

Wave-structure interaction for long wave models in the presence of a freely moving body on the bottom

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1 Setting of the problem

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2 The Boussinesq regime

- The Boussinesq system and the approximate solid equations
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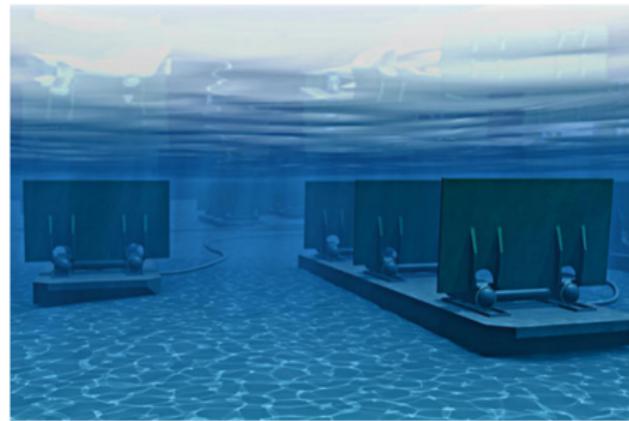
Motivation

Mathematical motivation : a better understanding of the water waves problem

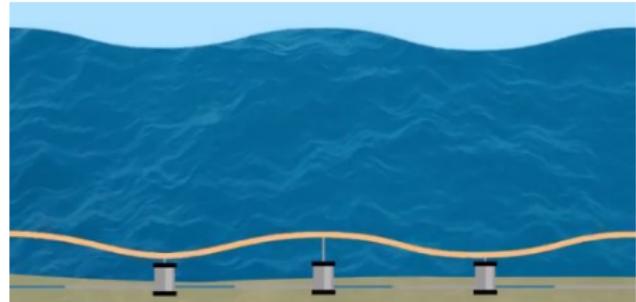
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Real life applications : Coastal engineering and wave energy converters

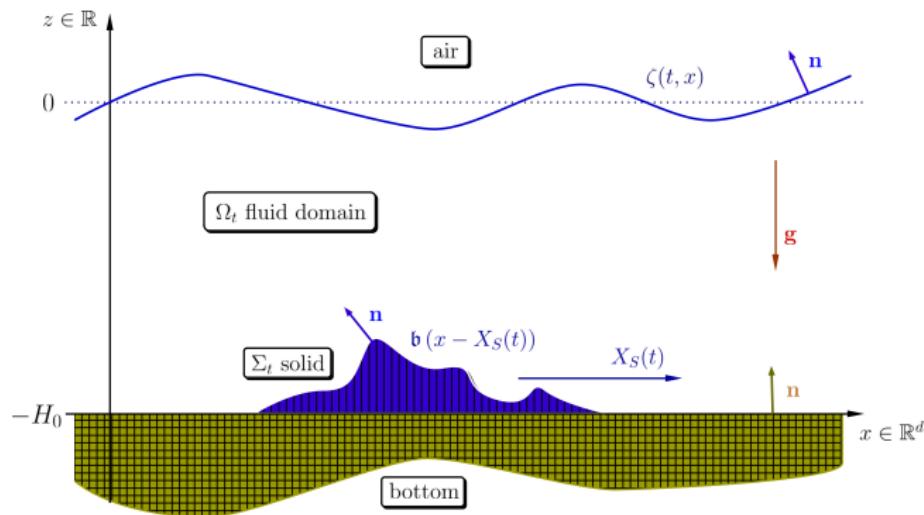


(a) Wave Roller



(b) Wave Carpet

The physical domain for the wave-structure interaction problem



$$\Omega_t = \{(x, z) \in \mathbb{R}^2 : -H_0 + b(x - X_S(t)) < z < \zeta(t, x)\}.$$

References

- In the case of a predefined evolution of the bottom topography :
 - T. Alazard, N. Burq, and C. Zuily, On the Cauchy Problem for the water waves with surface tension (2011),
 - F. Hiroyasu, and T. Iguchi, A shallow water approximation for water waves over a moving bottom (2015),
 - B. Melinand, A mathematical study of meteo and landslide tsunamis (2015);
- Fluid - submerged solid interaction :
 - G-H. Cottet, and E. Maitre, A level set method for fluid-structure interactions with immersed surfaces (2006),
 - P. Guyenne, and D. P. Nicholls, A high-order spectral method for nonlinear water waves over a moving bottom (2007),
 - S. Abadie et al., A fictitious domain approach based on a viscosity penalty method to simulate wave/structure interactions (2017).

The governing equations

Fluid dynamics

The water waves problem : Based on the Laplace equation

$$\begin{cases} \Delta\Phi = 0 & \text{in } \Omega_t \\ \Phi|_{z=\zeta} = \psi, \quad \sqrt{1 + |\partial_x b|^2} \partial_n \Phi_{\text{bott}} = \partial_t b. \end{cases}$$

An evolution equation for ζ , the surface elevation.

An evolution equation for ψ , the velocity potential on the free surface.

The governing equations

Fluid dynamics

$$\begin{cases} \partial_t \zeta + \partial_x(h\bar{V}) = \partial_t b, \\ \partial_t \psi + g\zeta + \frac{1}{2}|\partial_x \psi|^2 - \frac{(-\partial_x(h\bar{V}) + \partial_t b + \partial_x \zeta \cdot \partial_x \psi)^2}{2(1 + |\partial_x \zeta|^2)} = 0, \end{cases}$$

where

$$\bar{V} = \frac{1}{h} \int_{-H_0+b}^{\zeta} \partial_x \Phi(\cdot, z) dz.$$

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Solid mechanics

By Newton's second law :

$$\mathbf{F}_{\text{total}} = \mathbf{F}_{\text{gravity}} + \mathbf{F}_{\text{solid-bottom interaction}} + \mathbf{F}_{\text{solid-fluid interaction}}.$$

The governing equations

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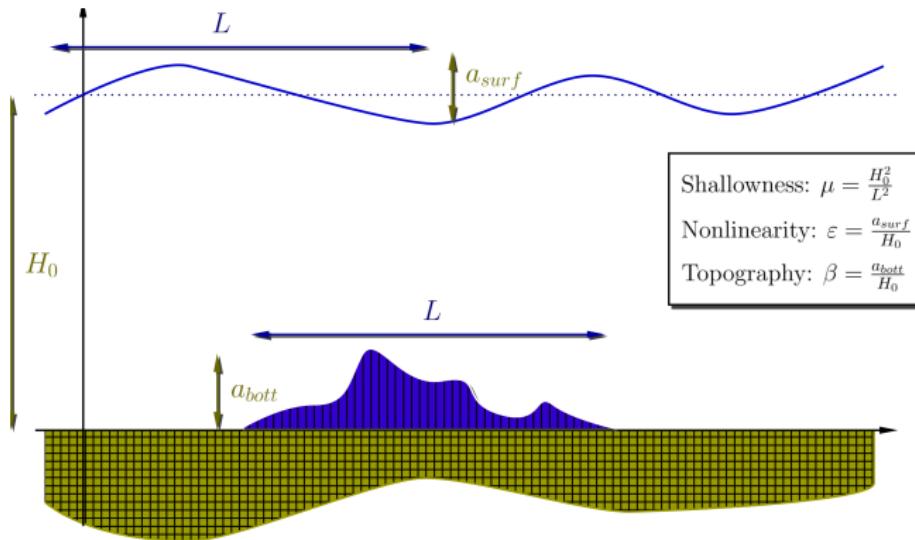
Solid mechanics

The equation of motion for the solid

$$M\ddot{X}_S(t) = -c_{fric} \left(Mg + \int_{I(t)} P_{bott} dx \right) \mathbf{e}_{tan} + \int_{I(t)} P_{bott} \partial_x b dx.$$

Characteristic scales of the problem

- L , the characteristic horizontal scale of the wave motion,
- H_0 , the base water depth,
- a_{surf} , the order of the free surface amplitude,
- a_{bott} , the characteristic height of the solid.



The coupled Boussinesq system

With an order $\mathcal{O}(\mu^2)$ approximation, we are going to work in the so called weakly nonlinear Boussinesq regime

$$0 \leq \mu \leq \mu_{\max} \ll 1, \quad \varepsilon = \mathcal{O}(\mu), \beta = \mathcal{O}(\mu). \quad (\text{BOUS})$$

The coupled Boussinesq system with an object moving at the bottom writes as

$$\begin{cases} \partial_t \zeta + \partial_x(h\bar{V}) = \frac{\beta}{\varepsilon} \partial_t b, \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2} |\partial_x \psi|^2 - \varepsilon \mu \frac{(-\partial_x(h\bar{V}) + \frac{\beta}{\varepsilon} \partial_t b + \partial_x(\varepsilon \zeta) \cdot \partial_x \psi)^2}{2(1 + \varepsilon^2 \mu |\partial_x \zeta|^2)} = 0, \\ \ddot{X}_S(t) = -\frac{c_{fric}}{\sqrt{\mu}} \left(1 + \frac{1}{\beta \tilde{M}} \int_{I(t)} P_{\text{bott}} dx \right) \mathbf{e}_{\tan} + \frac{1}{\tilde{M}} \int_{\mathbb{R}} P_{\text{bott}} \partial_x b dx. \end{cases}$$

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The coupled Boussinesq system with an object moving at the bottom writes as

$$\left\{ \begin{array}{l} \partial_t \zeta + \partial_x(h \bar{V}) = \frac{\beta}{\varepsilon} \partial_t \mathbf{b}, \\ \left(1 - \frac{\mu}{3} \partial_{xx}\right) \partial_t \bar{V} + \partial_x \zeta + \varepsilon \bar{V} \cdot (\partial_x \bar{V}) = -\frac{\mu}{2} \partial_x \partial_{tt} \mathbf{b}, \\ \ddot{X}_S(t) = -\frac{c_{fric}}{\sqrt{\mu}} \left(\frac{1}{\beta} c_{solid} + \frac{\varepsilon}{\beta \tilde{M}} \int_{I(t)} \zeta \, dx \right) \mathbf{e}_{tan} + \frac{\varepsilon}{\tilde{M}} \int_{\mathbb{R}} \zeta(t, x) \partial_x \mathbf{b} \, dx, \end{array} \right.$$

L^2 estimates

$$E_B(t) = \frac{1}{2} \int_{\mathbb{R}} \zeta^2 dx + \frac{1}{2} \int_{\mathbb{R}} h \bar{V}^2 dx + \frac{1}{2} \int_{\mathbb{R}} \frac{\mu}{3} h (\partial_x \bar{V})^2 dx + \frac{1}{2\varepsilon} |\dot{X}_S(t)|^2,$$

Proposition

Let $\mu \ll 1$ sufficiently small and let us take $s_0 > 1$. Any $\mathcal{U} \in \mathcal{C}^1([0, T] \times \mathbb{R}) \cap \mathcal{C}^1([0, T]; H^{s_0}(\mathbb{R}))$, $X_S \in \mathcal{C}^2([0, T])$ solutions to the coupled system, with initial data $\mathcal{U}(0, \cdot) = \mathcal{U}_{in} \in L^2(\mathbb{R})$ and $(X_S(0), \dot{X}_S(0)) = (0, v_{S_0}) \in \mathbb{R} \times \mathbb{R}$, verify

$$\sup_{t \in [0, T]} \left\{ e^{-\sqrt{\varepsilon} c_0 t} E_B(t) \right\} \leq 2E_B(0) + \mu T c_0 \|b\|_{H^3},$$

where

$$c_0 = c(\|\mathcal{U}\|_{T, W^{1, \infty}}, \|\mathcal{U}\|_{T, H^{s_0}}, \|b\|_{W^{4, \infty}}).$$

Long time existence for the Boussinesq system

Theorem

Let μ sufficiently small and $\varepsilon = \mathcal{O}(\mu)$. Let us suppose that the initial values ζ_{in} and b satisfy the minimal water depth condition.

If ζ_{in} and \bar{V}_{in} belong to $H^{s+1}(\mathbb{R})$ with $s \in \mathbb{R}$, $s > 3/2$, and that $X_{S_0}, v_{S_0} \in \mathbb{R}$, then there exists a maximal time $T > 0$ independent of ε such that there exists a solution

$$\begin{aligned} (\zeta, \bar{V}) &\in C\left([0, \frac{T}{\sqrt{\varepsilon}}]; H^{s+1}(\mathbb{R})\right) \cap C^1\left([0, \frac{T}{\sqrt{\varepsilon}}]; H^s(\mathbb{R})\right), \\ X_S &\in C^2\left([0, \frac{T}{\sqrt{\varepsilon}}]\right) \end{aligned}$$

of the coupled system

$$\begin{cases} D_\mu \partial_t \mathcal{U} + A(\mathcal{U}, X_S) \partial_x \mathcal{U} + B(\mathcal{U}, X_S) = 0, \\ \ddot{X}_S(t) = \mathcal{F}[\mathcal{U}](t, X_S(t), \dot{X}_S(t)). \end{cases}$$

with initial data $(\zeta_{in}, \bar{V}_{in})$ and (X_{S_0}, v_{S_0}) .

The numerical scheme

The discretization in space : Adapting a staggered grid finite difference scheme, based on the work of P. Lin and Ch. Man (*Appl. Math. Mod.* 2007).

- finite difference scheme,
- surface elevation and bottom is defined on grid points, averaged velocity is defined on mid-points,
- order 4 central difference scheme,
- third order Simpson method for calculating the integrals.

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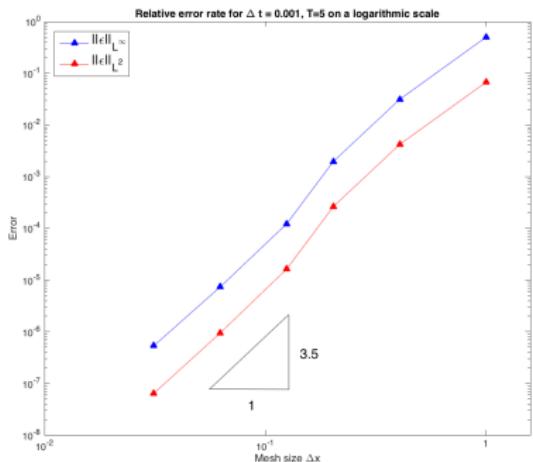
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The discretization in time :

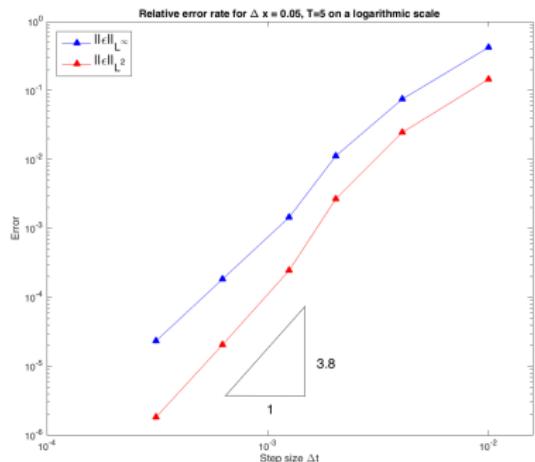
- Adams 4th order predictor-corrector algorithm for the fluid dynamics
- An explicit scheme for the solid equation : an adapted second order central scheme
- preserves the dissipative property due to the friction,

Convergence of the scheme

Convergence for frictional sliding ($c_{fric} = 0.1$)



(a) Spatial error, $\mu = \varepsilon = 0.1, \beta = 0.3$

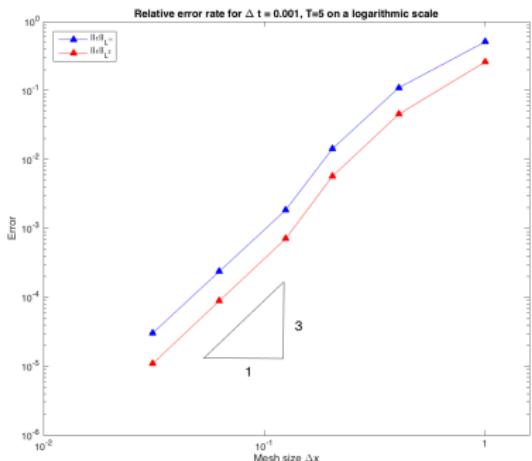


(b) Temporal error, $\mu = \varepsilon = 0.1, \beta = 0.3$

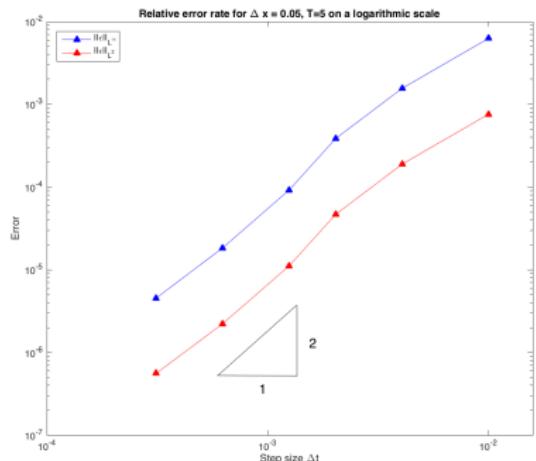
Almost of order 4, same convergence as the fixed/flat bottom scenario.

Convergence of the scheme

Convergence for frictionless sliding ($c_{fric} = 0.001$)



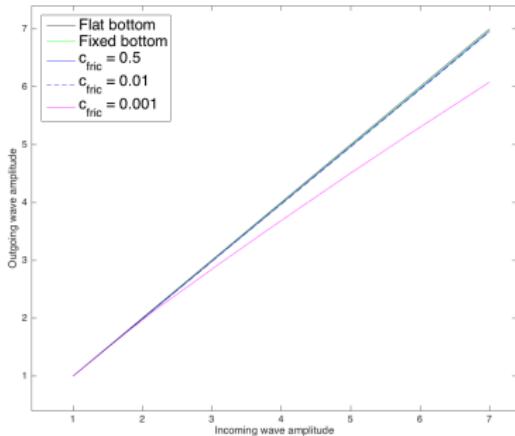
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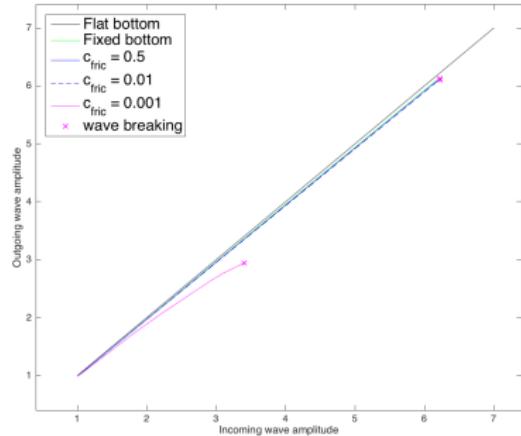
(b) Temporal error, $\mu = \varepsilon = 0.1, \beta = 0.3$

A reasonable convergence, loss due to the numerical scheme of the solid.

Amplitude variation for a passing wave

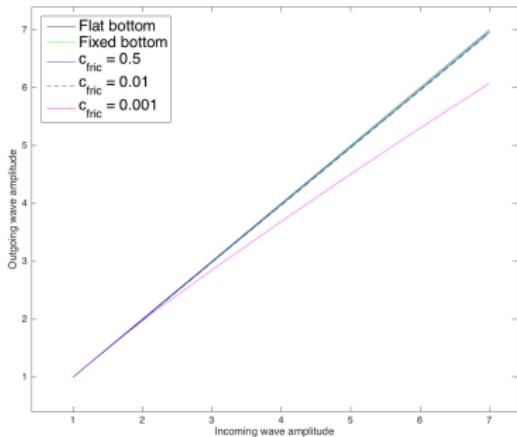
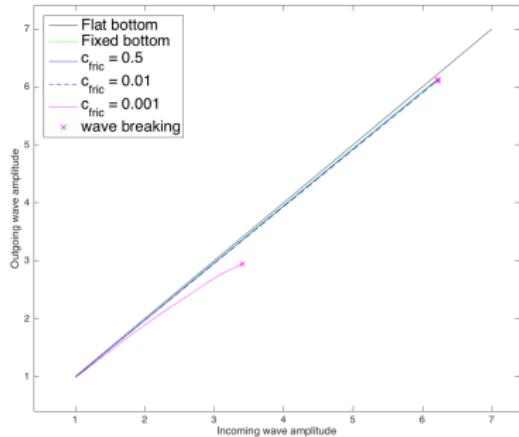


(a) Change in wave amplitude, $\mu = 0.25$, $\beta = 0.3$



(b) Change in wave amplitude, $\mu = 0.25$, $\beta = 0.5$

Amplitude variation for a passing wave

(a) Change in wave amplitude, $\mu = 0.25$, $\beta = 0.3$ (b) Change in wave amplitude, $\mu = 0.25$, $\beta = 0.5$

Noticeable attenuation for the moving solid.

Observe the wave-breaking for the relatively large solid.

The solid motion under the influence of the waves

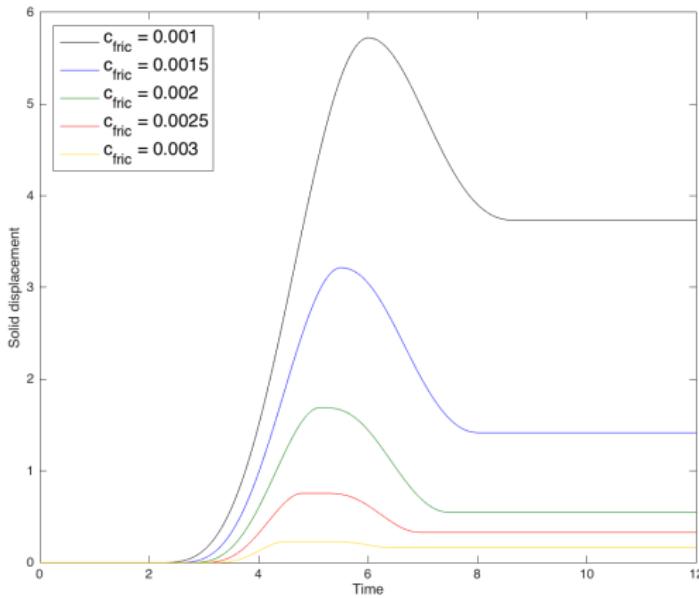
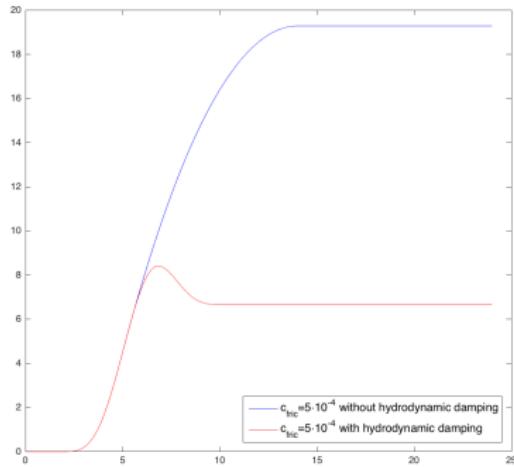


FIGURE – Solid position for varying coefficient of friction ($\mu = \varepsilon = 0.25$, $\beta = 0.3$)

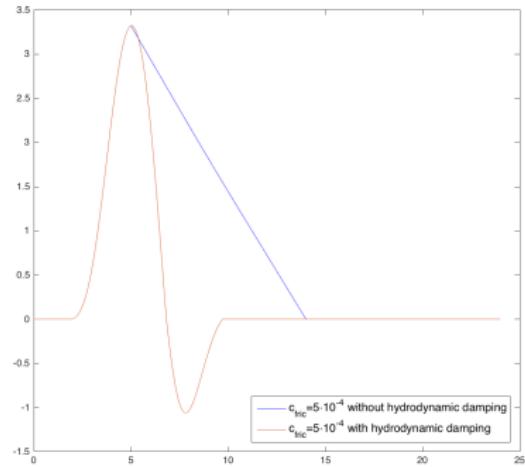
Observable : *hydrodynamic damping, frictional damping*.

The solid motion under the influence of the waves



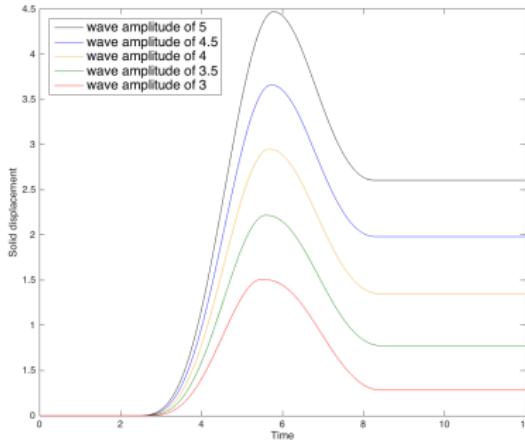
(a) Solid position, single wave, with and without hydrodynamic effects

Highlight : hydrodynamic damping effect

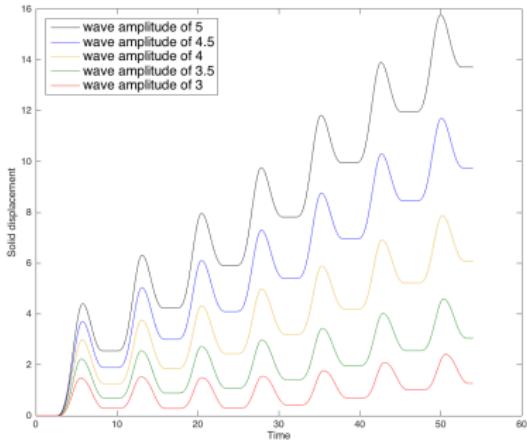


(b) Solid velocity, wavetrain, with and without hydrodynamic effects

The solid motion under the influence of the waves



(a) Solid position, single wave $\mu = 0.25$, $\beta = 0.3$, $c_{fric} = 0.001$



(b) Solid position, wavetrain $\mu = 0.25$, $\beta = 0.3$, $c_{fric} = 0.001$

Conclusions

What we did :

- characterise mathematically the physical setting of an object on the bottom of an "oceanographic fluid domain",
- establish the coupled system,
- analyse the order 2 asymptotic system in μ (weakly nonlinear Boussinesq setting),
- create an accurate finite difference scheme for the coupled model,
- highlight the effects of a free solid motion on wave transformation as well as the effects of friction on the system.

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What we still have to do :

- treat the case of a non-horizontal bottom,
- generalize the notion of friction to a more realistic physical interpretation,
- ...

Remerciements

Merci beaucoup pour votre attention !