Floating structures in shallow water: local well-posedness in the axisymmetric case

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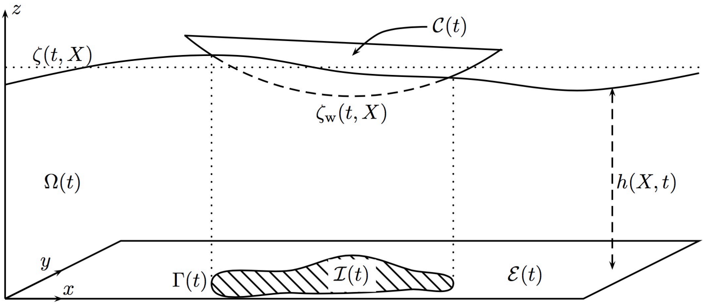
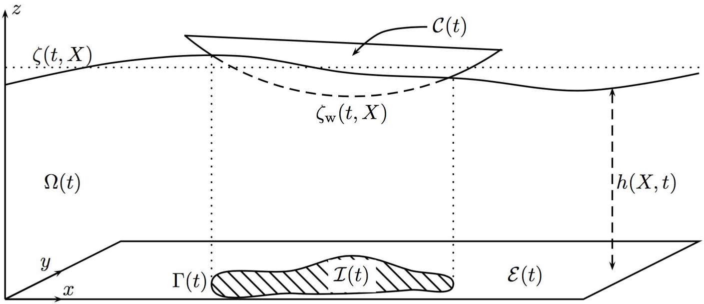


Two free boundary problem: surface elevation $\zeta(t, X)$ + contact line $\Gamma(t)$



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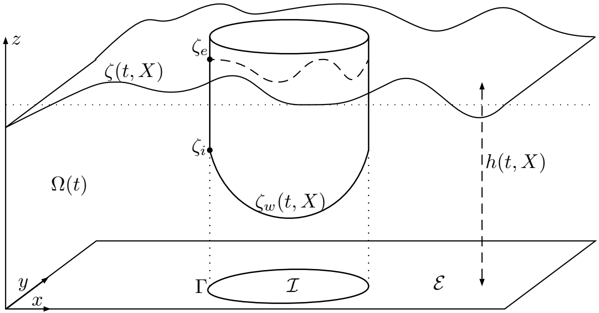
Assumptions on the solid

- Vertical side-walls
- Only vertical motion



One free boundary problem: surface elevation $\zeta(t,X)$

The contact line ~~$\Gamma(t)$~~ $\longrightarrow \Gamma$ does not depend on time!

Equations in the fluid domain $\Omega(t)$ for $\mathbf{U}$:

$$\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_{X,z} \mathbf{U} = -\frac{1}{\rho} \nabla_{X,z} P - g\mathbf{e_z} \quad \text{in} \quad \Omega_t \tag{1}$$

$$\text{div } \mathbf{U} = 0 \tag{2}$$

$$\text{curl } \mathbf{U} = 0 \tag{3}$$

Boundary conditions at the surface and the bottom:

$$z = \zeta, \qquad \partial_t \zeta - \mathbf{U} \cdot N = 0 \quad \text{with } N = \begin{pmatrix} -\nabla\zeta \\ 1 \end{pmatrix} \tag{4}$$

$$z = -h_0, \qquad \mathbf{U} \cdot \mathbf{e_z} = 0 \tag{5}$$

Pressure in $\mathcal{E}$:

$$\underline{P}_e = P_{atm} \tag{6}$$

Constraint in $\mathcal{I}$:

$$\zeta_i(t, X) = \zeta_w(t, X) \tag{7}$$

Jump at Γ:

$$\zeta_e(t, \cdot) \neq \zeta_i(t, \cdot) \tag{8}$$

$$\underline{P}_i(t, \cdot) = P_{atm} + \rho g(\zeta_e - \zeta_i) + P_{NH} \tag{9}$$

Continuity of the normal velocity at the vertical walls:

$$V \cdot \nu = V_C \cdot \nu \tag{10}$$

Analysis of the pb: state of art

- Linear model: John '49 (1d neglecting contact line time variations), Cummins '62 (2d and integro-differential equation for the solid motion), Ursell '64...
- Nonlinear model: Lannes '17 (modelisation and 1d explicit solid motion equation), Iguchi and Lannes '18 (1d sw well-posedness for moving contact line)

Proposition (Lannes '17)

Let us consider ζ *and* $\mathbf{U}$ *solutions of the constrained free surface Euler equations* (1)- (10)*. Then* ζ *and the horizontal discharge* Q*, with* $Q = \int_{-h_0}^{\zeta} V dz$*, solve*

$$\begin{cases} \partial_t h + \nabla \cdot Q = 0, \\ \partial_t Q + \nabla \cdot (\frac{1}{h} Q \otimes Q) + gh\nabla h + \nabla \cdot \mathbf{R}(h, Q) + h\mathbf{a}_{NH}(h, Q) = -\frac{h}{\rho} \nabla \underline{P}, \end{cases} \tag{11}$$

with the surface pressure $\underline{P}$ *given by*

$$\underline{P}_e = P_{atm} \quad \text{and} \quad \begin{cases} -\nabla \cdot (\frac{h_w}{\rho} \nabla \underline{P}_i) = -\partial_t^2 \zeta_w + \mathbf{a}_{FS}(h, Q) \quad \text{on } \mathcal{I} \\ \underline{P}_{i|_\Gamma} = P_{atm} + \rho g (\zeta_e - \zeta_i)_{|_\Gamma} + P_{NH}, \end{cases} \tag{12}$$

and with the transition condition at the contact line

$$Q_e \cdot \nu = Q_i \cdot \nu \quad \text{on} \quad \Gamma. \tag{13}$$

Shallow water regime: the wavelength L of the waves is larger than the depth h_0, *i.e.*

$$\mu = \frac{{h_0}^2}{L^2} \ll 1$$

At first order $O(\mu)$ we neglect the vertical variation of V in the quadratic term

- $\mathbf{R}(h, Q) \approx 0$
- $a_{NH} \approx 0$

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Hence the evolution equations become:

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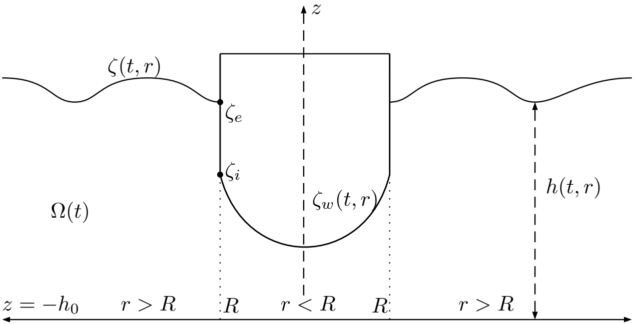
- Congested flow (g=0): traffic flows, compressible low-Mach coupling in gaz dynamics, hydrodynamics in pipes...

Cylindrical coordinates:

$\mathbf{U} = \mathbf{U}(t, r, \theta, z)$, $\mathbf{U} = (u_r, u_\theta, u_z) \Longrightarrow Q = Q(t, r, \theta)$, $Q = (q_r, q_\theta)$



We assume that the flow is axisymmetric without swirl, which means that the flow has no dependence on the angular variable θ and $u_\theta = 0$.

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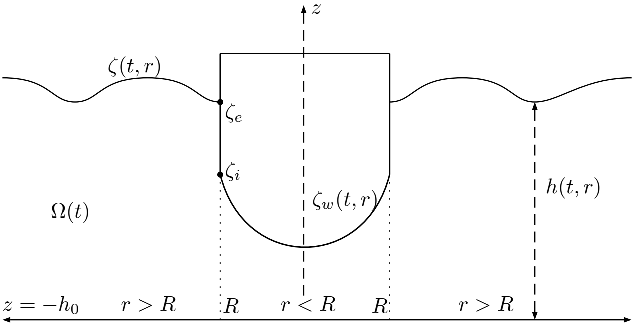

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$$\Longrightarrow Q(t, r) = (q_r, 0)$$

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with the surface pressure $\underline{P}$ given by

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and with the transition condition at the contact line

$$Q_e \cdot \nu = Q_i \cdot \nu \quad \text{on} \quad \Gamma.$$

$$\begin{cases} \partial_t h + \quad \partial_r q + \dfrac{q}{r} = 0, \\ \partial_t q + \partial_r \left(\dfrac{q^2}{h} \right) + \dfrac{q^2}{rh} + gh\partial_r h = -\dfrac{h}{\rho}\partial_r \underline{P}, \end{cases} \tag{14}$$

with the surface pressure $\underline{P}$ given by

$$\underline{P}_e = P_{atm} \quad \text{and} \quad \begin{cases} -\left(\partial_r + \dfrac{1}{r}\right)\left(\dfrac{h_w}{\rho}\partial_r \underline{P}_i\right) = -\partial_t^2 \zeta_w + \mathbf{a}_{FS} \quad \text{on} \quad r < R \\ \underline{P}_{i|_{r=R}} = P_{atm} + \rho g(\zeta_e - \zeta_i)_{|_{r=R}} + P_{cor}. \end{cases}$$

and with the transition condition at the contact line

$$q_e = q_i \quad \text{on} \quad r = R.$$

Exterior domain $\mathcal{E}$

Defining $u = (h_e, q_e)^T$ and adding the Cauchy datum we can write (14) in $\mathcal{E}$ as the following quasilinear hyperbolic initial boundary value problem

$$\begin{cases} \partial_t u + A(u)\partial_r u + B(u,r)u = 0 \\ \mathbf{e}_2 \cdot u_{|_{r=R}} = q_{i|_{r=R}} \\ u(t=0) = u_0 \end{cases} \tag{15}$$

with

$$A(u) = \begin{pmatrix} 0 & 1 \\ gh_e - \dfrac{q_e^2}{h_e^2} & \dfrac{2q_e}{h_e} \end{pmatrix}, \qquad B(u,r) = \begin{pmatrix} 0 & \dfrac{1}{r} \\ 0 & \dfrac{q_e}{rh_e} \end{pmatrix}$$

We study the linearized problem

$$\begin{cases} \partial_t u + A(\overline{u})\partial_r u + B(\overline{u}, r)u = f \\ \mathbf{e}_2 \cdot u_{|_{r=R}} = g \\ u(t=0) = u_0 \end{cases}$$

Subsonic regime $\det A < 0$ and uniform Lopatinskii condition $P_- \mathbf{e}_2^\perp \neq 0$ give the properties:

- The system is *Friedrichs symmetrizable*, *i.e.* there exists a symmetric matrix $S = S(\overline{u})$ such that $S(\overline{u})$ is uniformly positive definite on $L^2_r((R, +\infty))$ and $S(\overline{u})A(\overline{u})$ is symmetric.
- The boundary condition is *maximally dissipative*: $S(\overline{u})A(\overline{u})$ is negative definite on $\mathbf{e}_2^\perp$, where $\mathbf{e}_2^\perp$ is the orthogonal complement of $\mathbf{e}_2$.

- Functional space $X^k(T) := \bigcap_{j=0}^{k} C^j([0,T], H_r^{k-j}((R,+\infty)))$
- A priori estimate

$$\begin{aligned} &\|u(t)\|_{X^k}^2 + \|u_{|_{r=R}}\|_{H^k([0,t])}^2 \\ &\leqslant C_{s,T}(\overline{u})\left(\|u(0)\|_{X^k}^2 + \|g\|_{H^k([0,t])}^2 + \int_0^t \|f(\tau)\|_{X^k}^2 d\tau\right) \end{aligned} \tag{16}$$

Remark

The axisymmetry keeps the boundary condition maximally dissipative

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- A priori estimate

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Remark

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Theorem

Let k be an integer with $k \geqslant 1$. Suppose $u_0 \in H_r^k((R,+\infty))$, $g \in H^k([0,T])$ and $f \in H_r^k([0,T] \times (R,+\infty))$ satisfy the compatibility conditions up to the order $k-1$. Moreover assume $\overline{u} \in X^s(T)$ with $s = \max(k,2)$. Then there exists a unique solution $u \in X^k(T)$.

Solid: $G(t) = (0, 0, z_G(t))$, $\mathbf{U}_G(t) = (0, 0, w_G(t))$, $\omega = 0$

Define the displacement $\delta_G(t) := z_G(t) - z_{G,eq}$.

We can write $h_w(t, r) = h_{w,eq}(r) + \delta_G(t)$.

Recall that

$$\partial_t h_i + \partial_r q_i + \frac{q_i}{r} = 0 \quad \text{in} \quad r < R$$

From the contact constraint $\zeta_i = \zeta_w(h_i = h_w)$

$$\dot{\delta}_G + \partial_r q_i + \frac{q_i}{r} = 0 \quad \text{in} \quad r < R$$

and since we want $q_i \in L^2_r((0, R))$

$$q_i(t, r) = -\frac{r}{2}\dot{\delta}_G(t)$$

Free solid motion

Newton's law for the conservation of the linear momentum

$$m\ddot{\delta}_G = -mg + \int_{r<R} (\underline{P}_i - P_{atm})$$

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Using the elliptic equation on $\underline{P}_i$ (elementary potentials):

$$(m + m_a(\delta_G))\ddot{\delta}_G(t) = -\mathfrak{c}\delta_G(t) + \mathfrak{c}\zeta_e(t, R) + \left(\frac{\mathfrak{b}}{h_e^2(t, R)} + \beta(\delta_G) \right) (\dot{\delta_G}(t))^2$$

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- $m_a(\delta_G) > 0$ is the *added mass* (also for totally submerged solid [Glass, Sueur and Takahashi '12])

  $\partial_t\zeta_w = (\mathbf{U}_G + \omega \times \mathbf{r}_G) \cdot N_w \Longrightarrow \partial_t\zeta_w = w_G = \dot{\delta}_G(t)$

Fluid part (PDE)

$$\begin{cases} \partial_t u + A(u)\partial_r u + B(u,r)u = 0, \qquad r > R \\ \mathbf{e}_2 \cdot u_{|_{r=R}} = -\frac{R}{2}\dot{\delta}_G(t) \\ u(t=0) = u_0 := (h(0), q(0))^T, \end{cases} \tag{17}$$

Solid part (ODE)

$$\begin{cases} (m + m_a(\delta_G))\ddot{\delta}_G(t) \\ = -\mathfrak{c}\delta_G(t) + \mathfrak{c}(\mathbf{e}_1 \cdot u_{|_{r=R}} - h_0) + (\mathfrak{b}(u) + \beta(\delta_G))\,(\dot{\delta_G}(t))^2, \\ \delta_G(0) = \delta_0 := z_G(0) - z_{G,eq}, \\ \dot{\delta}_G(0) = \delta_1 := \dot{z}_G(0), \end{cases} \tag{18}$$

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Theorem (E.B.'18)

For $k \geqslant 2$, let $u_0 \in H^k_r((R, +\infty))$, δ_0 and δ_1 satisfy the compatibility conditions up to order $k-1$. Then the coupled problem (17) - (18) *admits a unique solution $(u, \delta_G) \in X^k(T) \times H^{k+1}([0,T])$.*

Proof:

Fixed point argument via a iterative scheme on the coupled system

$$\begin{cases} \partial_t u^n + A(u^{n-1})\partial_r u^n + B(u^{n-1}, r)u^n = 0, \\ \mathbf{e}_2 \cdot u^n_{|_{r=R}} = -\frac{R}{2}\dot{\delta}_G^{n-1}(t) \\ u^n(t=0) = u_0. \end{cases} \quad (19)$$

$$\begin{cases} (m + m_a(\delta_G^{n-1}))\ddot{\delta}_G^n(t) \\ = -\mathfrak{c}\delta_G^n(t) + \mathfrak{c}\left(\mathbf{e}_1 \cdot u^n_{|_{r=R}} - h_0\right) + \left(\mathfrak{b}(u^n) + \beta(\delta_G^{n-1})\right)\dot{\delta}_G^n(t)^2, \\ \delta_G^n(0) = \delta_0, \ \dot{\delta}_G^n(0) = \delta_1 \end{cases} \quad (20)$$

- Existence and uniqueness of u^n from the previous linear theory
- Control in $X^k([0,T]) \times H^{k+1}([0,T])$
- Convergence in $X^0([0,T]) \times H^1([0,T])$ + interpolation

Summary

- We do take into account nonlinear terms
- Axisymmetry keeps the boundary condition maximally dissipative even in 2d
- Validation of the shallow water approach to the floating body problem: several experimental data with an axisymmetric geometry

Perspectives

- One has to add horizontal motion and rotation:
  evolution of the contact line + no axisymmetric flow

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What's next?

The *return to equilibrium* problem: $u_0 = (h_0, 0)^T, \quad \delta_0 \neq 0, \quad \delta_1 = 0$

The compatibility conditions up to order 1 are not satisfied,

Different approach: linearized equations in the exterior domain - nonlinear equations in the interior domain $\rightarrow$ nonlinear integro-differential eq. on δ_G

THANK YOU FOR THE ATTENTION!