Equations nonlinéaire stochastiques du type Fokker–Planck

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Introduction

$$\begin{aligned} &dX - \operatorname{div}(DX)dt - \Delta\beta(X)dt = X\,dW \text{ in } (0,T)\times\mathbb{R}^d,\ T>0,\\ &X(0,\xi) = x(\xi),\ \xi\in\mathbb{R}^d,\ 1\le d<\infty, \end{aligned} \tag{1}$$

where W is a Wiener process in $H^{-1} := H^{-1}(\mathbb{R}^d)$ over a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with normal filtration $(\mathcal{F}_t)_{t\geq 0}$ of the form

$$W = \sum_{j=1}^{N} \mu_j e_j \beta_j. \tag{2}$$

Here $\{e_1, ..., e_N\}$ is an orthonormal system in $H^{-1}(\mathbb{R}^d)$ belonging to $C_b^2(\mathbb{R}^d) \cap W^{2,1}(\mathbb{R}^d)$, $\mu_j \in \mathbb{R}$ and $\{\beta_j\}_{j=1}^{\infty}$ are independent $(\mathcal{F}_t)$-Brownian motions on $(\Omega, \mathcal{F}, \mathbb{P})$. As regards the functions $D : \mathbb{R}^d \to \mathbb{R}^d$ and $\beta : \mathbb{R} \to \mathbb{R}$, we assume that

(i) $D \in C_b^1(\mathbb{R}^d;\mathbb{R}^d)$; $|D| \in L^1(\mathbb{R}^d)$, $\operatorname{div} D \in L^2(\mathbb{R}^d)$.

(ii) $\beta \in C(\mathbb{R}) \cap C^2(\mathbb{R}\setminus\{0\})$ *is monotonically nondecreasing*, $\beta(0)=0$, *and there are* $m \in [0,1]$, $a_i \in (0,\infty)$, $i=1,2,3$, *such that*

$$\begin{aligned} |\beta(r)| &\leq a_1|r|^m,\ \forall r \in \mathbb{R}, && (3)\\ |\beta''(r)r^2| + \beta'(r)|r| &\leq a_2|\beta(r)|,\ \forall r \in \mathbb{R}\setminus\{0\}, && (4)\\ \beta'(r) &\neq 0 \text{ and } \operatorname{sign} r\, \beta''(r) \leq 0,\ r \in \mathbb{R}\setminus\{0\}. && (5) \end{aligned}$$

(iii) *There exists a decreasing function* $\varphi : (0,1] \to (0,\infty)$ *such that*

$$\beta'(\lambda r) \leq \varphi(\lambda)\beta'(r),\ \forall r \in \mathbb{R}\setminus\{0\},\ \lambda \in (0,1]. \tag{6}$$

The deterministic Fokker–Planck equation (1) is related to the so-called correspondence principle in statistical mechanics which associates this equation to the entropy function

$$S(u) = \int_{\mathbb{R}} \Phi(u) d\xi,$$

where the function $\Phi \in C(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\})$ satisfies

$$\Phi'' < 0,\ \Phi' \geq 0,\ \Phi'(0) = +\infty, \tag{7}$$

and β is defined by

$$\beta(r) = \Phi(r) - r\Phi'(r),\ \forall r \geq 0. \tag{8}$$

For instance, if $\beta(r) \equiv a$ sign $(r) \log(1 + |r|)$, $a > 0$, and $\Phi(u) = -u \log u + (1 + u) \log(1 + u)$, then (1) is the classical boson equation in the Bose–Einstein statistics (see, e.g., [10]), while for $\beta(r) \equiv a|r|^{m-1} r$, one gets the so-called Plastino and Plastino model [11] in statistical mechanics.

Assumption (ii) leaves out the low diffusion case $m > 1$ which is relevant in porous media dynamics of low diffusion processes. (See, e.g., [3].)

However, for the examples in statistical mechanics mentioned above, the case $m > 1$ is not relevant. In fact, the entropy function corresponding to $\beta(u) = u^m$ is by (8) formally given in $1 - D$ by

$$S(u) = \frac{1}{1-m} \int_{\mathbb{R}} (u^m - u) d\xi, \ \Phi(u) = \frac{1}{1-m} (u^m - u),$$

for which the entropic conditions (7) are not satisfied if $m > 1$.

For vanishing drift D, equation (1) reduces to the fast diffusion stochastic porous media equation studied in [7] (see, also, [3]).

By the transformation

$$X(t) = e^{W(t)}y(t),\ t \geq 0, \tag{9}$$

equation (1) reduces, via Itô's formula, to the random differential equation (see, e.g., [4], [5], [6])

$$\begin{aligned} &\frac{\partial y}{\partial t} - e^{-W}\mathrm{div}(e^W Dy) - e^{-W}\Delta\beta(e^W y) + \frac{1}{2}\,\mu y = 0 \text{ in } (0,T)\times\mathbb{R}^d,\\ &y(0,\xi) = x(\xi),\ \xi\in\mathbb{R}^d, \end{aligned} \tag{10}$$

where

$$\mu = \sum_{j=1}^{N} \mu_j^2 e_j^2. \tag{11}$$

Let

$$D_0 = \{x \in L^1 \cap L^\infty \cap H^1;\ \beta(x) \in H^1,\ \Delta x \in L^1,\ \Delta\beta(x) \in L^1\}.$$

Theorem 1.1

Theorem

Under Hypotheses (i)–(iii)*, for each* $x \in D_0$*, equation* (10) *has, for each* $\omega \in \Omega$*, at least one strong solution*

$$y \in W^{1,2}([0,T]; H^{-1}) \cap L^\infty((0,T) \times \mathbb{R}^d) \cap L^\infty(0,T; L^1), \tag{12}$$

$$y \in L^2(0,T; H^1), \tag{13}$$

$$\beta(e^W y) \in L^2(0,T; H^1). \tag{14}$$

Moreover, if $x \geq 0$*, a.e. on* $\mathbb{R}^d$*, then* $y \geq 0$*, a.e. on* $(0,T) \times \mathbb{R}^d$*.*
If β *is locally Lipschitz on* $\mathbb{R}$ *and assumptions* (i)–(iii) *hold, then there is a unique strong solution* y *to* (10)*. This solution is* $(\mathcal{F}_t)$*–adapted, the map* $D_0 \ni x \to y(t,x)$ *is Lipschitz from* H^{-1} *to* $C([0,T]; H^{-1})$ *on balls in* $L^1 \cap L^\infty$ *and* y *extends by density to a strong solution to* (10)*, satisfying* (12), (14)*, for all* $x \in L^1 \cap L^\infty$*.*

A continuous $(\mathcal{F}_t)_{t\geq 0}$-adapted process $X : [0,T] \to H^{-1}$ is called strong solution to (1) if the following conditions hold:

$$X \in L^2([0,T]; L^2), \qquad \mathbb{P}\text{-a.s.}, \tag{15}$$

$$\beta(X) \in L^2(0,T; H^1), \qquad \mathbb{P}\text{-a.s.}, \tag{16}$$

$$X(t) - \int_0^t \operatorname{div}(DX(s))ds - \int_0^t \Delta\beta(X(s))ds = x + \int_0^t X(s)dW(s), \quad \forall t \in [0,T],\ \mathbb{P}\text{-a.s.} \tag{17}$$

Theorem 1.2

Theorem

If β *is locally Lipschitz on* $\mathbb{R}$ *and assumptions* (i)–(iii) *hold, then, for every* $x \in D_0$*, equation* (1) *has a unique strong solution* $X = e^W y$*, which satisfies*

$$Xe^{-W} \in W^{1,2}([0,T]; H^{-1}), \ \mathbb{P}\text{-}a.s., \tag{18}$$

and $X \geq 0$*, a.e. on* $(0,T) \times \mathbb{R}^d \times \Omega$ *if* $x \geq 0$*, a.e. on* $\mathbb{R}^d$.
Moreover, the map $x \mapsto X(t,x)$ *is* H^{-1}*-Lipschitz from balls in* $L^1 \cap L^\infty$ *to* $C([0,T]; H^{-1})$.

Proof of Theorem 1.1

Let $\beta_j^\varepsilon \subset C^1([0,T];\mathbb{R})$, $1 \le j \le N$, be defined by $\beta_j^\varepsilon(t) = (\mathbf{1}_{[0,\infty)}\beta_j * \rho_\varepsilon)(t)$, where $\rho_\varepsilon(t) \equiv \frac{1}{\varepsilon}\,\rho\left(\frac{t}{\varepsilon}\right)$ is a standard mollifier with $\rho \in C_0^\infty(\mathbb{R})$, $\rho \ge 0$.
We set

$$W_\varepsilon(t,\xi) = \sum_{j=1}^N \mu_j e_j(\xi)\beta_j^\varepsilon(t),\ t \ge 0,\ \xi \in \mathbb{R}^d.$$

$$(W_\varepsilon)_t \in C([0,T] \times \mathbb{R}^d)$$

$$\begin{aligned}\frac{\partial y_\varepsilon}{\partial t} &- e^{-W_\varepsilon}\mathrm{div}(e^{W_\varepsilon}Dy_\varepsilon) - e^{-W_\varepsilon}\Delta(\beta(e^{W_\varepsilon}y_\varepsilon) + \varepsilon e^{W_\varepsilon}y_\varepsilon)\\ &+\varepsilon e^{-W_\varepsilon}\beta(e^{W_\varepsilon}y_\varepsilon) + \frac{1}{2}\,\mu y_\varepsilon = 0 \text{ in } (0,T)\times\mathbb{R}^d, \qquad (19)\\ y_\varepsilon(0,\xi) &= x(\xi),\ \xi \in \mathbb{R}^d.\end{aligned}$$

Setting $z_\varepsilon = e^{W_\varepsilon} y_\varepsilon$, we get

$$\begin{aligned}
&\frac{\partial z_\varepsilon}{\partial t} - \Delta(\beta(z_\varepsilon) + \varepsilon z_\varepsilon) - \operatorname{div}(D z_\varepsilon) + \varepsilon\beta(z_\varepsilon) \\
&\qquad + \left(\frac{1}{2}\,\mu - (W_\varepsilon)_t\right) z_\varepsilon = 0 \text{ in } (0,T) \times \mathbb{R}^d, \\
&z_\varepsilon(0,\xi) = x(\xi),\ \xi \in \mathbb{R}^d.
\end{aligned} \tag{20}$$

Lemma 2.1

Lemma

Assume that $x \in H^1$ *such that* $\beta(x) \in H^1$.
Then, for each $\varepsilon \in (0,1]$, *equation* (19) *considered on* H^{-1} *has a unique strong solution* y_ε *satisfying*

$$y_\varepsilon \quad \in \quad W^{1,\infty}([0,T];H^{-1}) \cap L^\infty(0,T;H^1). \tag{21}$$

Moreover, if
$x \in D(A) = \{y \in H^1 \cap L^1;\ \beta(z) \in H^1 \cap L^1; \Delta(\beta(z) + \varepsilon z) \in L^1)$, *then* $y_\varepsilon \in C([0,T];L^1)$ *and* $z_\varepsilon = e^{W_\varepsilon} t_\varepsilon$, *obtained as the limit of a finite difference scheme is a mild solution to* (20) *in the space* L^1.

Lemma 2.2

Lemma

Let $x \in D(A)$*. Then* $y_\varepsilon \in L^\infty((0,T)\times\mathbb{R}^d)\cap L^\infty(0,T;L^1)$ *and*

$$\sup_{\varepsilon\in(0,1)}\{|y_\varepsilon|_{L^\infty((0,T)\times\mathbb{R}^d)}\} \le C(1+|x|_\infty), \tag{22}$$

$$\sup_{\varepsilon\in(0,1)}|y_\varepsilon|_{L^\infty(0,T;L^1)} \le C(|x|_1+1), \tag{23}$$

where C *is independent of* x*.*

Proof. Let $M = |x|_\infty + 1$ and $\alpha \in C^1[0,T]$ be such that $\alpha(0) = 0$, $\alpha' \geq 0$. Since y_ε is a strong solution of (19) in H^{-1}, we have

$$\begin{aligned}
&\frac{\partial}{\partial t}\,(y_\varepsilon - M - \alpha(t)) - e^{-W_\varepsilon}\Delta\left(\beta(e^{W_\varepsilon}y_\varepsilon) + \varepsilon e^{W_\varepsilon}y_\varepsilon\right)\\
&\qquad + e^{-W_\varepsilon}\Delta(\beta(e^{W_\varepsilon}(M+\alpha(t))) + \varepsilon e^{W_\varepsilon}(M+\alpha(t)))\\
&\qquad + \varepsilon e^{-W_\varepsilon}(\beta(e^{W_\varepsilon}y_\varepsilon) - \beta(e^{W_\varepsilon}(M+\alpha(t)))) \qquad\qquad (24)\\
&\qquad - e^{-W_\varepsilon}\mathrm{div}\left(e^{W_\varepsilon}D(y_\varepsilon - M - \alpha(t))\right)\\
&\qquad + \frac{1}{2}\,\mu(y_\varepsilon - M - \alpha(t)) = F_\varepsilon - \alpha'.
\end{aligned}$$

Now, we multiply (24) by $\mathrm{sign}(y_\varepsilon - M - \alpha(t))^+$ and integrate over $(0,t)\times\mathbb{R}^d$.

Lemma 2.3.

Lemma

Then there exists an increasing function $C : [0,\infty) \to (0,\infty)$ such that

$$\sup_{t\in[0,T]} |y_\varepsilon(t)|_2^2 + \int_0^T \int_{\mathbb{R}^d} |\nabla \beta(e^{W_\varepsilon} y_\varepsilon)|^2 ds\, d\xi \le C(|x|_\infty + |x|_1), \quad \forall \varepsilon \in (0,1], \tag{25}$$

for a constant $C > 0$, independent of $\varepsilon \in (0,1]$.

Proof of existence. Assume first that Hypotheses (i)–(iii) hold. Let $x \in D_0$. It follows, by Lemmas 2.2 and 2.3, that $\{\beta(e^{W_\varepsilon} y_\varepsilon)\}$ is bounded in $L^2(0,T;H^1)$, $\{y_\varepsilon\}$ is bounded in $L^\infty(0,T;L^2) \cap L^\infty((0,T)\times\mathbb{R}^d)$ and $\left\{\frac{dy_\varepsilon}{dt}\right\}$ is bounded in $L^2(0,T;H^{-1})$.

Moreover, taking into account that $\nabla\beta(e^{W_\varepsilon}y_\varepsilon)=\beta'(e^{W_\varepsilon}y_\varepsilon)\nabla(e^{W_\varepsilon}y_\varepsilon)$ and that by assumption (5) and estimate (22),

$$\beta'(e^{W_\varepsilon}y_\varepsilon)\geq\rho>0,\text{ a.e. in }(0,T)\times\mathbb{R}^d, \tag{26}$$

it follows that $\{y_\varepsilon\}$ is bounded in $L^2(0,T;H^1)$. As a matter of fact, we have

$$\begin{aligned}\sup_{\varepsilon\in(0,1]}\Bigg\{&\|y_\varepsilon\|_\infty+\|y_\varepsilon\|_{L^\infty(0,T;L^1)}+\|e^{W_\varepsilon}y_\varepsilon\|_{L^2(0,T;H^1)}+\|y_\varepsilon\|_{L^2(0,T;H^1)}\\ &+\|\beta(e^{W_\varepsilon}y_\varepsilon)\|_{L^2(0,T;H^1)}+\left\|\frac{dy_\varepsilon}{dt}\right\|_{L^2(0,T;H^{-1})}\Bigg\}\leq C^*(\omega),\ \omega\in\Omega,\end{aligned} \tag{27}$$

where C^* is $\mathcal{F}$–measurable and
$0<C^*(\omega)\leq Ce^{\|W\|_\infty}(\exp(\|\nabla W\|_\infty+\|\Delta W\|_\infty+1),\ \forall\omega\in\Omega.$

Hence, for fixed $\omega \in \Omega$ along a subsequence, again denoted $\{\varepsilon\}$, we have

$$\begin{array}{rcll} y_\varepsilon & \longrightarrow & y & \text{strongly in } L^2((0,T);L^2_{\text{loc}}(\mathbb{R}^d)) \\ & & & \text{weak-star in } L^\infty((0,T)\times\mathbb{R}^d) \\ & & & \text{weakly in } L^2(0,T;H^1), \\ \beta(e^{W_\varepsilon}y_\varepsilon) & \longrightarrow & \eta & \text{weakly in } L^2(0,T;H^1) \\ \dfrac{dy_\varepsilon}{dt} & \longrightarrow & \dfrac{dy}{dt} & \text{weakly in } L^2(0,T;H^{-1}) \\ W_\varepsilon & \longrightarrow & W & \text{in } C([0,T]\times\mathbb{R}^d), \end{array} \tag{28}$$

and so, letting $\varepsilon \to 0$ in equation (19), we see that

$$\begin{aligned} &\frac{dy}{dt} - e^{-W}\text{div}(De^W y) - e^{-W}\Delta\eta + \frac{1}{2}\,\mu y = 0 \text{ in } (0,T)\times\mathbb{R}^d, \\ &y(0) = x \ \text{ on } \mathbb{R}^d. \end{aligned} \tag{29}$$

To show that y is a solution to (10), it remains to be proven that $\eta = \beta(e^W y)$, a.e. in $(0,T) \times \mathbb{R}^d$. Since the map $z \to \beta(z)$ is maximal monotone in each $L^2((0,T) \times B_R)$, it is closed and so the latter follows by (28). Moreover, if the solution y to (10) is unique (we shall see later on that this happens if β is locally Lipschitz), it follows that the sequence $\{y_\varepsilon\}$ arising in (28) is independent of $\omega \in \Omega$, and so y is $(\mathcal{F}_t)_{t\geq 0}$–adapted.

Uniqueness.

Assume that, besides assumptions (i)-(iii), β is locally Lipschitz on $\mathbb{R}$ and consider y_1, y_2 to be two solutions to equation (10) satisfying (12)–(14) and let $z = y_1 - y_2$. We have

$$\begin{aligned} &\frac{\partial z}{\partial t} - e^{-W}\mathrm{div}(e^W Dz) - e^{-W}\Delta(\beta(e^W y_1) - \beta(e^W y_2)) + \frac{1}{2}\,\mu z = 0 \\ &\hspace{20em}\text{in } (0,T)\times\mathbb{R}^d, \\ &z(0) = 0 \ \text{ in } \mathbb{R}^d. \end{aligned} \tag{30}$$

$$\begin{aligned} \frac{\partial z}{\partial t} + (I-\Delta)(z\eta) &= e^{-W}\,\mathrm{div}(e^W Dz) - e^W\Delta(e^{-W})z\eta \\ &\quad - 2\nabla(e^{-W})\cdot\nabla(e^W z\eta) - \frac{1}{2}\,\mu z + z\eta, \end{aligned} \tag{31}$$

$$\eta = \begin{cases} \dfrac{\beta(e^W y_1) - \beta(e^W y_2)}{e^W z} & \text{on } [(\xi,t); z(t,\xi) \neq 0], \\ 0 & \text{on } [(\xi,t); z(t,\xi) = 0]. \end{cases}$$

We note that, by Hypothesis (ii) (5), we have, for some $\alpha_i = C^i(|x|_1 + |x|_\infty)$, $i = 0, 1$, where $C^i : [0, \infty) \to (0, \infty)$ are increasing continuous functions,

$$0 < \alpha_0 \le \eta \le \alpha_1, \text{ a.e. in } (0, T) \times \mathbb{R}^d. \tag{32}$$

We multiply (31) by $(I - \Delta)^{-1} z$ and integrate over $\mathbb{R}^d$ to get

$$\begin{aligned}
&\frac{1}{2}\,|z(t)|_{-1}^2 + \int_0^t \int_{\mathbb{R}^d} \eta z^2 ds\, d\xi \\
&= \frac{1}{2}\,|z(0)|_{-1}^2 + \int_0^t \int_{\mathbb{R}^d} e^{-W} \mathrm{div}(e^W Dz)(I - \Delta)^{-1} z ds\, d\xi \\
&- \int_0^t \int_{\mathbb{R}^d} e^W \Delta(e^{-W}) z\eta (I - \Delta)^{-1} z ds\, d\xi \\
&-2 \int_0^t \int_{\mathbb{R}^d} \nabla(e^{-W}) \cdot \nabla(e^W z\eta)(I - \Delta)^{-1} z ds\, d\xi \\
&+ \int_0^t \int_{\mathbb{R}^d} \left(-\frac{1}{2}\,\mu + \eta\right) z (I - \Delta)^{-1} z ds\, d\xi \\
&= \frac{1}{2}\,|z(0)|_{-1}^2 + \int_0^t (I_1 + I_2 + I_3 + I_4) ds.
\end{aligned} \tag{33}$$

Then, by (33), we obtain that

$$\frac{d}{dt}\,|z(t)|_{-1}^2 \le C_2|z(t)|_{-1}^2, \text{ a.e. } t > 0.$$

The stochastic equation with nonlinear drift

We consider here the equation

$$\begin{aligned} & dX - \operatorname{div}(a(X))dt - \Delta\beta(X)dt = X\,dW \text{ in } (0,T)\times\mathbb{R}^d, \\ & X(0,\xi) = x(\xi),\ \xi\in\mathbb{R}^d, \end{aligned} \tag{34}$$

where β and W are as in Section 1, while $a:\mathbb{R}\to\mathbb{R}^d$ satisfies the following assumption

(iv) *a is Lipschitzian and $a(0)=0$.*

The strong solution X to equation (34) is defined as for equation (1).

$$u_\xi = \nabla u,\ \ u_{\xi\xi} = \Delta u.$$

$$\begin{aligned} & \frac{\partial y}{\partial t} - e^{-W}\operatorname{div}(a(e^W y)) - e^{-W}(\beta(e^W y))_{\xi\xi} + \frac{1}{2}\,\mu y = 0 \text{ in } (0,T)\times\mathbb{R}^d, \\ & y(t,\xi) = x(\xi). \end{aligned} \tag{35}$$

Theorem 3.1

Theorem

If assumptions (ii), (iii), (iv) *hold and* β *is locally Lipschitz, for each* $x \in D_0$, *there is a unique strong solution* y *to equation* (35) *satisfying* (12)–(14). *Moreover, the process* y *is* $(\mathcal{F}_t)_{t\geq 0}$*-adapted and, if* $x \geq 0$, *a.e. on* $\mathbb{R}^d$, *then* $y \geq 0$, *a.e. on* $(0,T) \times \mathbb{R}^d$, *and the map* $D_0 \ni x \to y(\cdot, x)$ *is Lipschitz from* H^{-1} *to* $C([0,T], H^{-1})$ *on balls in* $L^1 \cap L^\infty$ *and extends to a strong solution to* (34) *satisfying* (12), (14), *for all* $x \in L^1 \cap L^\infty$.

We consider the approximating equation (see (19))

$$
\begin{aligned}
&\frac{\partial y_\varepsilon}{\partial t} - e^{-W_\varepsilon}\operatorname{div}(a(e^{W_\varepsilon}y_\varepsilon)) - e^{-W_\varepsilon}\beta(e^{W_\varepsilon}y_\varepsilon) - \varepsilon e^{-W_\varepsilon}(e^{W_\varepsilon}y_\varepsilon)_{\xi\xi} \\
&\qquad + \varepsilon e^{-W_\varepsilon}\beta(e^{W_\varepsilon}y_\varepsilon) + \frac{1}{2}\mu y_\varepsilon = 0 \text{ in } (0,T)\times\mathbb{R}^d, \qquad (36) \\
&y_\varepsilon(0,\xi) = x(\xi),\ \xi \in \mathbb{R}^d.
\end{aligned}
$$

Uniqueness. If β is locally Lipschitz and y_1, y_2 are solutions to (34), for $z = y_1 - y_2$, we get (see (30))

$$\frac{\partial z}{\partial t} - \mathrm{div}(a(e^W y_1) - a(e^W y_2)) - e^{-W}(\beta(e^W y_1) - \beta(e^W y_2))_{\xi\xi} + \frac{1}{2}\,\mu z = 0$$
$$z(0) = 0,$$

and, arguing as in the proof of uniqueness in Theorem 1, we get $z \equiv 0$. If $\beta \in L^1_{\mathrm{loc}}(\mathbb{R})$, then, multiplying scalarly in L^2 by $(I - \Delta)^{-1} z$ and using the local Lipschitzianity of β and a, we get as above the estimates (**??**)–(**??**). ■

By Theorem 3.1, we have

Corollary

If assumptions (ii), (iii), (iv) *hold and* β *is locally Lipschitz, then for each* $x \in D_0$ *there is a unique strong solution* X *to the stochastic equation* (34), *which satisfies*

$$Xe^{-W} \in W^{1,2}([0,T];H^{-1}),\ \mathbb{P}\text{-}a.s., \tag{37}$$

and $X \geq 0$, *a.e. on* $(0,T) \times \mathbb{R}^d \times \Omega$ *if* $x \geq 0$, *a.e. on* $\mathbb{R}^d$. *Moreover, the map* $x \mapsto X(t,x)$ *is* H^{-1}*-Lipschitz from balls in* $L^1 \cap L^\infty$ *to* $C([0,T];H^{-1})$.

Remark

If a is not Lipschitz, one cannot expect a strong solution for equation (34). In the deterministic case, if $\beta \equiv 0$, equation (34) reduces to a first order quasilinear equation previously studied by S. Kruzkov [13] (see, also, [8]), who introduced and proved existence of a generalized solution involving the so-called "entropy" conditions. (See also [2] for the case where β is present.) So, also in this case, one might expect to have a generalized solution in sense of Kruzkov, but this remains to be done.

References

1 Barbu, V., *Nonlinear Differential Equations of Monotone Type in Banach Spaces*, Springer, 2010.

2 Barbu, V., Generalized solutions to nonlinear Fokker–Planck equations, *J. Differential Equations*, 261 (2016), 2446-2471.

3 Barbu, V., Da Prato, G., Röckner, M., *Stochastic Porous Media Equations*, Lecture Notes in Mathematics, 2163, Springer, 2016.

4 Barbu, V., Röckner, M., On a random scaled porous media equation, *J. Diff. Equations*, 251 (9) (2011), 2494-2514.

5 Barbu, V., Röckner, M., Stochastic variational inequalities and applications to the total variation flow perturbed by linear multiplicative noise, *Arhive Rational Mech. Anal.*, 209 (2013), 797-834.

6 Barbu, V., Röckner, M., An operatorial approach to stochastic partial differential equations driven by linear multiplicative noise, *J. European Math. Soc.*, 17 (2015), 1789–1815.

7 Barbu, V., Röckner, M., Russo, F., Stochastic porous media equations in $\mathbb{R}^d$, *J. Math. Pures Appl.*, 103 (2015), 1024-1051.

8 Benilan, Ph., Kružkov, S.N., Conservation law with continuous flux functions, *NoDEA*, 3 (1996), 396-419.

9 P.H. Chavanis, Hamiltonian and Brownian systems with long-range interactions: V. Stochastic kinetic equations and theory of fluctuations, *Physica* A, 387(2008), 5716-5740.

10 P.H. Chavanis, Generalized stochastic Fokker-Planck equations, *Entropy*, 17(2015), 3205-3252.

11 Frank, T.D., *Nonlinear Fokker–Planck Equations*, Springer-Verlag, Berlin, 2005.

12 Frank, T.D., Daffertshofer, A., *H*-Theorem for nonlinear Fokker–Planck equations related to generalized thermostatistics, *Physica* A, 295 (2001), 455-474.

13 Kružkov, S.N., First order quasilinear equations in several independent variables, *Math.USSR – Sbornik*, 10 (1970), 217-243.

14 Plastino, A.R., Plastino, A., Nonextensive statistical mechanics and generalized Fokker–Planck equations, *Physica* A, 222 (1995), 347-355.

15 Prévot, C., Röckner, M., *A Concise Course on Stochastic Partial Differential Equations*, Springer, 2010.

16 Schwämmle, V., Nobre, F.D., Curado, E.M.F., Consequences of the *H* theorem from nonlinear Fokker–Planck equations, *Physical Review* E, 76 (2007), 041123.