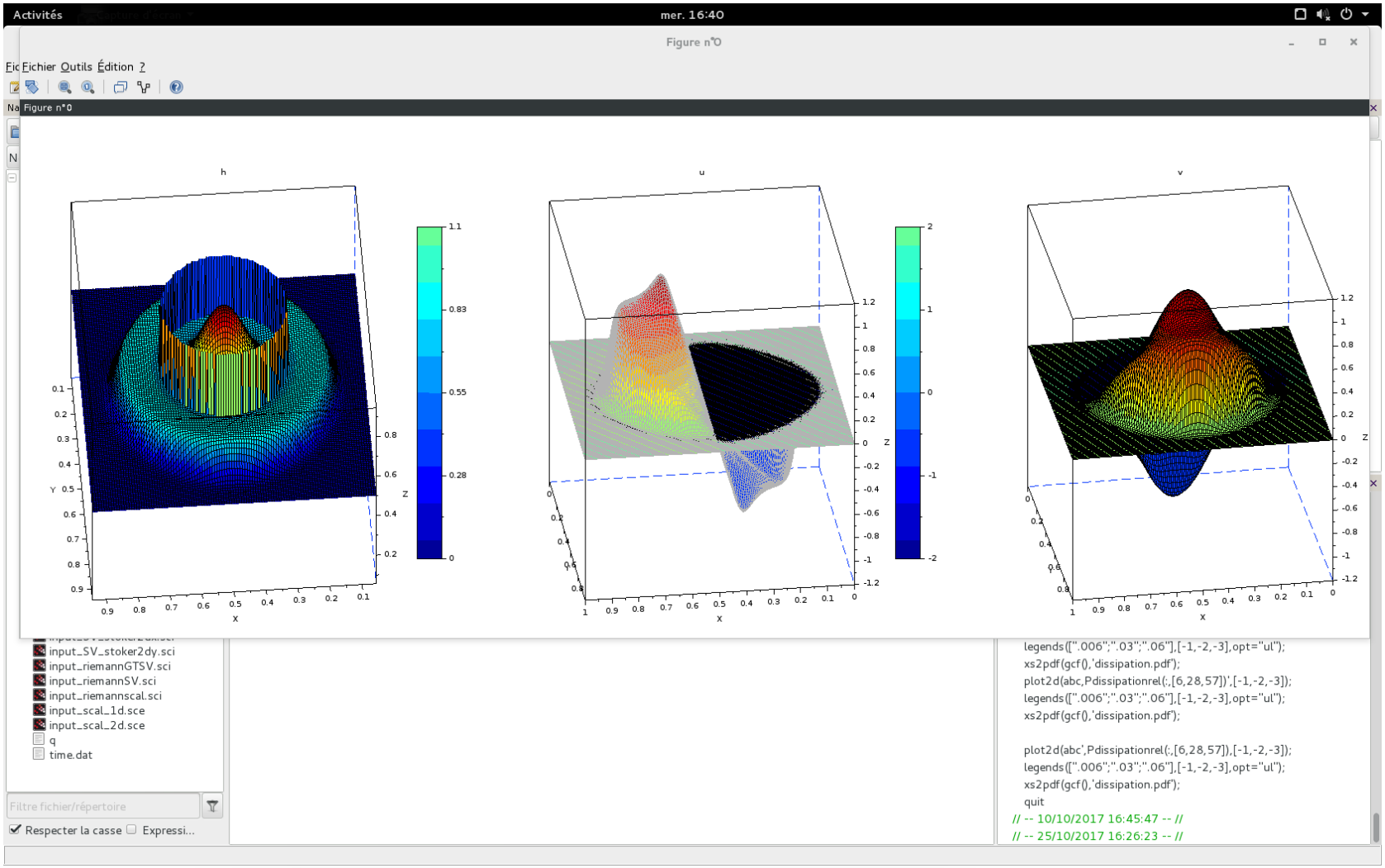
Results obtained with a Scilab FV code

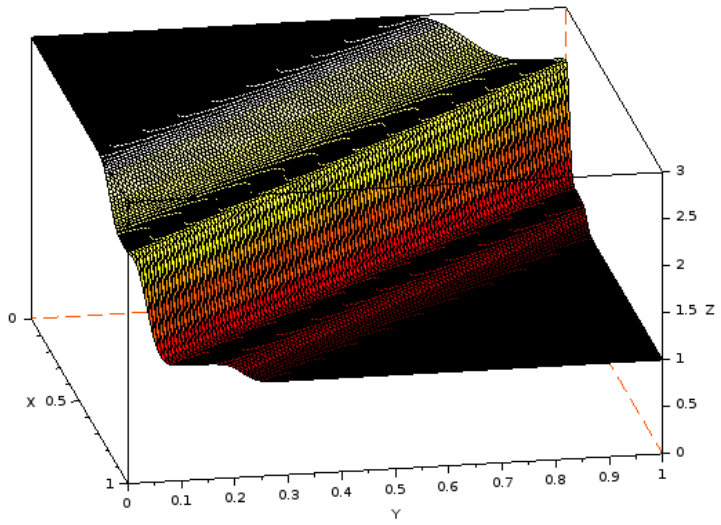


Stoker “dam-break” benchmark test case

Compute a solution for $t \in (0, .2)$ in $(x, y) \in [0, 1]^2$ starting from

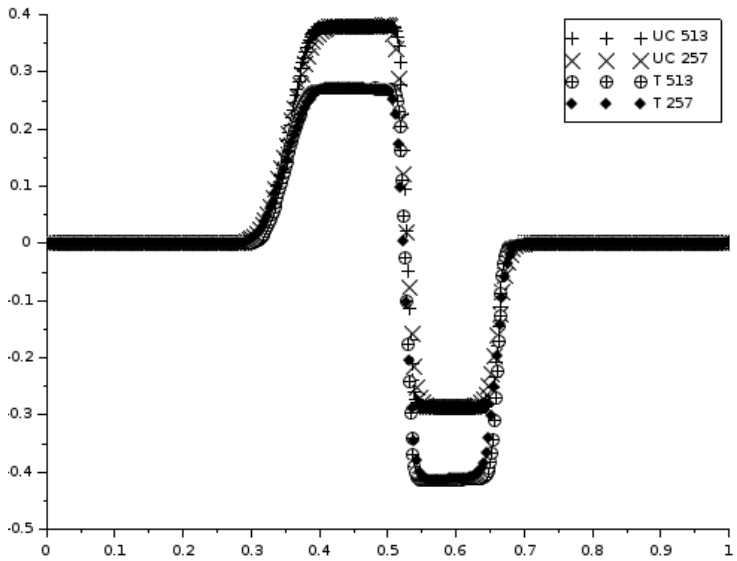
$$(H, U, V, C_{xx}, C_{yy}, C_{xy}, C_{zz}) = \begin{cases} (3, 0, 0, 1, 1, 0, 1) & x + y < 1 \\ (1, 0, 0, 1, 1, 0, 1) & x + y > 1 \end{cases}$$

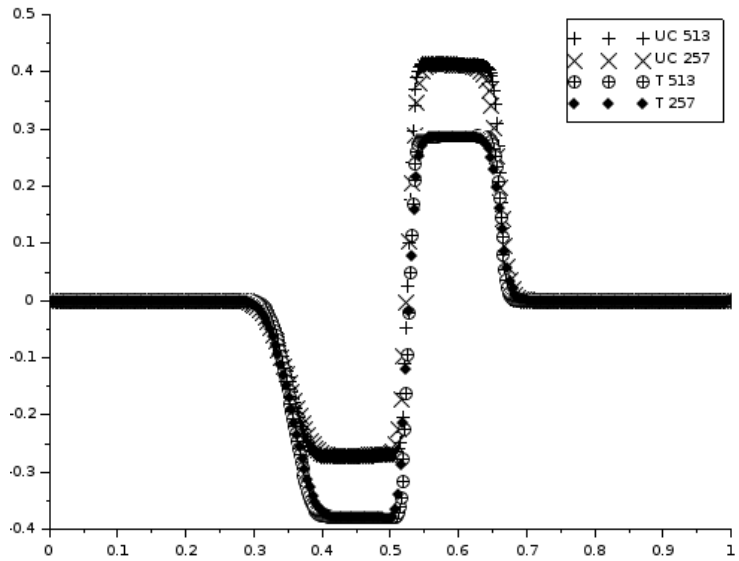
SV-UCM Depth H
$T = .2$
Froude $g^{-1/2} = .3$
Elasticity $G = 10 \approx g$
Weissenberg $\lambda = 1 \gg T$



Comparing SVTM and SVUCM

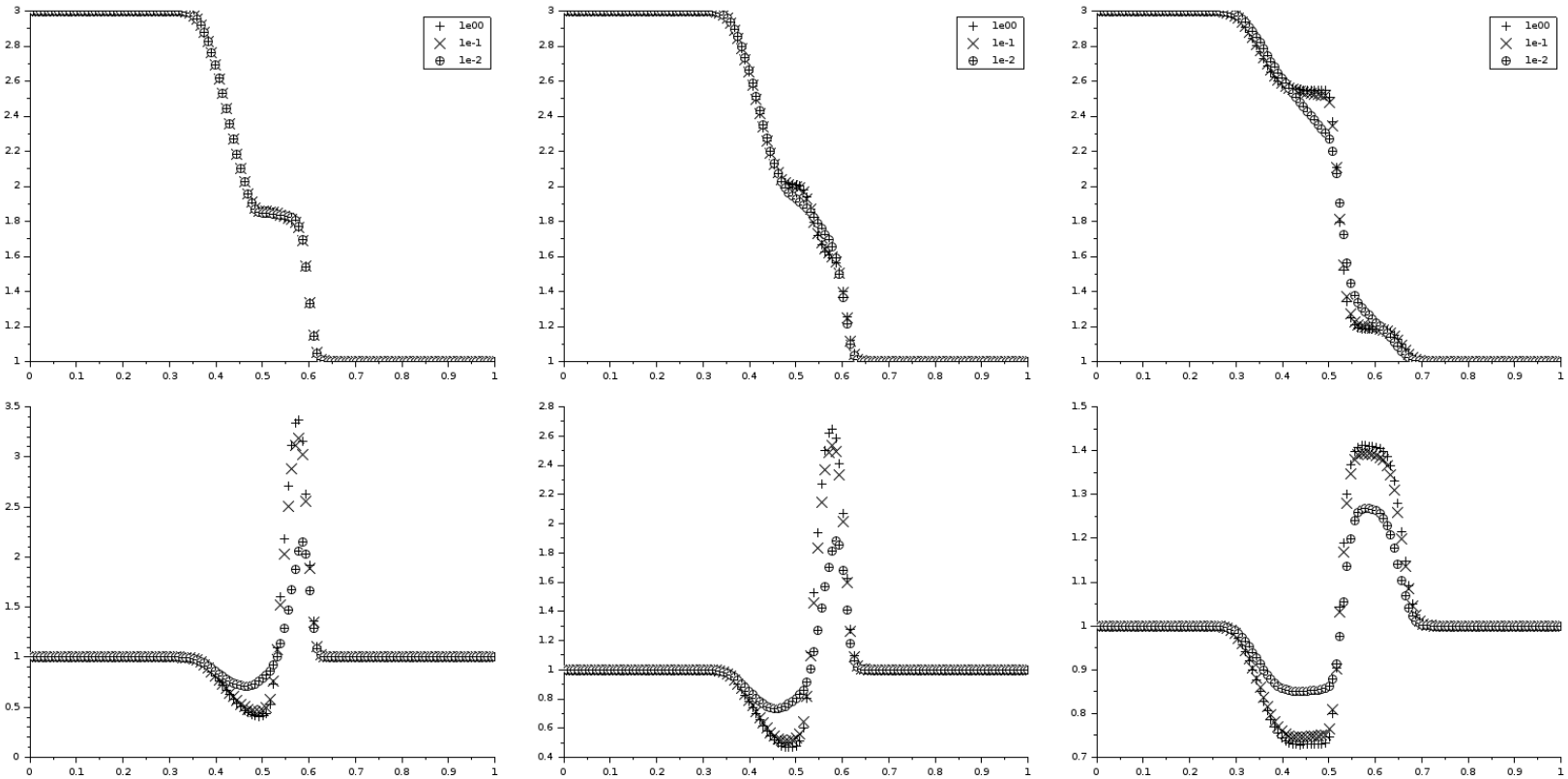
Stress components Σ_{nn}, Σ_{zz} (left/right) at $T = .2$ along normal $\boldsymbol{n}$ to $x + y = 1$ (G units)





Varying elasticity $G = .1, 1, 10$ at $g = 10$

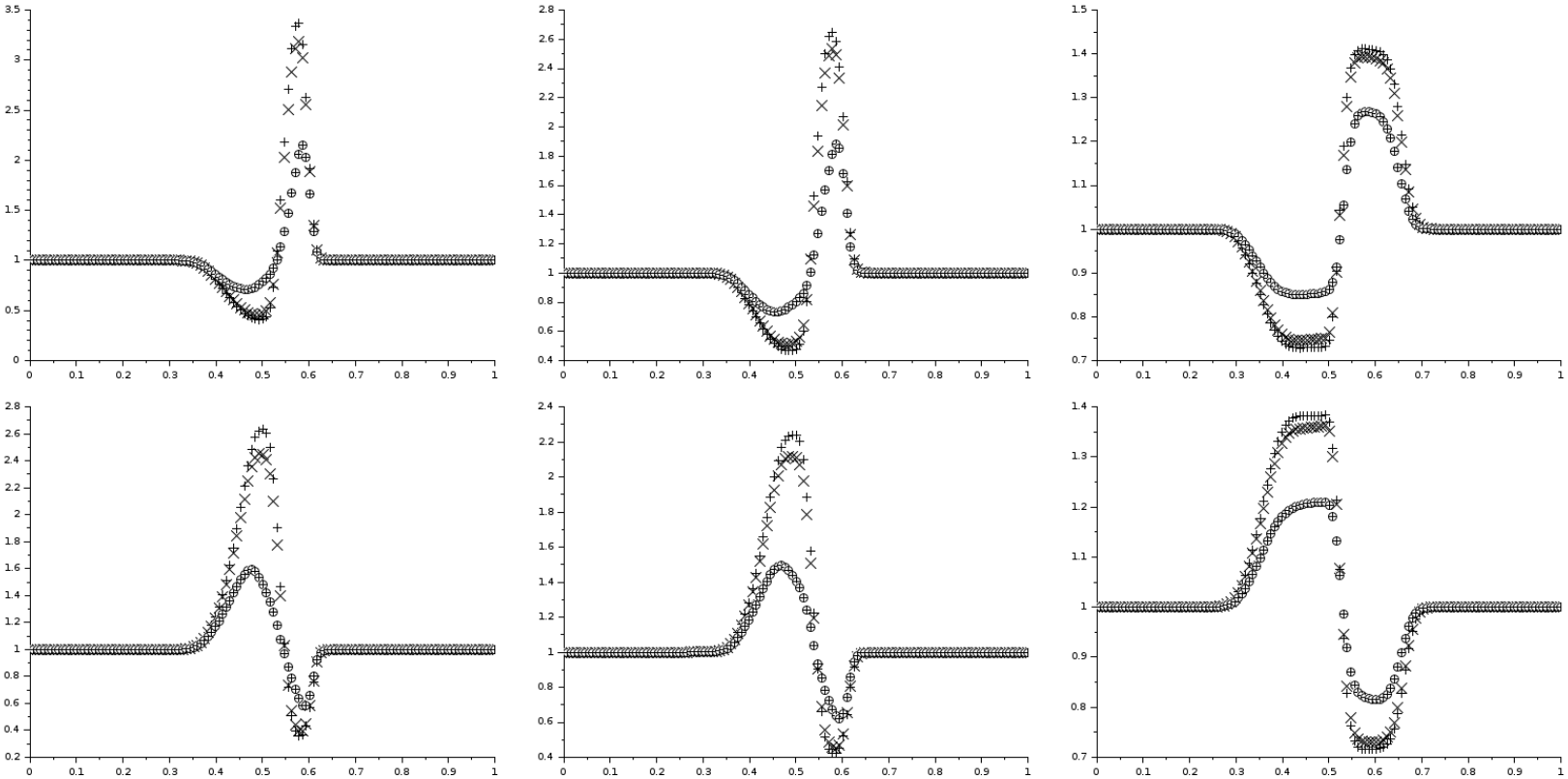
Depth H (top) and strain C_{xx} (bottom) at $T = .2$ for $\lambda = .01, .1, 1$



SVTM

Varying elasticity $G = .1, 1, 10$ at $g = 10$

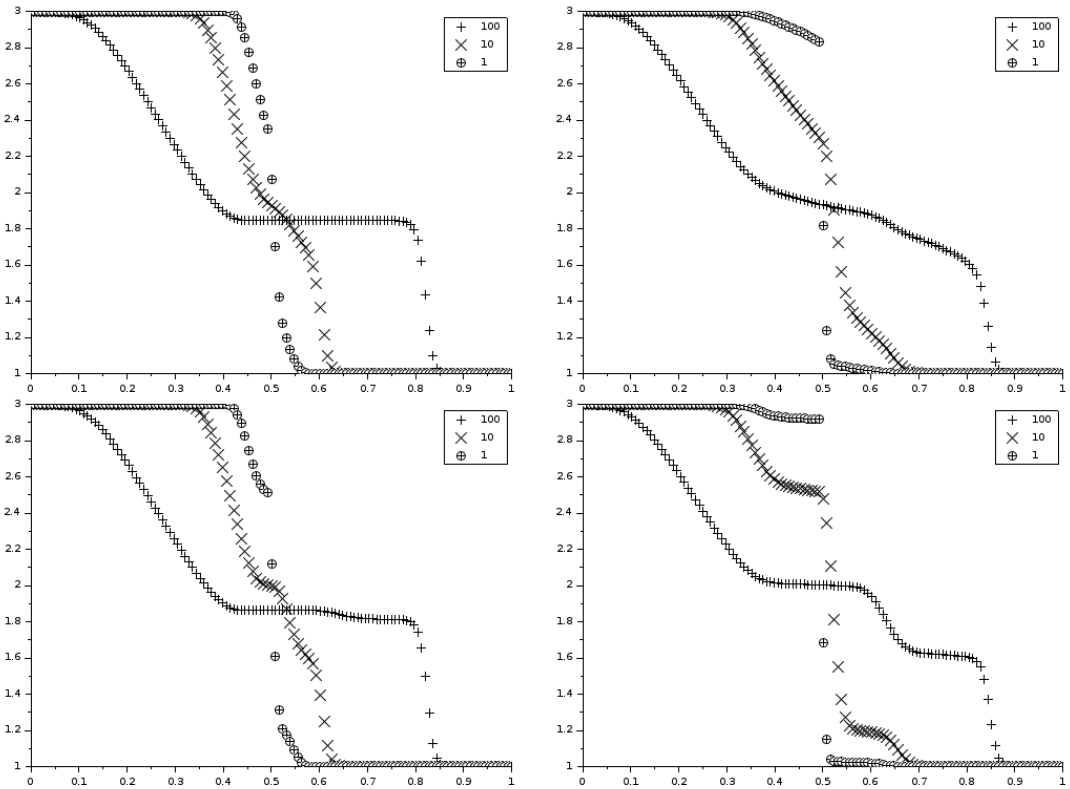
Strain C_{xx} (top) and C_{zz} (bottom) at $T = .2$ for $\lambda = .01, .1, 1$



SVTM

Varying Froude g

Depth H at $T = .2$ for $\lambda = .01$ (top) and .1 (bottom) with elasticity $G = 1$ (left) and 10 (right)



SVTM

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Entropy solutions for SV-UCM/SV-TM

Let's consider numerical approximations to Cauchy problems on $(t, \boldsymbol{x}) \in [0, T) \times \mathbb{R}^2$ for $H, C_{zz}, \boldsymbol{C}_h = \boldsymbol{C}_h^T > 0$

$$
\begin{aligned}
&\partial_t H + \operatorname{div}(H\boldsymbol{U}) = 0 \\
&\partial_t (H\boldsymbol{U}) + \operatorname{div}(H\boldsymbol{U} \otimes \boldsymbol{U} + H(gH/2 + \Sigma_{zz})\boldsymbol{I} - H\boldsymbol{\Sigma}_h) = -kH\boldsymbol{U} \\
&D_t \boldsymbol{C}_h \pm (\boldsymbol{\nabla U C}_h + \boldsymbol{\nabla U L}_h^T) = (\boldsymbol{I} - \boldsymbol{C}_h)/\lambda\,, \\
&D_t C_{zz} \mp 2 C_{zz} \operatorname{div} \boldsymbol{U} = (1 - C_{zz})/\lambda\,,
\end{aligned}
$$

with $\boldsymbol{\Sigma}_h = \pm G(\boldsymbol{I} - \boldsymbol{C}_h)$, $\Sigma_{zz} = \pm G(1 - C_{zz})$ and

$$\partial_t (HE) + \operatorname{div}(HE\boldsymbol{U} + H(P + \Sigma_{zz})\boldsymbol{U} - H\boldsymbol{\Sigma}_h \cdot \boldsymbol{U}) \leq -kH|\boldsymbol{U}|^2 - HD$$

$$2E = |\boldsymbol{U}|^2 + gH + G \operatorname{tr}(\boldsymbol{C} - \ln \boldsymbol{C} - \boldsymbol{I}),\ D = \frac{G}{2\lambda}(\operatorname{tr} \boldsymbol{C} + \operatorname{tr} \boldsymbol{C}^{-1} - 2 \operatorname{tr} \boldsymbol{I})$$

Finite-Volume (FV) – time-splitting

Consider *piecewise constant* FV approx. $q_h = \sum_i q_i(t) 1_{V_i}$ of state vector $q \in \mathbb{R}^7$ using polygonal cells V_i of a mesh of $\mathbb{R}^2$.

For each $[t^n, t^{n+1})$, first approximate the *homogeneous* system

$$q_i^{n+1,-} = q_i^n - \tau^n \sum_{\Gamma_{ij} \equiv \overline{V_i} \cap \overline{V_j} \neq \emptyset} \frac{|\Gamma_{ij}|}{|V_i|} \boldsymbol{F}_{i\to j}(q_i^n, q_j^n; \boldsymbol{n}_{i\to j}) \qquad (8)$$

in a *FV-consistent* sense for conservation laws

(i.e. it holds $\boldsymbol{F}^{H,H\boldsymbol{U}}_{i\to j}(q_i, q_j; \boldsymbol{n}_{i\to j}) + \boldsymbol{F}^{H,H\boldsymbol{U}}_{j\to i}(q_j, q_i; \boldsymbol{n}_{j\to i}) = 0$
and $\boldsymbol{F}^{H,H\boldsymbol{U}}_{i\to j}(q_i, q_j; \boldsymbol{n}_{i\to j}) = \boldsymbol{F}^{H,H\boldsymbol{U}}(q) \cdot \boldsymbol{n}_{i\to j}$ when $q = q_i = q_j$).

Rotation invariance: $\boldsymbol{F}_{i\to j}(q_i, q_j; \boldsymbol{n}_{i\to j}) = \boldsymbol{O}_{i,j} \tilde{\boldsymbol{F}}(\boldsymbol{O}_{i,j}^{-1} q_i, \boldsymbol{O}_{i,j}^{-1} q_j)$

with a flux $\tilde{\boldsymbol{F}}$ for *1D Riemann* problems that is consistent

$$\tilde{\boldsymbol{F}}_{i\to j}(q_l, q_r) = \boldsymbol{F}(q_l)\boldsymbol{n} - \int_{-\infty}^{0} (R(\xi, q_l, q_r) - q_l)\, d\xi$$

Sub-step 1: CFL and entropy-consistency

Under CFL on $\tau^n \equiv t^{n+1} - t^n$, sub-step 1 is a convex combination

$$q_i^{n+1,-} = \sum_j \frac{|\Gamma_{ij}|}{|V_i|} \int_{-\left(\sum_j \frac{|\Gamma_{ij}|}{|V_i|}\right)^{-1}}^{0} \boldsymbol{O}_{i,j} R(x/\tau^n, \boldsymbol{O}_{i,j}^{-1} q_i^n, \boldsymbol{O}_{i,j}^{-1} q_j^n) dx \,,$$

- which preserves domains convex in q if R does, and
- which is consistent with $\partial_t HE + \text{div}\, \boldsymbol{G} \leq 0$

$$HE(q_i^{n+1,-}) - HE(q_i^n) + \tau^n \sum_j \frac{|\Gamma_{ij}|}{|V_i|} \tilde{G}(\boldsymbol{O}_{i,j}^{-1} q_i^n, \boldsymbol{O}_{i,j}^{-1} q_j^n) \leq 0 \tag{9}$$

if i) HE is convex in q, and if ii) R is "entropic":

$$\exists \tilde{G} \text{ such that } \tilde{G}(\boldsymbol{O}_{i,j}^{-1} q, \boldsymbol{O}_{i,j}^{-1} q) = \boldsymbol{G}(\boldsymbol{O}_{i,j}^{-1} q) \cdot \boldsymbol{e}_x = \boldsymbol{G}(q) \cdot \boldsymbol{n}_{i \to j}$$

$$\boldsymbol{G}(q_r) \cdot \boldsymbol{e}_x + \int_0^{+\infty} \left(HE\left(R(\xi, q_l, q_r)\right) - HE(q_r)\right) d\xi \leq \tilde{G}(q_l, q_r) \leq \boldsymbol{G}(q_l) \cdot \boldsymbol{e}_x - \int_{-\infty}^{0} \left(HE\left(R(\xi, q_l, q_r)\right) - HE(q_l)\right) d\xi \,.$$

Sub-step 2: Backward-Euler for sources

$$\boldsymbol{U}(q_i^{n+1}) - \boldsymbol{U}(q_i^{n+1,-}) = -\tau^n k \boldsymbol{U}(q_i^{n+1})$$
$$\boldsymbol{C}_h(q_i^{n+1}) - \boldsymbol{C}_h(q_i^{n+1,-}) = \tau^n \left(\boldsymbol{I} - \boldsymbol{C}_h(q_i^{n+1}) \right) / \lambda$$
$$C_{zz}(q_i^{n+1}) - C_{zz}(q_i^{n+1,-}) = \tau^n \left(1 - C_{zz}(q_i^{n+1}) \right) / \lambda$$

given $H_i^{n+1} = H_i^{n+1,-}$ and τ^n finally yields an explicit *admissible* state $\boldsymbol{U}(q_i^{n+1}), \boldsymbol{C}_h(q_i^{n+1}), C_{zz}(q_i^{n+1})$ satisfying

$$HE(q_i^{n+1}) - HE(q_i^n) + \tau^n \sum_j \frac{|\Gamma_{ij}|}{|V_i|} \tilde{G}(\boldsymbol{O}_{i,j}^{-1} q_i^n, \boldsymbol{O}_{i,j}^{-1} q_j^n)$$
$$\leq -\tau^n k |\boldsymbol{U}_i^{n+1}|^2 - \tau^n D(q_i^{n+1}) \quad (10)$$

using (9), and *HE* convexity [Barrett B. 2011]

1D Riemann solver construction (sub-step 1)

We build entropy-consistent *approximate* (simple) solutions to 1D Riemann problems for the nonconservative system

$$\begin{cases} \partial_t h + \partial_x(hu) = 0 \\ h\left(\partial_t u + u\partial_x u\right) + \partial_x\left(gh + G(c_{xx} - c_{zz})\right) = 0 \\ h\left(\partial_t v + u\partial_x v\right) + \partial_x\left(Gc_{xy}\right) = 0 \\ \partial_t c_{xx} + u\partial_x c_{xx} + 2c_{xx}\partial_x u = 0 \\ \partial_t c_{yy} + u\partial_x c_{yy} + 2c_{xy}\partial_x v = 0 \\ \partial_t c_{xy} + u\partial_x c_{xy} + c_{xy}\partial_x u + c_{xx}\partial_x v = 0 \\ \partial_t c_{zz} + u\partial_x c_{zz} - 2c_{zz}\partial_x u = 0 \end{cases}$$

rewritten with the variable

$$\tilde{q} := \left(h, hu, h^{-2}c_{xx}, h^2 c_{zz}, h(c_{yy} - c_{xy}^2/c_{xx}), hc_{xy}/\sqrt{c_{xx}}, hv\right)$$

A 5-wave simple solver

We build a 5-wave Riemann solver of Suliciu type for

$$\begin{cases} \partial_t h + \partial_x(hu) = 0 \\ \partial_t(hu) + \partial_x(hu^2) + \partial_x(gh^2/2) + \partial_x\left(Gh(c_{xx} - c_{zz})\right) = 0 \\ \partial_t(h^{-2}c_{xx}) + u\partial_x(h^{-2}c_{xx}) = 0 \\ \partial_t(h^2 c_{zz}) + u\partial_x(h^2 c_{zz}) = 0 \\ \partial_t(c_{yy} - c_{xy}^2/c_{xx}) + u\partial_x(c_{yy} - c_{xy}^2/c_{xx}) = 0 \\ \partial_t(hv) + \partial_x(huv) + \partial_x\left(Ghc_{xy}\right) = 0 \\ \partial_t(h^{-1}c_{xy}) + u\partial_x(h^{-1}c_{xy}) + (h^{-2}c_{xx})h\partial_x v = 0 \end{cases}$$

in Lagrange coordinates, ensuring entropy consistency,
then map it for *Euler coordinates* [Gallice 2001, Bouchut 2004]

Suliciu relaxation of pressures

The 4×4 subsystem for (h, u, c_{xx}, c_{zz}) is closed

$$\begin{cases} \partial_t h^{-1} - \partial_x u = 0 \\ \partial_t u + \partial_x (g h^2/2 + G h (c_{xx} - c_{zz})) = 0 \\ \partial_t (h^{-2} c_{xx}) = 0 \quad \partial_t (h^2 c_{zz}) = 0 \\ \partial_t (c_{yy} - c_{xy}^2 / c_{xx}) = 0 \\ \partial_t v + \partial_x (G h c_{xy}) = 0 \\ \partial_t (c_{xy}/h) + c_{xx} \partial_x v = 0 \end{cases}$$

it looks like *Euler gas dynamics* with entropy $(h c_{xx}^{-1/2}, h c_{zz}^{+1/2})$

A 3-wave solver *relaxing* $\pi_{||} \approx P_{||} := gh + G(c_{xx} - c_{zz})$

$$h(\partial_t \pi_{||} + u \partial_x \pi_{||}) + c_{||}^2 \partial_x u = 0$$

was proposed in [Bouchut B. 2013] for 1D models.

A linearly degenerate system

We propose an *extension* with 5 waves, in Euler coordinates:

$$
\begin{cases}
\partial_t h - \partial_x(hu) = 0 \\
\partial_t(hu) + \partial_x(hu^2 + \pi_{\|}) = 0 \\
\partial_t(h\pi_{\|}) + \partial_x(hu\pi_{\|} + c_{\|}^2 u) = 0 \\
\partial_t(h^{-1}c_{xx}) + \partial_x(h^{-1}c_{xx}u) = 0 \\
\partial_t(h^3 c_{zz}) + \partial_x(h^3 c_{zz}u) = 0 \\
\partial_t c_{xy} + \partial_x(huc_{xy} + a^2 v) = 0 \\
\partial_t(hv) + \partial_x(huv + \pi_{\perp}) = 0 \\
\partial_t(h\pi_{\perp}) + \partial_x(hu\pi_{\perp} + bu + c_{\perp}^2 v) = 0 \\
\partial_t\left(h(c_{yy} - c_{xy}^2/c_{xx})\right) + \partial_x\left(h(c_{yy} - c_{xy}^2/c_{xx})u\right) = 0
\end{cases}
\tag{11}
$$

One can choose parameters $c_{\|}^2 \approx h^2(gh + G(3c_{xx} + c_{zz}))$, $a^2 \approx c_{xx}$, $b \approx 2Gh^2 c_{xy}$ and $c_{\perp}^2 \approx Gh^2 c_{xx}$ for consistency with

$$\partial_t \left(h(E_{\|} + E_{\perp})\right) + \partial_x \left(hu(E_{\|} + hE_{\perp} + uP_{\|} + vP_{\perp})\right) \leq 0$$

$$E_{\|} = \frac{u^2}{2} + e_{\|}\,, \quad e_{\|} = \frac{gh}{2} + \frac{G(c_{xx} + c_{zz})}{2} - \frac{G\ln(c_{xx}c_{zz})}{2}$$

$$E_{\perp} = \frac{v^2}{2} + e_{\perp}\,, \quad e_{\perp} = \frac{Gc_{xy}^2/c_{xx}}{2} + \frac{G(c_{yy} - c_{xy}^2/c_{xx})}{2} - \frac{G\ln(c_{yy} - c_{xy}^2/c_{xx})}{2}$$

$$P_{\|} = -\partial_{h^{-1}}\big|_{hc_{xx}^{-1/2}, hc_{zz}^{+1/2}} e_{\|} = \frac{gh^2}{2} + Gh(c_{xx} - c_{zz}) \qquad P_{\perp} = Ghc_{xy}$$

under 2×3 conditions, recalling the Riemann solver structure:

$$e_{\|}(q_o^*) \leq (\hat{e}_{\|})_o^* \quad e_{\perp}(q_o^*) \leq (\hat{e}_{\perp})_o^* \quad e_{\perp}\left(q_o^{\sharp}\right) \leq (\hat{e}_{\perp})_o^{\sharp}$$

$$\partial_t \left(h(u^2/2 + \hat{e}_{\|})\right) + \partial_x \left(hu(u^2/2 + \hat{e}_{\|}^2) + \pi_{\|} u\right) = 0$$

$$\partial_t \left(h(v^2/2 + \hat{e}_{\perp})\right) + \partial_x \left(hu(v^2/2 + \hat{e}_{\perp}) + \pi_{\perp} v\right) = 0$$

To sum up...

Finally, for SVTM, we use:

1. $\left(H, HU, HV, HC_{xx}, HC_{yy}, HC_{xy}/\sqrt{C_{xx}C_{xy}}, HC_{zz}\right)$ as one possible state vector q (HE convex in q)
2. the analytical Riemann solution *integrated* for flux

$$\frac{1}{\tau^n}\int_{-\left(\sum_j \frac{|\Gamma_{ij}|}{|V_i|}\right)^{-1}}^{0} R(x/\tau^n, q_l, q_r)dx$$

$$= \bar{s}_i^n q_l + \left(\xi_{-1,-} - \xi_{-2,-}\right)\left(q_l^* - q_l\right) + \left(\xi_{0,-} - \xi_{-1,-}\right)\left(q_l^\sharp - q_l\right)$$

$$+\left(\xi_{1,-} - \xi_{0,-}\right)\left(q_r^\sharp - q_l\right) + \left(\xi_{2,-} - \xi_{1,-}\right)\left(q_r^* - q_l\right) + \left(-\xi_{2,-}\right)\left(q_r - q_l\right)$$

For SVUCM: the same approach holds – it is simpler ($b \equiv 0$) !

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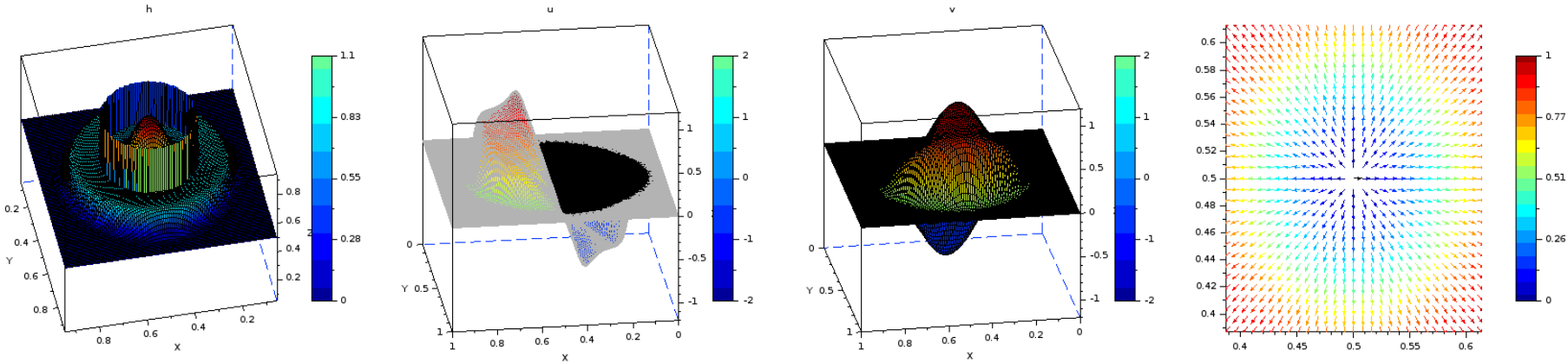
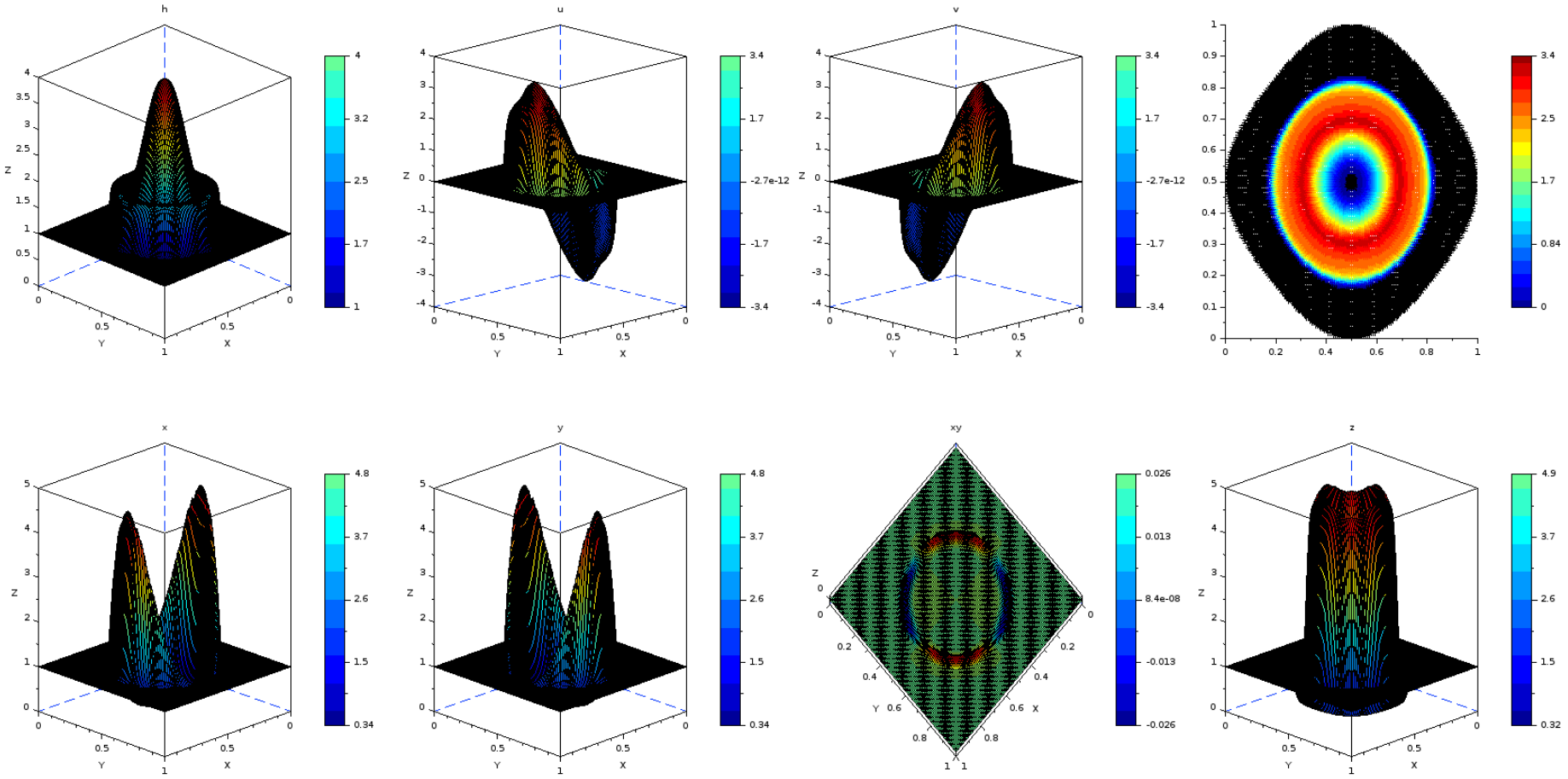
2D column “circular dam-break” benchmark

Solution at $T = .2$ in $(x, y) \in [0, 1]^2$ starting from

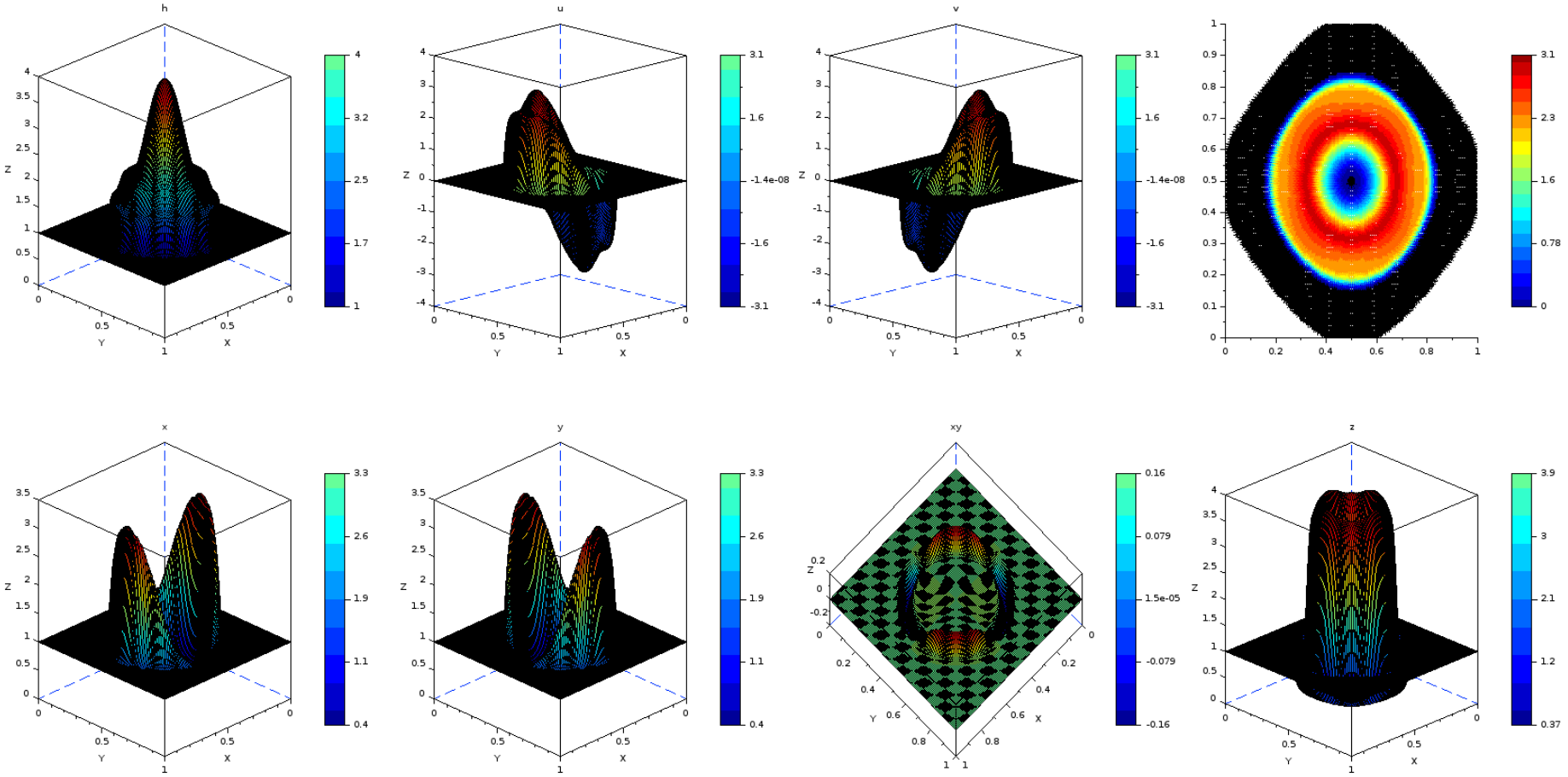
$$(H, U, V, C_{xx}, C_{yy}, C_{xy}, C_{zz}) = \begin{cases} (3, 0, 0, 1, 1, 0, 1) & (x - .5)^2 + (y - .5)^2 < .2 \\ (1, 0, 0, 1, 1, 0, 1) & (x - .5)^2 + (y - .5)^2 > .2 \end{cases}$$



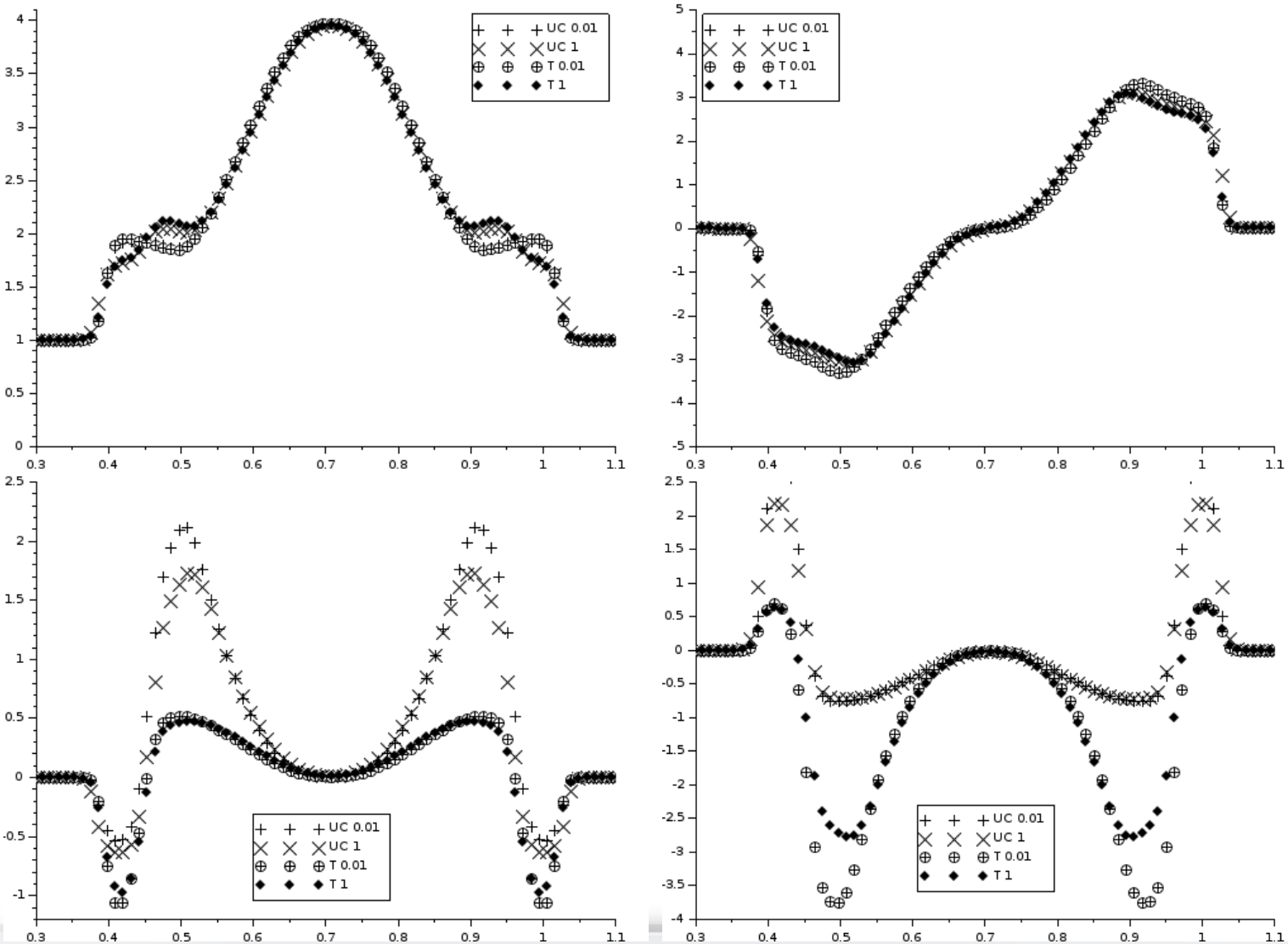
SVTM: $g = 10$, $G = 0.01$, $\lambda = 1$ at $T = .2$



SVTM: $g = 10$, $G = 1$, $\lambda = 1$ at $T = .2$



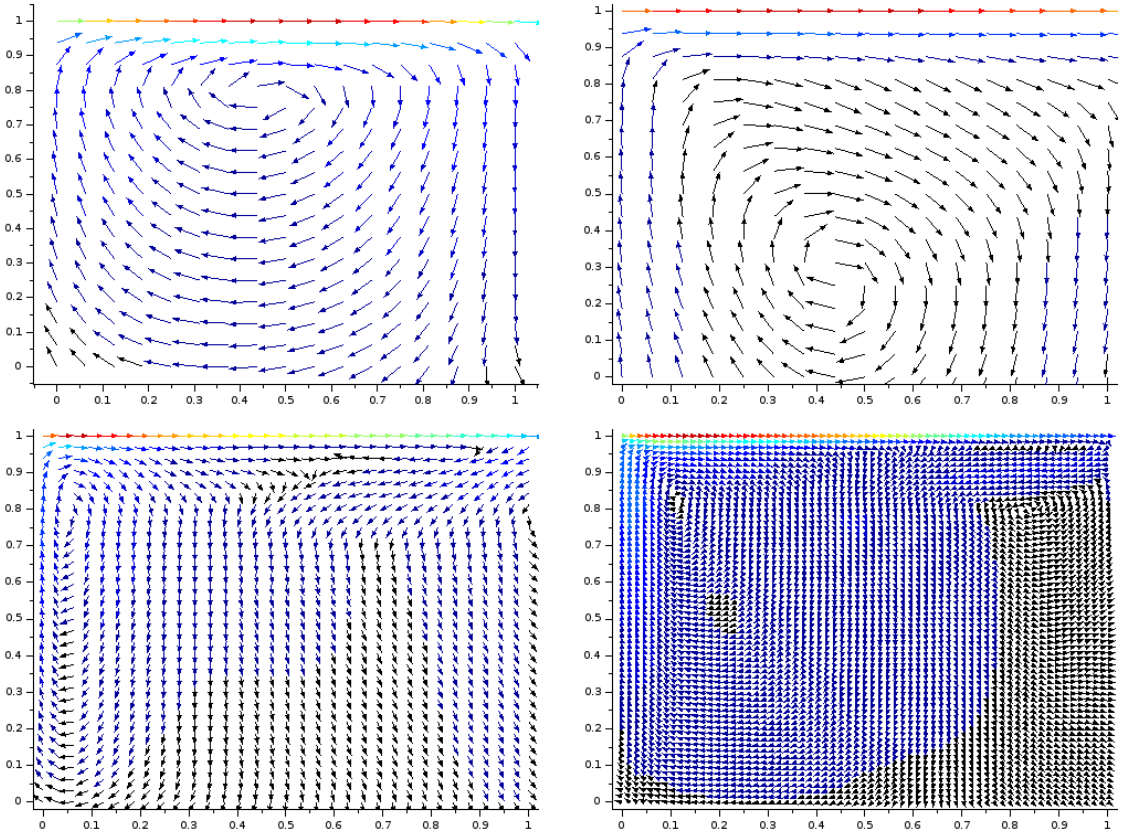
SVTM vs. SVUCM: H, U_r and Σ_{rr}, Σ_{zz}



HWNP ?

- So far we have considered “simple” free-shear flows.
- HWNP seem triggered by boundary no-slip constraints (as usual for a non-zero retardation time)
- We add a viscous extra-stress with small $\nu_s = 10^{-1} \leq G\lambda$ that can be treated in sub-step 2 of our time-splitting
- We also choose a small Froude $g^{-1/2} = 10^{-3/2}$

Lid-driven cavity $\boldsymbol{U} = (U, V)$: $G = 1 \rightarrow 10$



Top: SVTM ($\nu_s = .1$ or $.01$; $\lambda = .1$ or 1) converges
Bottom: SVUCM ($\nu_s = .01$, $\lambda = .1$) does not.

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Conclusion

- Saint-Venant generalized to Maxwell *viscoelastic* fluids
- Two hyperbolic models for *non-Newtonian* (2D) flows which preserve mass, energy and have a second-principle
- *Entropy-consistent* FV numerical schemes

Much remains to be done. . .

- Has J.C. Maxwell (1831-1879) ever met A.J.C. de Saint-Venant (1797-1886) ?

Probably no: my title was a fake news.
Though, it seems Saint-Venant as well as Boussinesq did influence Maxwell . . . from a philosophical viewpoint !

Discussion

Do you have any interest in Fixt Fate, Free Will and c. If so Boussinesq [of hydrodynamical reputation] "Conciliation du véritable déterminisme mécanique avec l'existence de la vie et de la liberté morale" (Paris, 1878) does the whole business by the theory of the singular solutions of the dif.eq. of motion. Two other Frenchmen have been working on the same or similar tracks. Cournot (now dead) and De St. Venant [of elastics reputation, Torsion of Prism and c.].

J. C. Maxwell, letter to Francis Galton, Feb. 1879
(in T. M. Porter 1986, The rise of statistical thinking 1820-1900)

Thanks for listening