Some nice features of AP-schemes

Anisotropic transport equations

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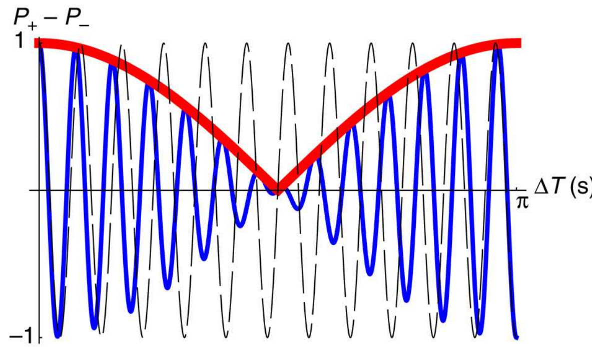
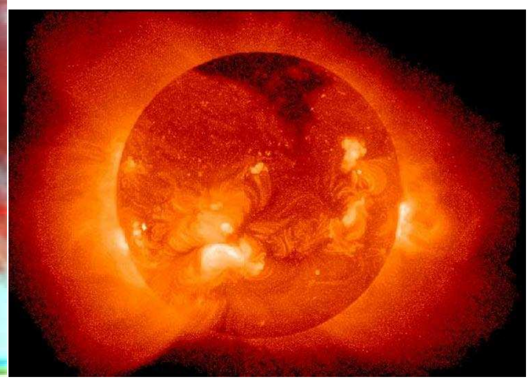
Objective: Numerical study of highly anisotropic, multi-scale problems

- Many pb. in nature exhibit multi-scale behaviours, which can be rather different in character
- Typical: occurence of one or several small/large parameters (Reynolds, Peclet, Mach nbr. etc)
- General, unified treatment is impossible



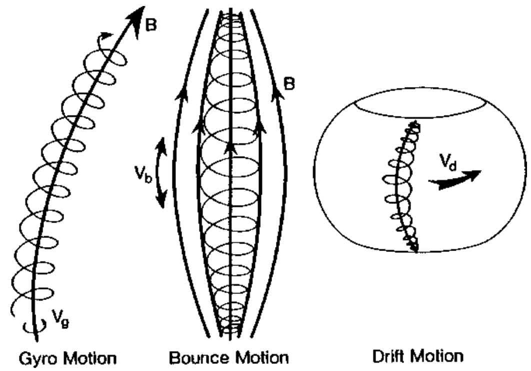


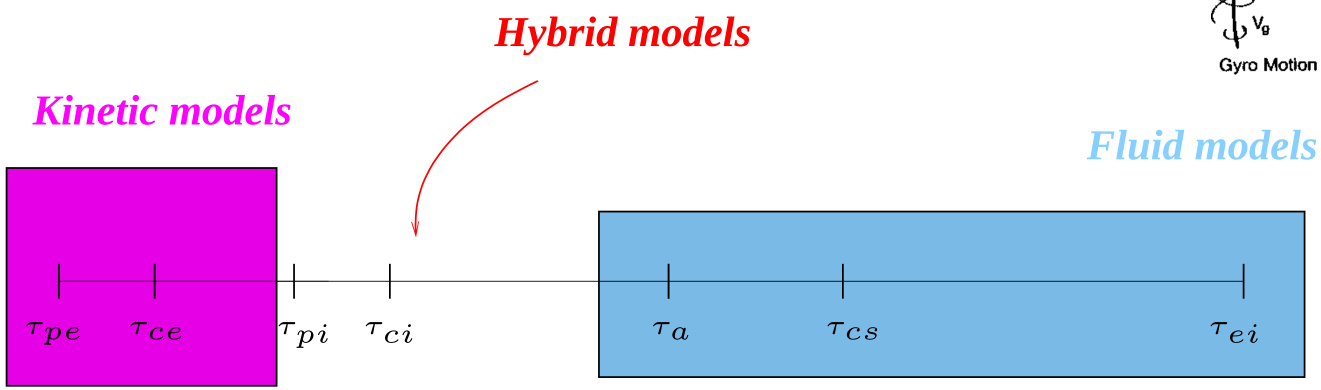
Multi-scale plasma dynamics

Plasma dynamics is characterized by multi-scale phenomena

$\Rightarrow$ Strong magn. fields create anisotropies

$\Rightarrow$ Particles gyrate around the field lines





$\tau_{pe,pi}$: Inv. electr./ion plasma freq.

$\tau_{ce,ci}$: Electr./ion cyclotron period

τ_a: Alfen wave period

τ_{cs}: Ion sound period

τ_{ei}: Electr-ion collision time

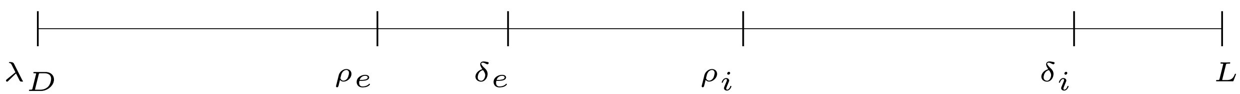
λ_D: Debye length

$\rho_{e,i}$: Electr./ion Larmor radius

$\delta_{e,i} = c/\omega_{pe,pi}$: Electr./ion skin depth

$\omega_{pe,pi}$: Electr./ion plasma frequency

c: sound speed



A small-scale numerical simulation is out of reach

- requires mesh-sizes dependent on small scale param. $\varepsilon \ll 1$
- excessive computational time and memory space are needed to capture small scales

It is not always of interest to resolve the details at the small scale. Multi-scale strategies are much more adequate!

- homogeneisation, domain decomposition, multi-grids, multi-scale methods based on wavelets or finite elements, multi-scale variational methods

Essential feature of these methods

- capture efficiently the large scale behavior of the solution, without resolving the small scale features

Difficulty: Resolution of multiscale pb. can be very difficult, if the pb. becomes singular, as one of the parameters $\varepsilon \to 0$

- (P^ε) sing. perturbed pb. with sol. f_ε;
- the seq. f_ε converges towards f_0, sol. of a limit pb. (P^0);
- the limit pb. (P^0) is different in type from the initial (P^ε);
- standard schemes would require $\Delta t, \Delta x \sim \varepsilon$ for stability.

Definition: A scheme $P^{\varepsilon,h}$ is AP iff it is convergent for $h \to 0$ uniformely in ε, *i.e.*

$$\begin{array}{ccc} P^{\varepsilon,h} & \xrightarrow{h \to 0} & P^{\varepsilon} \\ \downarrow{\scriptstyle \varepsilon \to 0} & & \downarrow{\scriptstyle \varepsilon \to 0} \\ P^{0,h} & \xrightarrow{h \to 0} & P^{0} \end{array}$$

AP-procedure:

- requires that the limit problem (P^0) is identified and well-posed;
- consists in trying to mimic at discrete level the asymptotic behaviour of the sing. perturbed pb. sol. f_ε;
- requires a sufficient degree of implicitness (not obvious).

Advantages:

- gives accurate and stable results, with no restrictions on the computational mesh;
- enables to capture automatically the Limit model P^0, if $\varepsilon \to 0$ (micro-macro transition);
- no more coupling needed, if $\varepsilon(x)$ is variable.

Fundamental kinetic model: Vlasov/Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f + \frac{q}{m}(E + v \times B) \cdot \nabla_v f = Q(f)$$

Several small scales/parameters occur, leading to diff. regimes:

- Hydrodynamic scaling [Filbet/Jin; Dimarco/Pareschi]

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} Q(f)$$

➠ $0 < \varepsilon \ll 1$: mean free path (Knudsen nbr.)

➠ in the limit $\varepsilon \to 0$, one gets the compressible Euler eq.

➠ AP-scheme: Decomposition of the source term in stiff- and non-stiff part

$$\frac{Q(f)}{\varepsilon} = \frac{Q(f) - P(f)}{\varepsilon} + \frac{P(f)}{\varepsilon}$$

- Drift-Diffusion scaling [Klar; Lemou/Mieussens]

$$\partial_t f + \frac{1}{\varepsilon}(v \cdot \nabla_x f + E \cdot \nabla_v f) = \frac{1}{\varepsilon^2} Q(f)$$

  - $0 < \varepsilon \ll 1$: mean free path; long-time asymp.
  - in the limit $\varepsilon \to 0$, one gets the Drift-Diffusion model
  - AP-scheme: Micro-Macro decomp. $f = \rho M + \varepsilon g$

- High-field limit, strong magn. fields [Bostan, Frenod, Golse, Saint-Raymond]

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f + \frac{1}{\varepsilon}(v \times B) \cdot \nabla_v f = 0$$

  - $0 < \varepsilon \ll 1$: cyclotronic period; strong B-field;
  - in the limit $\varepsilon \to 0$, one gets the gyro-kinetic model.

Kinetic models and specific limit regimes

- Adiabatic scaling [Negulescu,...]

$$\partial_t f + \frac{1}{\varepsilon} v \cdot \nabla_x f - \frac{1}{\varepsilon}(E + \frac{1}{\varepsilon} v \times B) \cdot \nabla_v f = \frac{1}{\varepsilon} Q(f)$$

➠ $0 < \varepsilon \ll 1$: small electron/ion mass ratio, collisionality, strong B-fields

➠ in the limit $\varepsilon \to 0$, one gets the electr. Boltzmann rel.

Diff. regimes, Diff. kind of asymptotic behaviour as $\varepsilon \to 0$:

➠ diffusive behaviour (HD,DD)

➠ highly oscillating behaviour (BE)

$\Rightarrow$ different kinds of num. schemes required !

Evolution pb. with stiff transport terms

- Aim: Efficient num. resolution of multi-scale pb. of the type:

$$\partial_t f^\varepsilon + \frac{\mathbf{b}}{\varepsilon} \cdot \nabla f^\varepsilon + \mathcal{L} f^\varepsilon = 0$$

- Motivating physical models:

➠ Anisotropic Fokker-Planck eq. in the gyro-kinetic scaling:

$$\partial_t f_i^\varepsilon + v \cdot \nabla_x f_i^\varepsilon + E \cdot \nabla_v f_i^\varepsilon + \frac{1}{\varepsilon}(v \times B) \cdot \nabla_v f_i^\varepsilon = \eta \nabla_{\mathbf{v}} \cdot [\mathbf{v} f_i^\varepsilon + \nabla_{\mathbf{v}} f_i^\varepsilon] \,.$$

➠ Vlasov-Poisson eq. in the long-time asymptotics:

$$\begin{cases} \partial_t f_e + \frac{1}{\varepsilon} v\, \partial_x f_e - \frac{1}{\varepsilon} E(t,x)\, \partial_v f_e = 0 \,, & \forall t \in \mathbb{R}^+ \,, \;\; \forall (x,v) \in \Omega \subset \mathbb{R}^2 \\ -\partial_{xx}\varphi = 1 - n_e \,, \qquad n_e(t,x) = \int_{\mathbb{R}} f_e(t,x,v)\, dv \,, & E = -\partial_x \varphi \,. \end{cases}$$

➠ Euler 2D eq. / Vorticity eq. in the long-time asymptotics:

$$\partial_t \omega^\varepsilon + \frac{1}{\varepsilon} \mathbf{u}^\varepsilon \cdot \nabla \omega^\varepsilon = 0 \,, \qquad -\Delta \Psi^\varepsilon = \omega^\varepsilon \,, \qquad \mathbf{u}^\varepsilon =^\perp \nabla \Psi^\varepsilon \,.$$

I. Vlasov eq. in the gyro-kinetic regime

Work based on:

[1] B. Fedele, C. Negulescu, *Numerical study of an anisotropic Vlasov equation arising in plasma physics*, to appear in KRM (Kinetic and Related Models), 2018.

Starting model: Anisotropic Vlasov eq. (gyro-kinetic regime)

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon + E \cdot \nabla_v f^\varepsilon + \frac{1}{\varepsilon}(v \times B) \cdot \nabla_v f^\varepsilon = 0$$

Aim: design efficient numerical scheme

- accuracy and stability independent on ε (AP property);
- rapid, not time and memory consuming simulations;
- simple implementation, practical scheme.

Important questions:

- what is the asymptotic behaviour of the solution f^ε as $\varepsilon \to 0$?
- what does one want to see in the asymptotic limit? All microscopic information or only the macroscopic information?

- Magnetic field $\mathbf{B}$: direction $\mathbf{b}(\mathbf{x}) := \frac{\mathbf{B}(\mathbf{x})}{|\mathbf{B}(\mathbf{x})|}$; magnitude $\mathfrak{M}(\mathbf{x}) := |\mathbf{B}(\mathbf{x})|$; $\nabla \cdot \mathbf{B} = 0$.

- Dominant operator: $\mathcal{T} := (\mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}}$

$$\mathcal{T} : D(\mathcal{T}) \to L^2(\Omega \times \mathbb{R}^3)\,, \qquad D(\mathcal{T}) := \left\{ f \in L^2(\Omega \times \mathbb{R}^3) \ / \ \mathcal{T} f \in L^2(\Omega \times \mathbb{R}^3) \right\}\,.$$

- Characteristics: $\mathcal{C}_{\mathbf{x},\mathbf{v}} := \{(X(s;\mathbf{x},\mathbf{v}), V(s;\mathbf{x},\mathbf{v})),\ s \in \mathbb{R}\}$

$$\begin{cases} \dfrac{dX}{ds} = 0\,, \\[2mm] \dfrac{dV}{ds} = \mathfrak{M}(X(s))\, V(s) \times \mathbf{b}(X(s))\,, \end{cases}$$

$$X(s;\mathbf{x},\mathbf{v}) = \mathbf{x}\,, \quad V(s;\mathbf{x},\mathbf{v}) = \cos(\mathfrak{M}(\mathbf{x})\, s)\, \mathbf{v}_\perp + \sin(\mathfrak{M}(\mathbf{x})\, s)\, {}^\perp\mathbf{v} + \mathbf{v}_{||}\,, \quad \forall s \in \mathbb{R}$$

➡ periodic trajectories with period $T_c(\mathbf{x}) := \frac{2\,\pi}{\mathfrak{M}(\mathbf{x})}$;

➡ invariants: $\mathbf{x}$, $|\mathbf{v}_\perp|$ and $\mathbf{v}_{||}$;

➡ $\ker \mathcal{T} := \{f \in L^2(\Omega \times \mathbb{R}^3) \ / \ \exists\, g : \Omega \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ st. $f(\mathbf{x},\mathbf{v}) = g(\mathbf{x}, v_{\mathbf{b}}, |\mathbf{v}_\perp|)\}$.

Asymptotic behaviour as $\varepsilon \to 0$

- Cylindrical coordinates with respect to $\mathbf{b}$: $\mathbb{S}^1_{\mathbf{b}} := \{\varpi \in \mathbb{R}^3 \,/\, |\varpi| = 1,\ \varpi \cdot \mathbf{b} = 0\}$

$$\mathbf{v} = \mathbf{v}_{||} + \mathbf{v}_{\perp} = v_{\mathbf{b}}\, \mathbf{b} + r\, \varpi\,, \quad r := |\mathbf{v}_{\perp}|\,, \quad \varpi := \frac{\mathbf{v}_{\perp}}{|\mathbf{v}_{\perp}|} \in \mathbb{S}^1_{\mathbf{b}}\,.$$

- Gyro-average operator: $\mathcal{J} : L^2(\Omega \times \mathbb{R}^3) \to \ker(\mathcal{T})$ (orthog. proj. on $\ker(\mathcal{T})$)

$$\begin{aligned} \mathcal{J}(f)(\mathbf{x},\mathbf{v}) &:= \frac{1}{T_c(\mathbf{x})} \int_0^{T_c(\mathbf{x})} f(X(s;\mathbf{x},\mathbf{v}), V(s;\mathbf{x},\mathbf{v}))\, ds \\ &= \frac{1}{2\,\pi} \int_{\mathbb{S}^1_{\mathbf{b}}} f(\mathbf{x}, v_{\mathbf{b}}\, \mathbf{b} + |\mathbf{v}_{\perp}|\, \varpi)\, d\varpi\,. \end{aligned}$$

- Decomposition : $L^2(\Omega \times \mathbb{R}^3) = \ker(\mathcal{T}) \oplus^{\perp} \ker(\mathcal{J})$ $\quad f = \mathcal{J}(f) + f'$

$$\mathcal{T} : D(\mathcal{T}) \cap \ker(\mathcal{J}) \to \ker(\mathcal{J})\,, \quad \text{bij. map}$$

- Limit model for $\varepsilon \to 0$:

$$\begin{cases} f^0 \in \ker(\mathcal{T}) \quad \text{i.e.} \quad (v \times B) \cdot \nabla_v f^0 = 0 \\ \partial_t f^0 + \mathcal{J}(v \cdot \nabla_x f^0) + \mathcal{J}(E \cdot \nabla_v f^0) = 0\,. \end{cases}$$

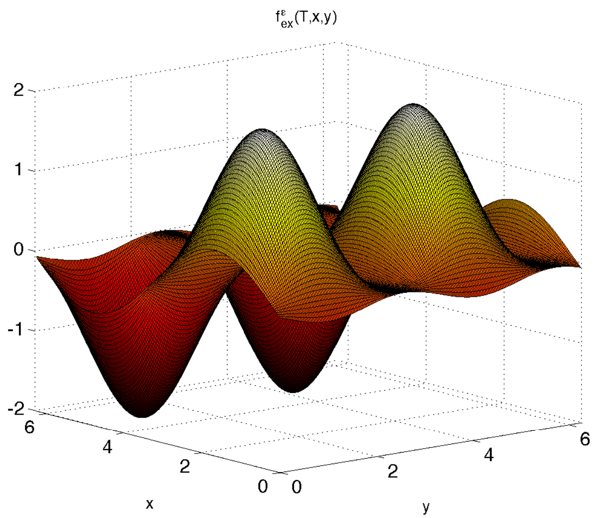
Simple toy model

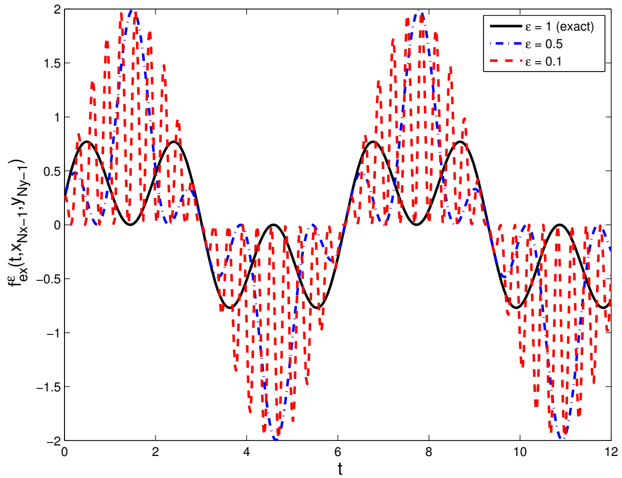
- Simplified toy model:

$$(V)_\varepsilon \quad \begin{cases} \partial_t f^\varepsilon + a\,\partial_x f^\varepsilon + \dfrac{b}{\varepsilon}\,\partial_y f^\varepsilon = 0\,, \quad \forall (t,x,y) \in [0,T] \times [0,L_x] \times [0,L_y]\,, \\ f^\varepsilon(0,x,y) = f_{in}(x,y) = \sin(x)\big(\cos(2y)+1\big)\,. \end{cases}$$

- Exact solution:

$$f^\varepsilon_{ex}(t,x,y) = f_{in}(x-at, y-\frac{b}{\varepsilon}t) = \sin\left(x-at\right)\left[\cos\left(2\left(y-\frac{b}{\varepsilon}t\right)\right)+1\right],$$





Simple toy model

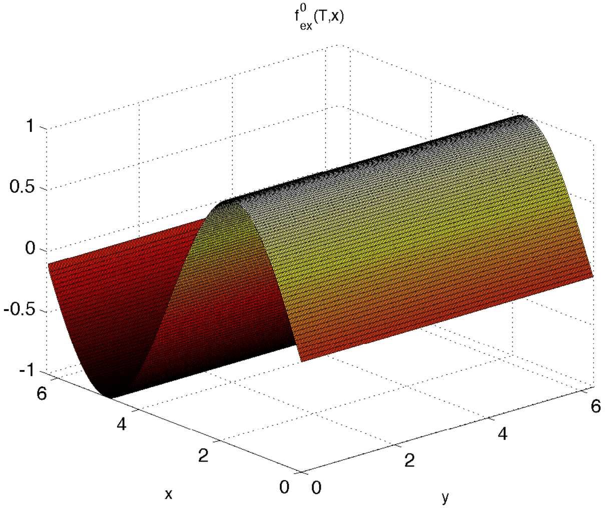
- Limit model:

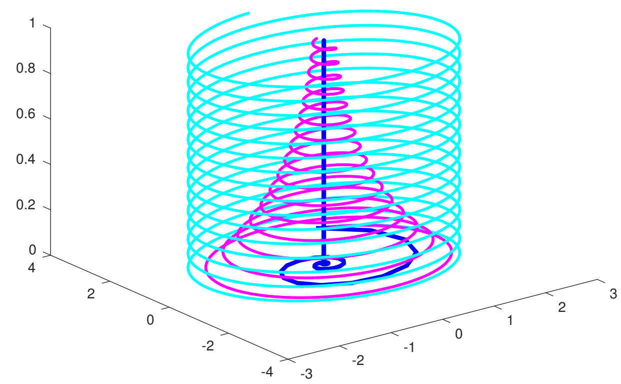
$$\begin{cases} \partial_t f^0 + a\partial_x f^0 = 0\,, & \forall (t,x) \in [0,T] \times [0,L_x], \\ f^0(0,x) = \bar{f}_{in}(x)\,, & \forall x \in [0,L_x]\,, \end{cases}$$

Average or Projection: $\bar{f}^\varepsilon(t,x) := \frac{1}{L_y}\int_0^{L_y} f^\varepsilon(t,x,y)dy\,.$

- Exact limit solution:

$$f^0(t,x) = \bar{f}_{in}(x-at) = \sin\left(x-at\right).$$





Numerical schemes

- Field aligned or NOT-field aligned configuration
- Micro-Macro scheme (Fourier schemes)
- IMEX or Implicit schemes
- Lagrange-Multiplier scheme (NEW SCHEME!)

Aligned versus Not-Aligned configuration

Model $$\partial_t f + \mathbf{v} \cdot \nabla_x f + \Big[\mathbf{E} + \frac{1}{\varepsilon}(\mathbf{v} \times \mathbf{B})\Big] \cdot \nabla_v f = 0\,, \qquad \mathbf{B} = \mathbf{e}_z\,.$$

$$\mathbf{v}_{||} = (0,0,v_z)^t\,, \quad \mathbf{v}_\perp = (v_x, v_y, 0)^t\,, \quad {}^\perp\mathbf{v} := (v_y, -v_x, 0)^t = \mathbf{v} \times \mathbf{B}$$

$$v = (v_x, v_y, v_z) \Leftrightarrow (r, \theta, v_z)\,, \quad \begin{cases} v_x := r\cos(\theta) & , \quad \theta \in [0, 2\pi) \\ v_y := r\sin(\theta) & \quad r \geq 0 \end{cases} .$$

$$\partial_t F + v_z \partial_z F + E_z \partial_{v_z} F + (E_x \cos\theta + E_y \sin\theta)\,\partial_r F - \frac{1}{r}(E_x \sin\theta - E_y\cos\theta)\,\partial_\theta F$$
$$+ r\,(\cos\theta \partial_x F + \sin\theta \partial_y F) - \frac{1}{\varepsilon}\partial_\theta F = 0\,.$$

- First toy model - Polar, field-aligned configuration: ($E \equiv 0$, ...)

$$\partial_t F + r\,\cos\theta\,\partial_x F - \frac{1}{\varepsilon}\,\partial_\theta F = 0\,.$$

- Second toy model - Cartesian, not field-aligned configuration: ($E \equiv 0$, ...)

$$\partial_t f + \frac{v_y}{\varepsilon}\partial_{v_x} f - \frac{v_x}{\varepsilon}\partial_{v_y} f = 0\,.$$

First toy model:

$$(V)_\epsilon \quad \begin{cases} \partial_t f^\epsilon + a\,\partial_x f^\epsilon + \dfrac{b}{\epsilon}\,\partial_y f^\epsilon = 0\,, \quad \forall (t,x,y) \in [0,T] \times [0,L_x] \times [0,L_y]\,, \\ f^\epsilon(0,x,y) = f_{in}(x,y)\,, \end{cases}$$

Micro-Macro decomposition:

$$H^\epsilon(t,x) := \frac{1}{L_y} \int_0^{L_y} f^\epsilon(t,x,y)\,dy\,, \quad h^\epsilon(t,x,y) := f^\epsilon(t,x,y) - H^\epsilon(t,x)\,, \quad \bar{h^\epsilon} = 0\,.$$

$$(MM)_\epsilon \quad \begin{cases} \partial_t H^\epsilon + a\partial_x H^\epsilon = 0\,, & \forall (t,x) \in [0,T] \times [0,L_x] \\ \partial_t h^\epsilon + a\partial_x h^\epsilon + \dfrac{b}{\epsilon}\partial_y h^\epsilon = 0\,, & \forall (t,x,y) \in [0,T] \times \Omega \\ \bar{h^\epsilon} = 0\,, & \forall (t,x) \in [0,T] \times [0,L_x]\,. \end{cases}$$

Advantages/Disadvantages:

- AP-scheme; well-posed in the limit $\varepsilon \to 0$;
- the average procedure can be cumbersome in general configurations.

IMEX-schemes

▶ First toy model :

$$(IMEX)_\varepsilon \quad \frac{f^{\varepsilon,n+1}-f^{\varepsilon,n}}{\Delta t} + a\,\partial_x f^{\varepsilon,n} + \frac{b}{\varepsilon}\,\partial_y f^{\varepsilon,n+1} = 0\,, \quad \forall n \geq 0\,.$$

▶ Second toy model :

$$(IMP)_\varepsilon \quad \frac{f^{\varepsilon,n+1}-f^{\varepsilon,n}}{\Delta t} + \frac{y}{\varepsilon}\,\partial_x f^{\varepsilon,n+1} - \frac{x}{\varepsilon}\,\partial_y f^{\varepsilon,n+1} = 0, \qquad \forall n \geqslant 0\,.$$

▶ General gyro-kinetic model :

$$(IMEX)_\varepsilon \quad \frac{f^{\varepsilon,n+1}-f^{\varepsilon,n}}{\Delta t} + \mathbf{v}\cdot\nabla_x f^{\varepsilon,n} + \mathbf{E}\cdot\nabla_v f^{\varepsilon,n} + \frac{1}{\varepsilon}\,(\mathbf{v}\times\mathbf{B})\cdot\nabla_v f^{\varepsilon,n+1} = 0\,.$$

Advantages/Disadvantages:

- very simple scheme;
- still ill-posed in the limit $\varepsilon \to 0$, not AP, will not capture the adequate limit, breaks-down for $\varepsilon \ll 1$;
- big difference between field-aligned and NOT field-aligned configuration.

Lagrange-Multiplier scheme

IDEA: $f^\varepsilon = p^\varepsilon + \varepsilon q^\varepsilon$ such that $\mathcal{T} p^\varepsilon = 0$

- First toy model :

$$(La)_\varepsilon \quad \begin{cases} \partial_t f^\varepsilon + a\,\partial_x f^\varepsilon + b\,\partial_y q^\varepsilon = 0\,, & \forall (t,x,y) \in [0,T]\times\Omega \\ \partial_y f^\varepsilon = \varepsilon\,\partial_y q^\varepsilon\,, & \forall (t,x,y) \in [0,T]\times\Omega \\ q^\varepsilon_{|\Gamma_{in}} = 0\,, & \text{(inflow constraint to fix } q^\varepsilon) \end{cases}$$

- Second toy model :

$$(La)_\varepsilon \quad \begin{cases} \partial_t f^\varepsilon + y\,\partial_x q^\varepsilon - x\,\partial_y q^\varepsilon = 0, \\ y\partial_x f^\varepsilon - x\,\partial_y f^\varepsilon = \varepsilon\Big(y\,\partial_x q^\varepsilon - x\,\partial_y q^\varepsilon\Big) - (\Delta x \Delta y)^\gamma\, q^\varepsilon\,, \quad \text{(regularization to fix } q^\varepsilon) \end{cases}$$

- General gyro-kinetic model :

$$(La)_\varepsilon \quad \begin{cases} \partial_t f^\varepsilon + \mathbf{v}\cdot\nabla_x f^\varepsilon + \mathbf{E}\cdot\nabla_v f^\varepsilon + \mathcal{T}\, q^\varepsilon = 0, \\ \mathcal{T}\, f^\varepsilon = \varepsilon \mathcal{T}\, q^\varepsilon - (\Delta x \Delta y)^\gamma\, q^\varepsilon\,. \end{cases}$$

Lagrange-Multiplier scheme

▶ First toy model :

$$(La)_\varepsilon \quad \begin{cases} \frac{f^{\varepsilon,n+1}-f^{\varepsilon,n}}{\Delta t} + a\,\partial_x f^{\varepsilon,n} + b\,\partial_y q^{\varepsilon,n+1} = 0 \\ \partial_y f^{\varepsilon,n+1} = \varepsilon\,\partial_y q^{\varepsilon,n+1} \\ q^{\varepsilon,n+1}_{|\Gamma_{in}} = 0\,. \end{cases}$$

▶ Second toy model :

$$(La)_\varepsilon \quad \begin{cases} \dfrac{f^{\varepsilon,n+1}-f^{\varepsilon,n}}{\Delta t} + y\,\partial_x q^{\varepsilon,n+1} - x\,\partial_y q^{\varepsilon,n+1} = 0, \\ y\,\partial_x f^{\varepsilon,n+1} - x\,\partial_y f^{\varepsilon,n+1} = \varepsilon\Big(y\,\partial_x q^{\varepsilon,n+1} - x\,\partial_y q^{\varepsilon,n+1}\Big) - (\Delta x \Delta y)^\gamma\, q^{\varepsilon,n+1}\,. \end{cases}$$

Advantages/Disadvantages:

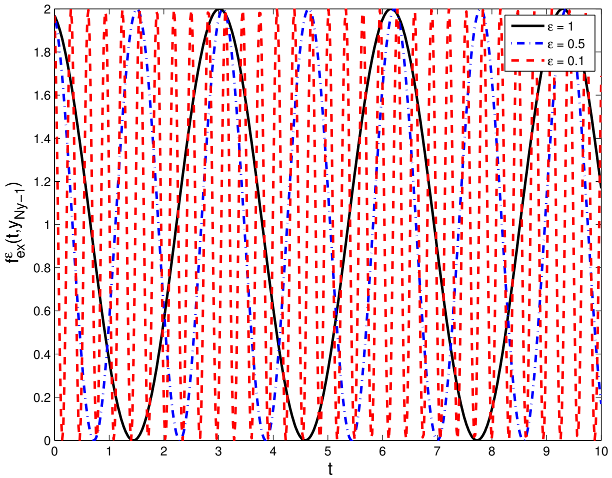
- the scheme is Asymptotic-Preserving, choice of the grid indep. on ε;
- Two unknowns $(f^\varepsilon, q^\varepsilon)$ instead of only one in standard schemes;
- regularization procedure to fix q^ε on the field lines, is delicate.

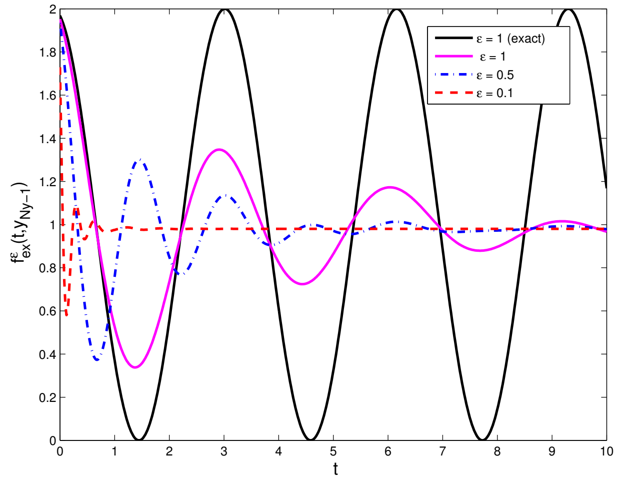
Numerical results

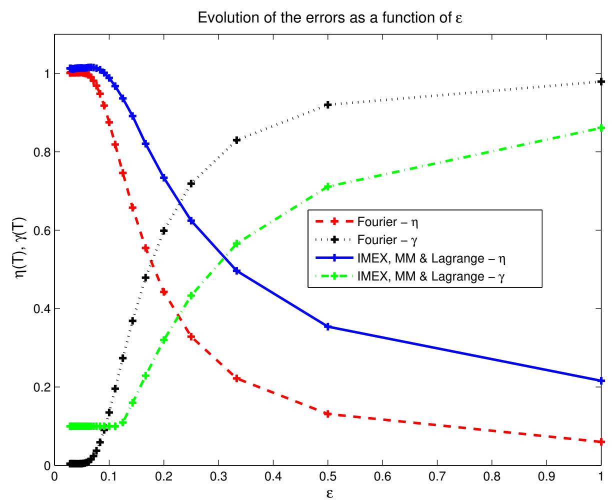
- IMEX versus Lagrange-Multiplier scheme
- Field aligned and NOT field-aligned configuration
- Error analysis (truncation and round-off errors)

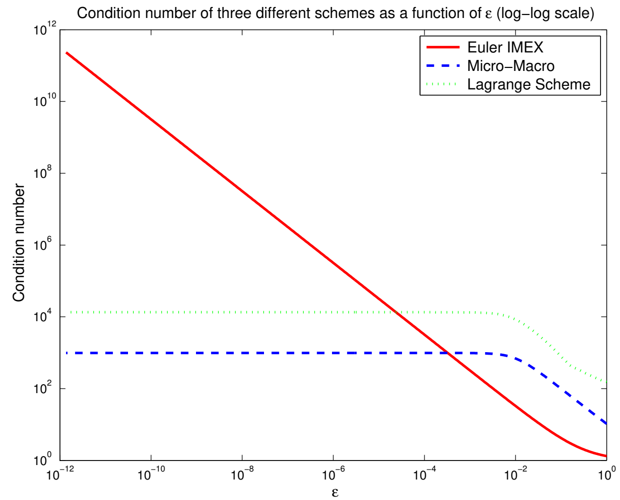
IMEX versus Lagrange-Multiplier: First toy model

$$\partial_t f^\varepsilon + a\,\partial_x f^\varepsilon + \frac{b}{\varepsilon}\,\partial_y f^\varepsilon = 0$$









- Global errors: $$||\mathcal{F}_{ex} - \mathcal{F}_{num}|| \leq ||\mathcal{F}_{ex} - \mathcal{F}|| + ||\mathcal{F} - \mathcal{F}_{num}||$$

$$\mathcal{A}\,\mathcal{F}_{ex} = \mathcal{B} + \varepsilon\mathcal{T}\,, \qquad \mathcal{A}\,\mathcal{F} = \mathcal{B}\,, \qquad (\mathcal{A} + \delta\mathcal{A})\,\mathcal{F}_{num} = \mathcal{B} + \delta\mathcal{B}$$

- Truncation errors IMEX/La:

$$\mathcal{T}_{I/L} = -\nabla \cdot (D_{I/L} \nabla f^{\varepsilon}) + \mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta x^2) + \mathcal{O}(\Delta y^2)$$

$$D_I := \begin{pmatrix} \frac{a\Delta x}{2}(1-\alpha) & 0 \\ 0 & \frac{b\Delta y}{2\varepsilon}\left(1 + \frac{\beta}{\varepsilon}\right) \end{pmatrix}, \quad \alpha := \frac{a\Delta t}{\Delta x}, \quad \beta := \frac{b\Delta t}{\Delta y}.$$

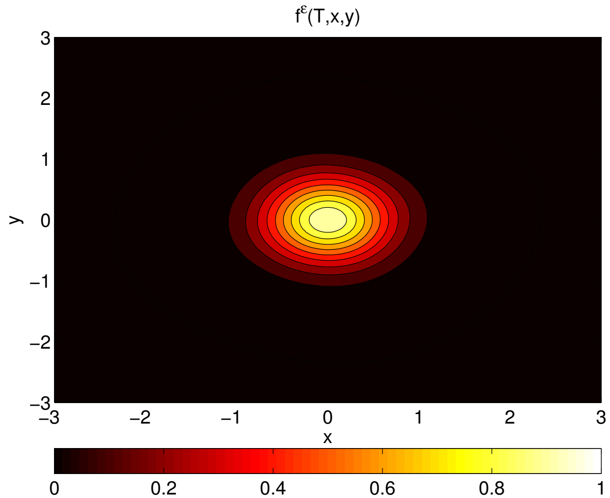
$$D_L := \begin{pmatrix} \frac{a\Delta x}{2}(1-\alpha) & 0 \\ 0 & \frac{b\Delta y}{2\varepsilon}\left(1 + \frac{\beta}{\varepsilon}\right) \end{pmatrix}, \quad \alpha := \frac{a\Delta t}{\Delta x}, \quad \beta := \frac{b\Delta t}{\Delta y}.$$

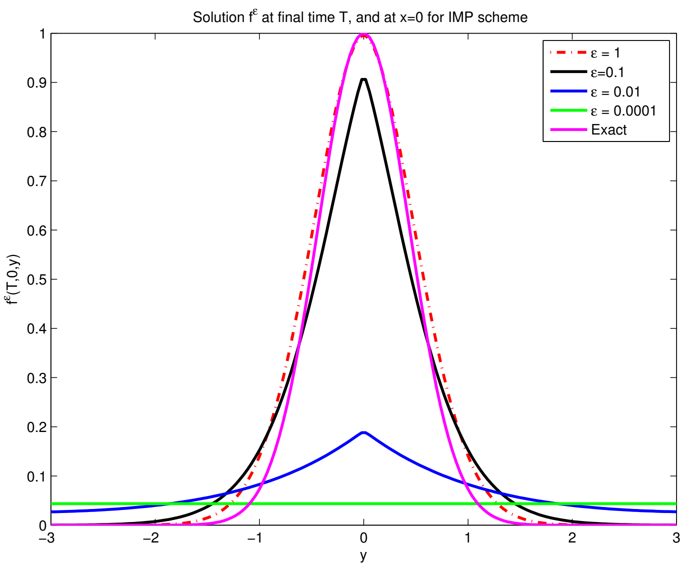
- Round of errors :

$$\frac{||\mathcal{F} - \mathcal{F}_{num}||}{||\mathcal{F}||} \leq \frac{\mathrm{cond}(\mathcal{A})}{1 - ||\mathcal{A}^{-1}||\,||\delta\mathcal{A}||} \left(\frac{||\delta\mathcal{A}||}{||\mathcal{A}||} + \frac{||\delta\mathcal{B}||}{||\mathcal{B}||} \right)$$

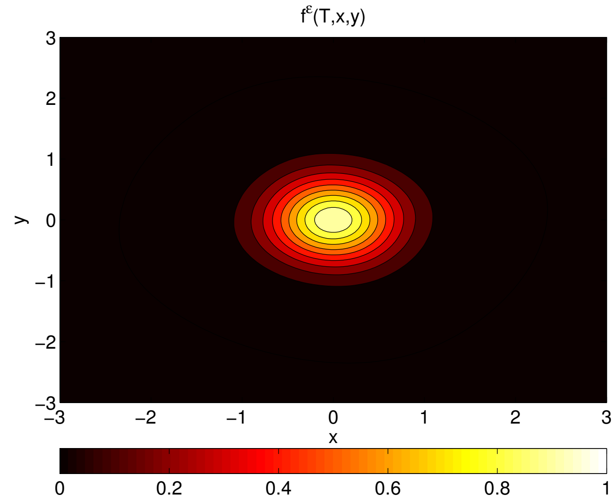
IMEX versus Lagrange-Multiplier: Second toy model

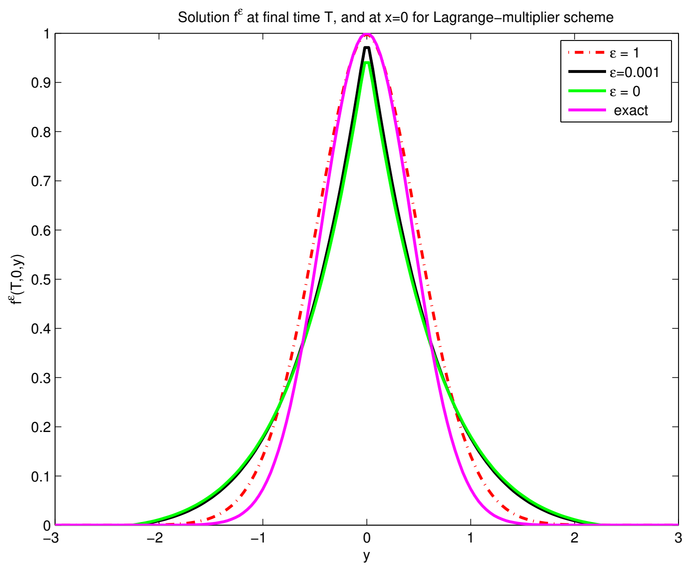
IMEX



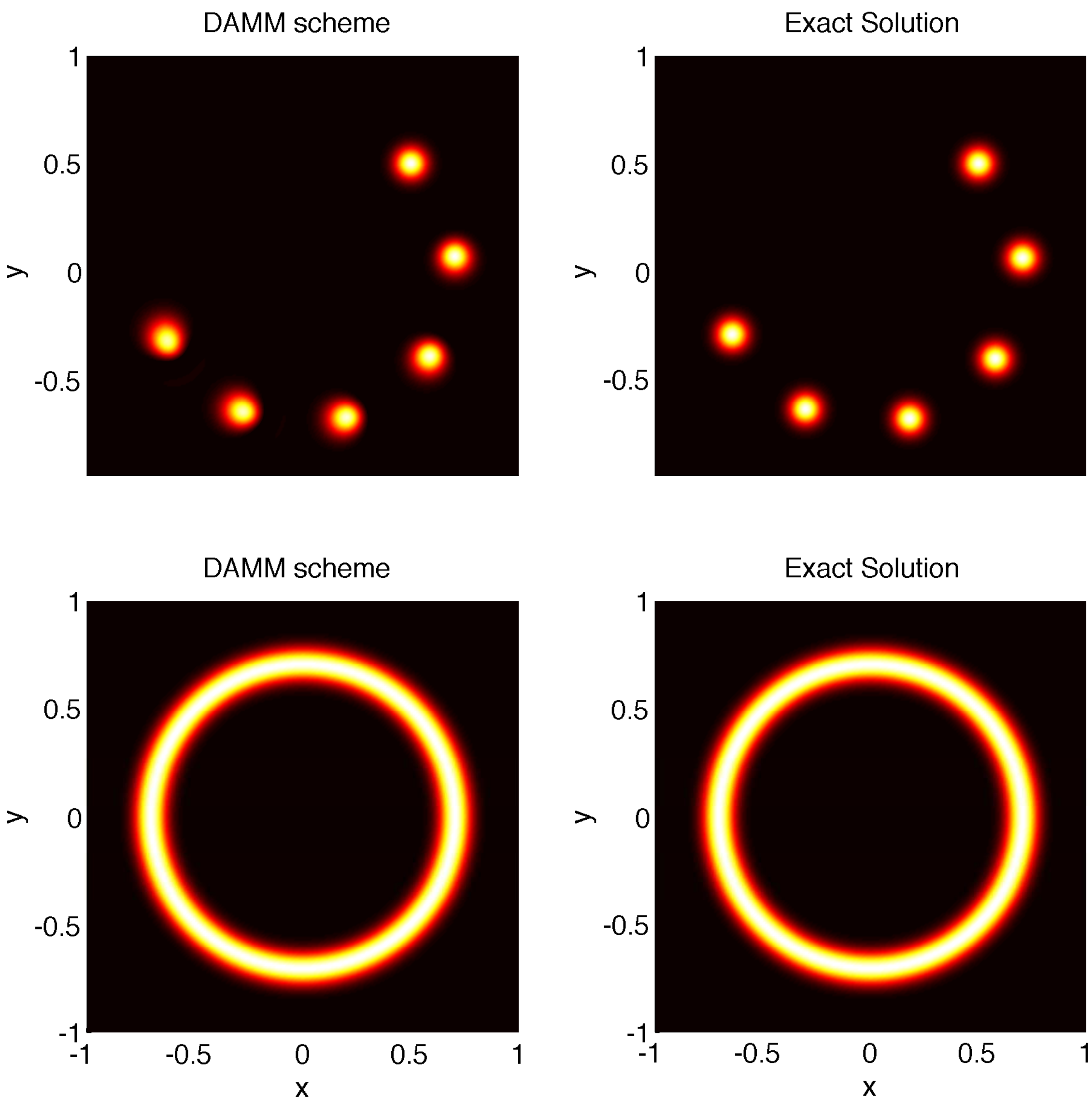


Lagrange-Multiplier





IMEX versus Lagrange-Multiplier: Second toy model



Error analysis

- Global errors: $||\mathcal{F}_{ex} - \mathcal{F}_{num}|| \leq ||\mathcal{F}_{ex} - \mathcal{F}|| + ||\mathcal{F} - \mathcal{F}_{num}||$

$$\mathcal{A}\,\mathcal{F}_{ex} = \mathcal{B} + \varepsilon \mathcal{T}\,, \qquad \mathcal{A}\,\mathcal{F} = \mathcal{B}\,, \qquad (\mathcal{A} + \delta\mathcal{A})\,\mathcal{F}_{num} = \mathcal{B} + \delta\mathcal{B}$$

- Truncation errors IMEX/La:

$$D_I := \frac{1}{\varepsilon} \begin{pmatrix} \frac{y_j \Delta x}{2}\left(1 + \frac{\alpha_j}{\varepsilon}\right) & \frac{-x_i y_j \Delta t}{2\varepsilon} \\ \frac{-x_i y_j \Delta t}{2\varepsilon} & \frac{x_i \Delta y}{2}\left(1 + \frac{\beta_i}{\varepsilon}\right) \end{pmatrix}, \quad \alpha_j := \frac{y_j \Delta t}{\Delta x}, \quad \beta_i := \frac{x_i \Delta t}{\Delta y}.$$

$$\begin{pmatrix} \mathcal{T}_{L1} \\ \mathcal{T}_{L2} \end{pmatrix} = \begin{pmatrix} \nabla\cdot & 0 \\ 0 & \nabla\cdot \end{pmatrix} \begin{pmatrix} 0 & D_{L_1} \\ D_{L_2} & -\varepsilon D_{L_2} \end{pmatrix} \begin{pmatrix} \nabla f^\varepsilon \\ \nabla q^\varepsilon \end{pmatrix} + \mathcal{O}(\Delta t^2, \Delta x^2, \Delta y^2)$$

$$D_{L1} := \begin{pmatrix} \frac{y_j \Delta x}{2}\left(1 + \frac{\alpha_j}{\varepsilon}\right) & \frac{-x_i y_j \Delta t}{2\varepsilon} \\ \frac{-x_i y_j \Delta t}{2\varepsilon} & \frac{x_i \Delta y}{2}\left(1 + \frac{\beta_i}{\varepsilon}\right) \end{pmatrix}, \quad D_{L2} = \begin{pmatrix} \frac{y_j \Delta x}{2} & 0 \\ 0 & \frac{x_i \Delta y}{2} \end{pmatrix}.$$

II. Vlasov-Poisson syst. in the long-time asymptotics

$$
\begin{cases}
\partial_t f_e + \frac{1}{\varepsilon} v\, \partial_x f_e - \frac{1}{\varepsilon} E(t,x)\, \partial_v f_e = 0 \\
-\partial_{xx}\varphi = 1 - n_e\,, \qquad n_e(t,x) = \int_{\mathbb{R}} f_e(t,x,v)\, dv\,, \qquad E = -\partial_x \varphi\,.
\end{cases}
$$

Work based on:

[2] B. Fedele, C. Negulescu, S. Possanner, *Asymptotic-Preserving scheme for the resolution of evolution equations with stiff transport terms*, submitted.

- Starting linear, stiff transport model:

$$(V)^\varepsilon \quad \begin{cases} \partial_t f^\varepsilon + \dfrac{\mathbf{b}}{\varepsilon} \cdot \nabla f^\varepsilon = 0\,, \qquad \mathbf{b} := (v, -E(t,x)) \\ f^\varepsilon(0,\mathbf{x}) = f_{in}(\mathbf{x})\,. \end{cases}$$

- Stabilized AP-reformulation: $f^\varepsilon = p^\varepsilon + \varepsilon\, q^\varepsilon$ with $\mathbf{b} \cdot \nabla f^\varepsilon = \varepsilon\, \mathbf{b} \cdot \nabla q^\varepsilon$

$$(MM)^\sigma_\varepsilon \quad \begin{cases} \partial_t f^{\varepsilon,\sigma} + \mathbf{b} \cdot \nabla q^{\varepsilon,\sigma} = 0 \\ \mathbf{b} \cdot \nabla f^{\varepsilon,\sigma} = \varepsilon\, \mathbf{b} \cdot \nabla q^{\varepsilon,\sigma} - \sigma\, q^{\varepsilon,\sigma} \qquad \text{(Stabilization)} \end{cases}$$

- $\varepsilon \to 0$ Limit-model:

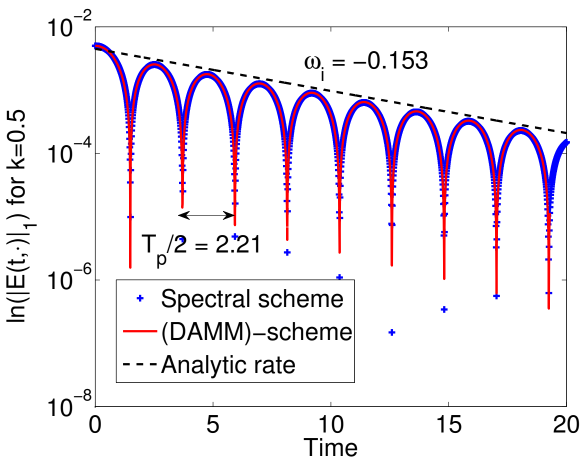
$$(MM)^\sigma_0 \quad \begin{cases} \partial_t f^{0,\sigma} + \mathbf{b} \cdot \nabla q^{0,\sigma} = 0\,, \\ \mathbf{b} \cdot \nabla f^{0,\sigma} + \sigma\, q^{0,\sigma} = 0\,. \end{cases}$$

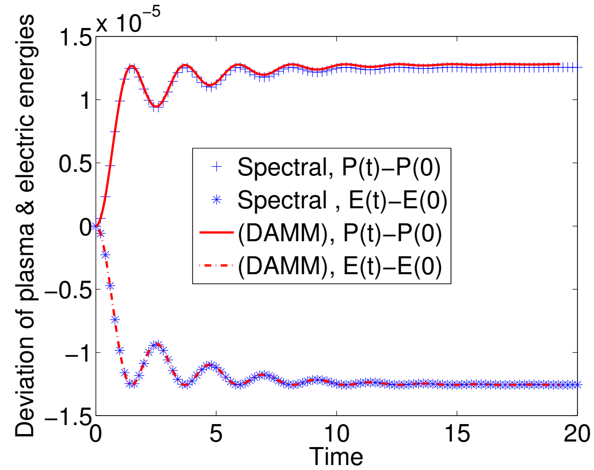
- Equivalent to degenerate diffusion equation:

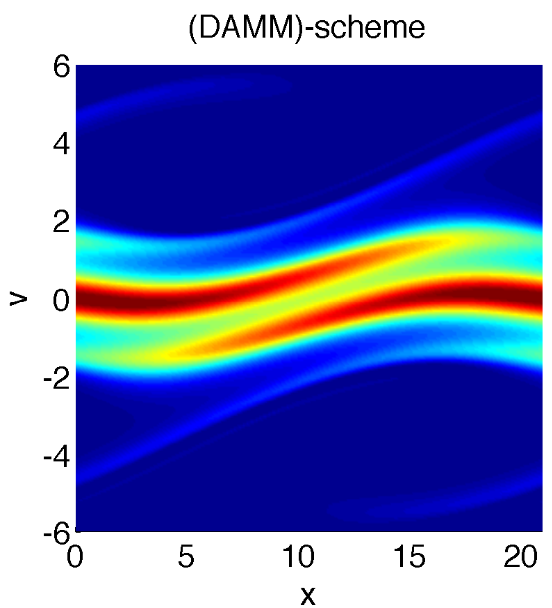
$$\partial_t f^{0,\sigma} - \frac{1}{\sigma} \nabla \cdot \left[(\mathbf{b} \otimes \mathbf{b})\, \nabla f^{0,\sigma} \right] = 0$$

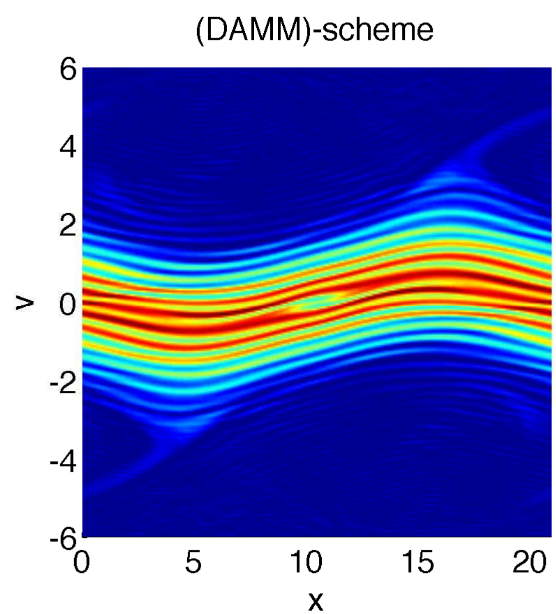
Landau damping test case ($\varepsilon = 1$)

$$f_{in}(x,v) = \frac{1}{\sqrt{2\pi}}\,(1+\gamma\cos(kx))\,e^{-v^2/2}$$

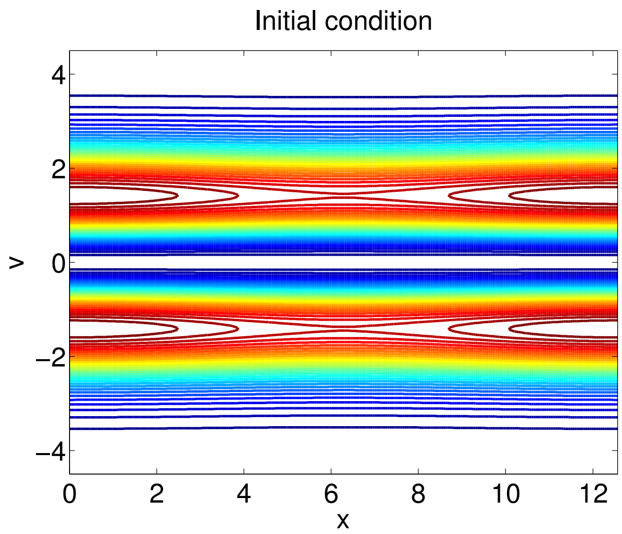


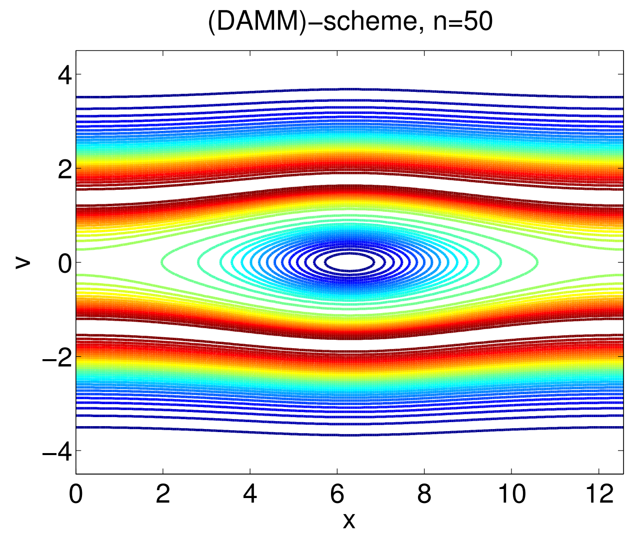


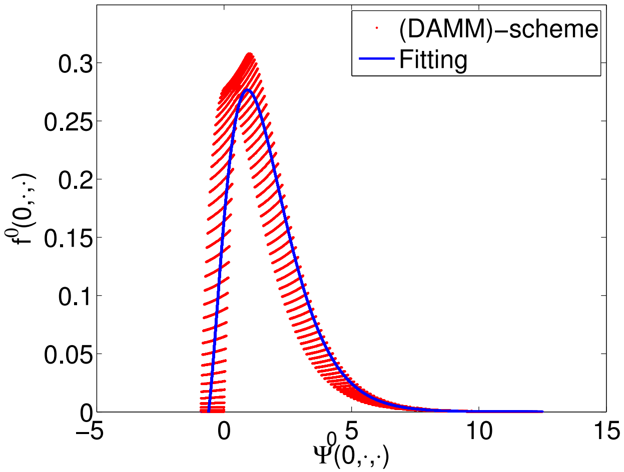


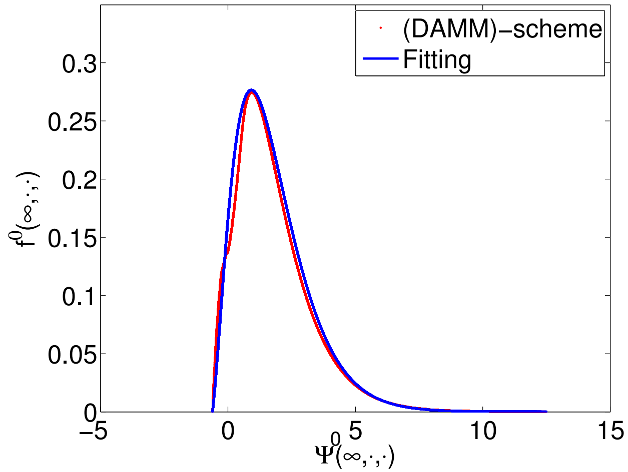


$$f_{in}(x,v) = \frac{1}{\sqrt{2\,\pi}}\, v^2 e^{-v^2/2}(1+\gamma\,\cos(k\,x))$$









III. Euler 2D/ Vorticity eq. in the long-time asymptotics

$$\begin{cases} \partial_t \omega^\varepsilon + \dfrac{\mathbf{u}^\varepsilon}{\varepsilon} \cdot \nabla \omega^\varepsilon = \nu \, \Delta \omega^\varepsilon \,, \\ -\Delta \Psi^\varepsilon = \omega^\varepsilon \,, \qquad \mathbf{u}^\varepsilon =^\perp \nabla \Psi^\varepsilon \,. \end{cases}$$

Work based on:

[3] B. Fedele, C. Negulescu, M. Ottaviani *Long-time asymptotics of the vorticity equation and its numerical study*, in preparation.

- Starting linear, stiff transport model:

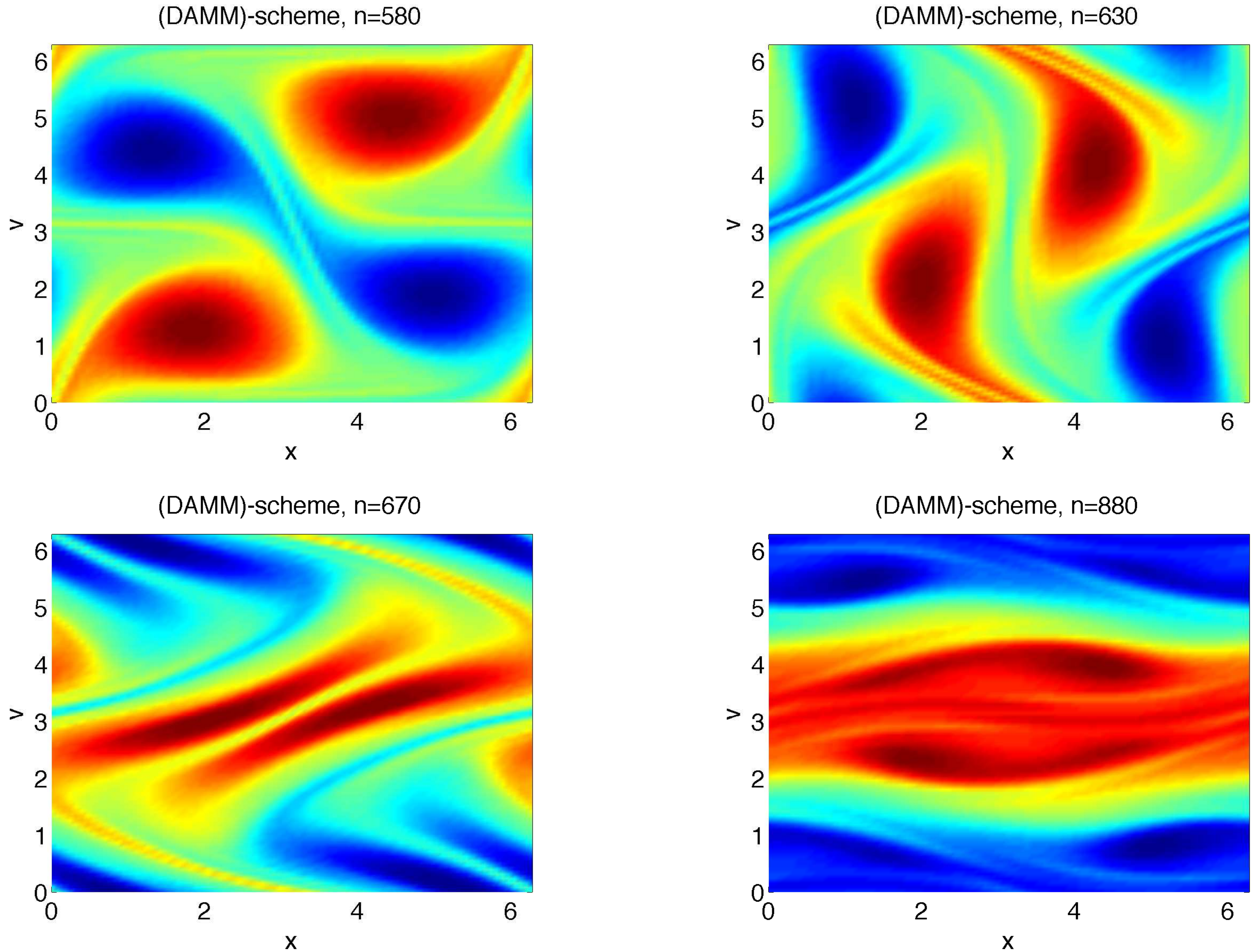
$$(V)^\varepsilon \quad \begin{cases} \partial_t \omega^\varepsilon + \dfrac{\mathbf{u}}{\varepsilon} \cdot \nabla \omega^\varepsilon = \nu \, \Delta \omega^\varepsilon \\ \omega^\varepsilon(0, \mathbf{x}) = \omega_{in}(\mathbf{x}) \, . \end{cases}$$

- Stabilized AP-reformulation: $\omega^\varepsilon = \chi^\varepsilon + \varepsilon \, \theta^\varepsilon$ with $\mathbf{u} \cdot \nabla \omega^\varepsilon = \varepsilon \, \mathbf{u} \cdot \nabla \theta^\varepsilon$

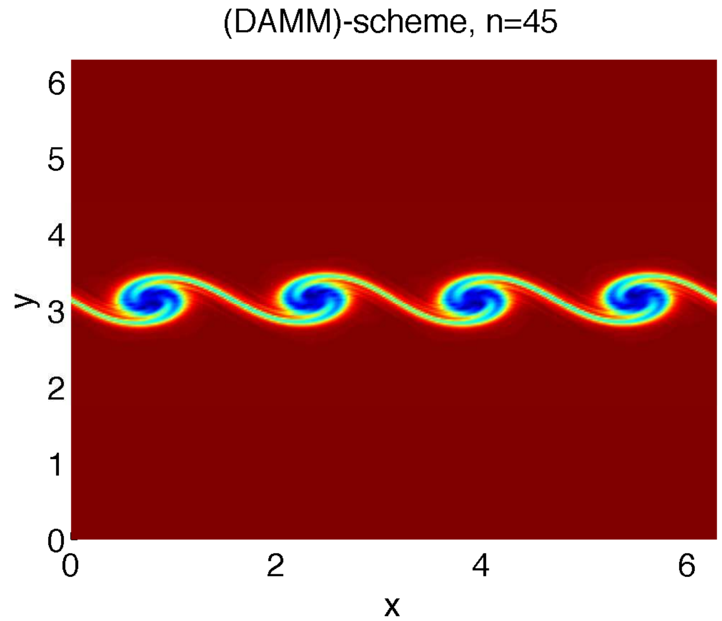
$$(MM)^\sigma_\varepsilon \quad \begin{cases} \partial_t \omega^{\varepsilon,\sigma} + \mathbf{u} \cdot \nabla \theta^{\varepsilon,\sigma} = \nu \, \Delta \omega^\varepsilon \\ \mathbf{u} \cdot \nabla \omega^{\varepsilon,\sigma} = \varepsilon \, \mathbf{u} \cdot \nabla \theta^{\varepsilon,\sigma} - \sigma \, \theta^{\varepsilon,\sigma} \qquad \text{(Stabilization)} \end{cases}$$

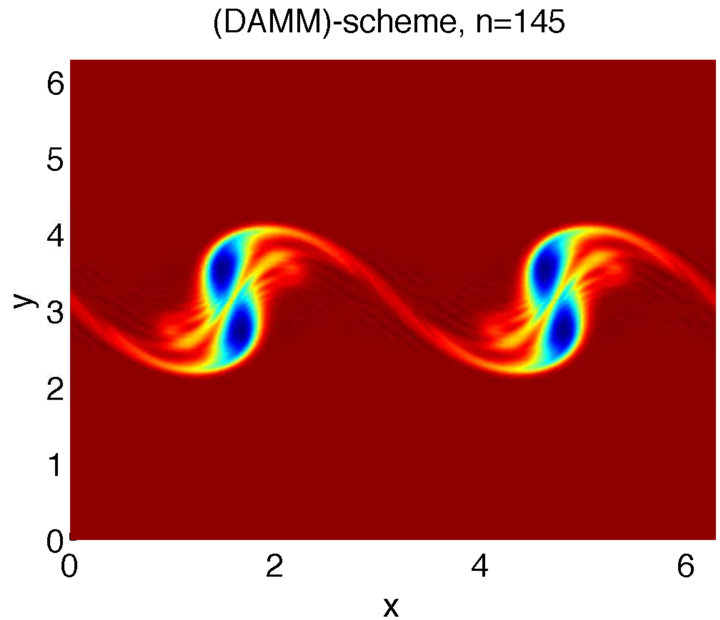
- Validation of the scheme for $\varepsilon \in [0, 1]$:
  - ➠ Kelvin-Helmholtz instability, for $\nu \equiv 0$
  - ➠ Two-stream shear-flow, for $\nu \geq 0$
  - ➠ Poiseuille flow, for $\nu \geq 0$

Some nice pictures (Taylor-Green instab.)

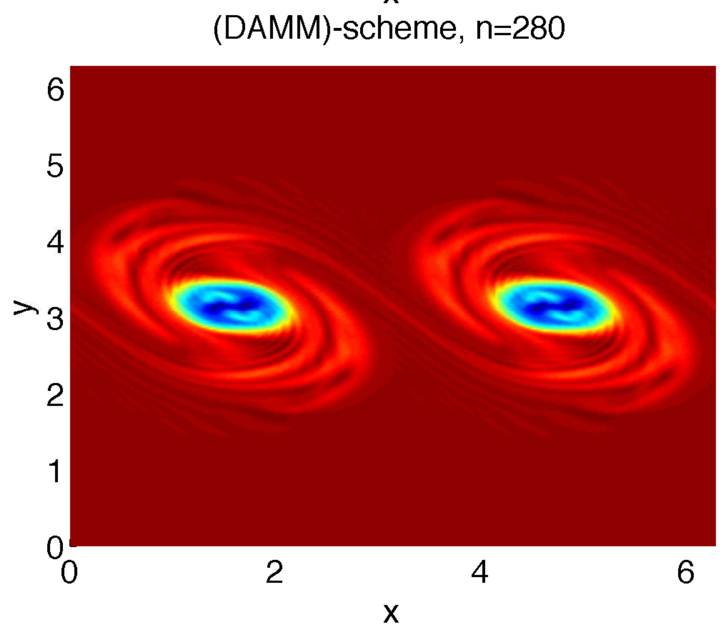


Some more nice pictures





- Kelvin-Helmholtz instability
- Two-stream instability, ...



Conclusions

For the resolution of the anisotropic Vlasov eq. in the gyro-kinetic regime, two strategies can be adopted:

- Field aligned configuration:
  - ➠ IMEX-scheme is the most appropriate scheme;
  - ➠ Change of coordinate system can be cumbersome.
- Cartesian, NOT-aligned configuration:
  - ➠ Lagrange-Multiplier scheme is the most appropriate scheme;
  - ➠ IMEX-scheme leads rapidly to erroneous results;
  - ➠ More time-consuming for the same grid; However AP-scheme $\Rightarrow$ ε-independent grid.

Thank's