4d $\mathcal{N} = 2$ theories and chiral algebras

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Nobody (even the typist) proof-read these notes, so there may be obvious mistakes: tell BLF.

Abstract

We discuss chiral algebras obtained by considering the cohomology of a supercharge in 4d $\mathcal{N} = 2$ theories. These are lecture notes for the 2018 IHÉS summer school on *Supersymmetric localization and exact results.*

These lecture notes assume familiarity with supersymmetry at the level of the first few chapters of the book by Wess and Bagger.

1 Lecture 1, July 27

This course is based mainly on one paper: "Infinite chiral symmetry in four dimensions" by Beem–Lemos–Liendo–Peelaers–Rastelli–van Rees.

Sorry in advance:

- there will be too many Q, so don't confuse Q, $\mathbb{Q}$, $\mathbf{Q}$, Q etc, especially when hand-written;
- indices will not be completely consistent;
- we will talk a lot about the representation theory of the superconformal algebra, which will make it easier to understand the rest of the chiral algebra: this is hopefully worthwhile because familiarity with the representation theory is useful more broadly than just in the study of chiral algebras.

1.1 Context

We will consider 4d $\mathcal{N} = 2$ SCFTs at the conformal point (not away on the Coulomb branch). We will focus on correlators of local operators. We will not use the Lagrangian (for now).

The local operators $\mathcal{O}_I(x)$ transform in representations of the superconformal algebra $\mathfrak{su}(2,2|2)$ whose bosonic subalgebra is $\mathfrak{so}(4,2) \oplus \mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_r$, namely the conformal algebra of $3+1$ dimensions, and R-symmetry (notice that Zohar Komargodski uses S and R instead of our R and r respectively).

Recall the conformal algebra has generators

- $M_\alpha{}^\beta$ and $M^{\dot\alpha}{}_{\dot\beta}$ (we distinguished the self-dual and anti-self-dual two-forms);
- $P_{\alpha\dot\alpha}$ (remember a vector is equivalent to a bispinor);
- $K^{\dot\alpha\alpha}$;
- D.

Among commutators there is

$$[D, P_{\alpha\dot\alpha}] = P_{\alpha\dot\alpha} \tag{1}$$
$$[D, K^{\dot\alpha\alpha}] = -K^{\dot\alpha\alpha} \tag{2}$$
$$[K^{\dot\alpha\alpha}, P_{\beta\dot\beta}] = \delta^\alpha_\beta \delta^{\dot\alpha}_{\dot\beta} D + \delta^\alpha_\beta M^{\dot\alpha}{}_{\dot\beta} + \delta^{\dot\alpha}_{\dot\beta} M^\alpha{}_\beta. \tag{3}$$

We denote generators of $\mathfrak{su}(2)_R$ as $R^\pm$ and R, and the generator of $\mathfrak{u}(1)_r$ by r. Besides the conformal algebra and the R-symmetry algebra we have the supercharges:

	Q^I_α	$\widetilde{Q}_{I\alpha}$	$S^{\dot\alpha}_I$	$\widetilde{S}^{I\dot\alpha}$
$\mathfrak{su}(2)_R$	**2**	**2**	**2**	**2**
$\mathfrak{u}(1)_r$	1/2	−1/2	−1/2	1/2

with

$$\{Q^I_\alpha, \widetilde{Q}_{J\dot\alpha}\} = \delta^I_J P_{\alpha\dot\alpha} \tag{4}$$
$$\{\widetilde{S}^{I\dot\alpha}, S^\alpha_J\} = \delta^I_J K^{\dot\alpha\alpha} \tag{5}$$
$$\{Q^I_\alpha, S^\beta_J\} = \frac{1}{2}\delta^I_J \delta^\alpha_\beta D + \delta^I_J M_\alpha{}^\beta - \delta^\beta_\alpha R^I_J. \tag{6}$$

Here R^I_J is appropriately defined in terms of $R^\pm$ and R:

$$\begin{pmatrix} R^1{}_1 & R^1{}_2 \\ R^2{}_1 & R^2{}_2 \end{pmatrix} = \begin{pmatrix} \frac{r}{2} + R & R^+ \\ R^- & \frac{r}{2} - R \end{pmatrix}. \tag{7}$$

1.2 Intermezzo: representations of the conformal algebra

Note: in radial quantization there exists a state-operator correspondence

$$|\mathcal{O}_I\rangle = \mathcal{O}_I(0)|\Omega\rangle \tag{8}$$

where $|\ \rangle$ denotes a state (a vector in the Hilbert space) on S^3, and $|\Omega\rangle$ is the vacuum, namely $G|\Omega\rangle = 0$ for all generators G.

A primary operator is $\widehat{\mathcal{O}}_I(x)$ that

- has a definite dimension Δ, namely $[D, \widehat{\mathcal{O}}_I(0)] = \Delta\widehat{\mathcal{O}}_I(0)$, hence $D\widehat{\mathcal{O}}_I(0)|\Omega\rangle = \Delta\widehat{\mathcal{O}}_I(0)|\Omega\rangle$;

- is killed by special conformal generators K, namely $[K^{\alpha\dot\alpha}, \widehat{\mathcal{O}}_I(0)] = 0$, hence $K^{\alpha\dot\alpha}\widehat{\mathcal{O}}_I(0)|\Omega\rangle = 0$.

Starting from this lowest-dimension state, we can act with some P_μ to build "descendants":

$$\widehat{\mathcal{O}}_I(0)|\Omega\rangle \xrightarrow{P_\mu} \partial_\mu\widehat{\mathcal{O}}_I(0)|\Omega\rangle \xrightarrow{P_\nu} \partial_\mu\partial_\nu\widehat{\mathcal{O}}_I(0)|\Omega\rangle \cdots \tag{9}$$

Note that it was important to have put the operator at zero: $[K, \widehat{\mathcal{O}}(x)]$ is only guaranteed to vanish when $x = 0$ and $\widehat{\mathcal{O}}$ is a primary.

Example: free massless boson Consider $\phi(x)$ with $\Box\phi(x) = 0$. Exercise: show that $\phi(x)$ and $:\phi^n:(x)$ and

$$T_{\mu\nu} = :\Big(\partial_\mu\phi\partial_\nu\phi - \frac{1}{d}\eta_{\mu\nu}\partial^\rho\phi\partial_\rho\phi + (\frac{2}{d} - 1)\phi\partial_\mu\partial_\nu\phi\Big): \tag{10}$$

are primary operators[1]. Similarly, show that $\partial_\mu\phi(x)$ and $:\phi\partial_\mu\phi:$ are descendants.

Note that sometimes a descendant is *null*, meaning that its norm vanishes and it decouples from the theory. Once we quotient by (the sub-representation spanned by) this state, the representation that remains is then called *short*. Null descendants only exist if the quantum numbers of the primary operator obey some constraints. For instance

- $\Box\phi$ is null only if ϕ has dimension $(d-2)/2$;
- $\partial_\mu j^\mu = 0$ only if the current has $\Delta = d - 1$;
- $\partial_\mu T^{\mu\nu} = 0 = \partial_\nu T^{\mu\nu}$ only if the tensor has $\Delta =$.

Exercise: using the relation between P_μ and the stress tensor, show that the stress tensor is a primary operator.

1.3 Representations of $\mathfrak{su}(2,2|2)$

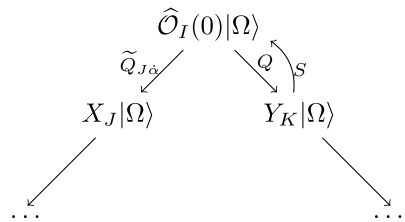
A superconformal primary $\widehat{\mathcal{O}}_I(0)|\Omega\rangle$ is

- an eigenvector of D, J^3_L and J^3_R (space-time rotations $M_\alpha{}^\beta$ and $M^\alpha{}_\beta$), and R-symmetries R and r, with eigenvalues denoted (Δ, j_1, j_2, R, r),
- killed by all K, S and $\widetilde{S}$.

We can act on the superconformal primary with supercharges to get new conformal primaries (actually we may need to project out conformal descendants). We

[1] In 2d, operators that are primary with respect to the finite-dimensional subalgebra we are considering here are called quasi-primary operators.

can draw things as



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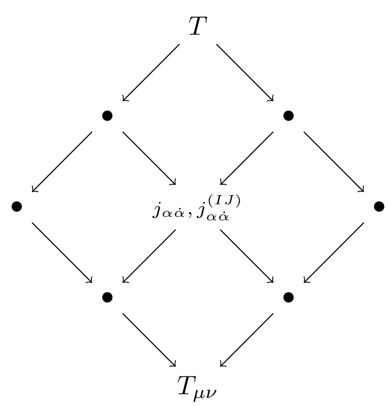
We describe some representations, first in the notations of Dolan and Osborn, then in the notations of Cordova, Dumitrescu and Intrilligator.

- $A^{\Delta}_{R,r,(j_1,j_2)}$ also called $L\overline{L}[j_1,j_2]^{(R,r)}_{\Delta}$ is a long multiplet, we can act on the superconformal primary with Q's 4 times, and 4 times with $\widetilde{Q}$.
- $\widehat{C}_{0(0,0)}$ also called $A_l\overline{A}_l[0,0]^{(0,0)}$ contains the stress tensor; the superconformal primary has $\Delta = 2$, $j_1 = j_2 = 0$, $R = r = 0$



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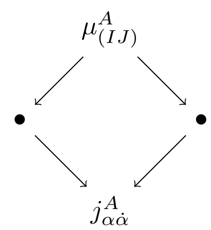
  where $j_{\alpha\dot{\alpha}}, j^{(IJ)}_{\alpha\dot{\alpha}}$ are $\mathfrak{u}(1) \times \mathfrak{su}(2)$ R-symmetry currents of dimension $\Delta = 3$, and we have supercurrents at dimension $\Delta = 7/2$ and the stress tensor $T_{\mu\nu}$ at $\Delta = 4$.

- $\widehat{B}_R$ also called $B_1\overline{B}_1[0,0]^{(R,0)}$ whose superconformal primary has $\Delta = 2R$ and $j_1 = j_2 = r = 0$.
  - In particular for $R = 1/2$ we have the usual hypermultiplet, with the hypermultiplet scalars Q_I at the top (obeying $\Box Q_I = 0$) and the fermions $\widetilde{\lambda}_{\dot{\alpha}}$ and λ_α have $\Delta = 3/2$ (hence are in the second row of the diagram).

– An other example is for $R = 1$ we get



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where $j^A_{\alpha\dot\alpha}$ is a conserved flavour symmetry current (A is an adjoint index for the flavour symmetry). It has dimension 3 while $\mu^A_{(IJ)}$ has dimension 2. In the case of a Lagrangian theory with N_f hypermultiplets in the representation R of the gauge algebra $\mathfrak{g}$:

$$\mu^A_{(IJ)} = :\widetilde{A}^{\overline{a}}_{I\overline{m}} Q^b_{Jn} :\delta^{\overline{m}n} T^A_{b\overline{a}} \tag{14}$$

where $\overline{m}$ and n are gauge indices, $\overline{a}$ and b are flavour indices.

1.4 Chiral algebras — pedestrian approach

All of the structure we will outline now will have a nice cohomological explanation.

Consider an n-point function of local operators

$$\Big\langle \mathcal{O} \quad \mathcal{O} \quad \dots \quad \mathcal{O} \quad \Big\rangle. \tag{15}$$

We only consider *Schur* operators[2], nmaely $\Delta = j_1 + j_2 + 2R$. These are always in short representations of the superconformal algebra. They always have $R > 0$. They are always a highest weight of $\mathfrak{su}(2)_R \times \mathfrak{su}_{\text{left}} \times \mathfrak{su}(2)_{\text{right}}$, namely can be written as

$$\mathcal{O}^{1\cdots 1}_{+\cdots+\dot{+}\cdots\dot{+}} \tag{16}$$

with $2R > 0$ (one or more) R-symmetry indices (superscripts) and zero or more of each kind of Lorentz spinor indices. We need the R-symmetry multiplet of the Schur operators:

$$\mathcal{O}^{I_1\cdots I_{2R}}_{+\cdots+\dot{+}\cdots\dot{+}}(x) \tag{17}$$

and we will restrict x to the two-plane $x_1 = x_2 = 0$ and set $z = x_3 + ix_4$ and $\overline{z} = x_3 - ix_4$, then contract each R-symmetry index with a position-dependent vector

$$u_I(\overline{z}) = \begin{pmatrix} 1 \\ \overline{z} \end{pmatrix}. \tag{18}$$

A typical correlator we consider is

$$u_I(\overline{z}_1) u_J(\overline{z}_2) u_K(\overline{z}_3) u_L(\overline{z}_3) \Big\langle \mathcal{O}^I_{+\dot{+}}(z_1, \overline{z}_1) \mathcal{O}^J_{++}(z_2, \overline{z}_2) \mathcal{O}^{KL}(z_3, \overline{z}_3) \Big\rangle. \tag{19}$$

[2]These operators are called Schur operators because they contribute to the Schur index, which is named like that because of its link with Schur polynomials.

The claim is that all of these correlators are meromorphic: no $\overline{z}$ dependence!

$$\begin{aligned}&\frac{\partial}{\partial\overline{z}_i}\Big(u_I(\overline{z}_1)u_J(\overline{z}_2)u_K(\overline{z}_3)u_L(\overline{z}_3)\Big\langle\mathcal{O}^I_{+\dot{+}}(z_1,\overline{z}_1)\mathcal{O}^J_{++}(z_2,\overline{z}_2)\mathcal{O}^{KL}(z_3,\overline{z}_3)\Big\rangle\Big)\\&=\text{contact terms, like }\partial_{\overline{z}}\frac{1}{z}\simeq\delta^{(2)}(z,\overline{z}).\end{aligned}\tag{20}$$

The strategy is then

- determine residues at all of its poles (where operators collide);
- determine the behaviour at infinity;
- deduce the whole correlation function thanks to meromorphicity.

Trivial example: free hypermultiplet The free hypermultiplet's scalars Q_I and $\widetilde{Q}_J$ have $\Delta = 1$ and $R = 1/2$ and $r = j_1 = j_2 = 0$ so they are Schur ($\Delta = j_1 + j_2 + 2R$). We compute

$$\begin{aligned}&u^I(\overline{z})u^J(\overline{w})\Big\langle Q_I(z,\overline{z})\widetilde{Q}_J(w,\overline{w})\Big\rangle\\&=\frac{\epsilon_{IJ}}{|z-w|^2}u^I(\overline{z})u^J(\overline{w})=\frac{\overline{w}-\overline{z}}{|z-w|^2}=\frac{1}{w-z}\end{aligned}\tag{21}$$

which is meromorphic. The other Schur operators in this free theory are normal-ordered products

$$:Q\cdots Q\cdots\widetilde{Q}\cdots\widetilde{Q}\cdots\partial_{+\dot{+}}\cdots Q\cdots\widetilde{Q}\cdots\partial_{+\dot{+}}\cdots:\tag{22}$$

Note that $\partial_{+\dot{+}} = \partial_z$.

Constructing the chiral algebra Introduce a notation for each Schur operator contracted with $u_I(\overline{z})$:

$$q(z) \coloneqq [u_I(\overline{z})Q^I(z,\overline{z})]_\chi, \qquad \widetilde{q}(z) \coloneqq [u_I(\overline{z})\widetilde{Q}^I(z,\overline{z})]_\chi\tag{23}$$

where the subscript χ just mean "in the chiral algebra". All other Schur operators contracted with u_I are obtained by normal-ordered products from $q(z)$ and $\widetilde{q}(z)$ like

$$:q\partial\widetilde{q}:(z) = \lim_{w\to z}\big(q(w)\partial\widetilde{q}(z) - \text{singular terms in OPE}\big).\tag{24}$$

Notice a sleigh of hands: we had used the normal-ordered product in 4d and now we are using the 2d normal-ordered product.

Correlators of these operators are meromorphic. All the axioms of vertex operator algebras as considered by mathematicians are obeyed.

In the free hypermultiplet case, the chiral algebra is generatred by $q(z)$ and $\widetilde{q}(z)$ with singular OPE

$$q(z)\widetilde{q}(w) \sim \frac{-1}{z-w}, \qquad q(z)q(w)\sim 0, \qquad \widetilde{q}(z)\widetilde{q}(w)\sim 0,\tag{25}$$

where $\sim$ means "up to regular terms", so ~ 0 means that there is no short-distance singularity. This chiral algebra has a name: it is the chiral algebra of symplectic bosons with $h_q = h_{\tilde{q}} = 1/2$.

Notice that the chiral algebra is not unitary, even though the original 4d theory was. This is not particularly surprising because we are computing some complexified correlators in the 4d theory anyways.

1.5 Chiral algebras — proper approach

Let $\mathbb{Q} = Q^1_- + \widetilde{S}^{2\dot{-}}$ and $\mathbb{Q}^\dagger = S_1^- + \widetilde{Q}_{2\dot{-}}$. Then $\mathbb{Q}^2 = 0$ and

$$\{\mathbb{Q}, \mathbb{Q}^\dagger\} = D - M_+{}^+ - M^{\dot{+}}_{\dot{+}} - 2R. \tag{26}$$

We deduce that for all states in a unitary theory $\Delta - j_1 - j_2 - 2R \geq 0$. We also deduce that an operator is Schur (equality case) if and only if $\mathbb{Q}\mathcal{O}(0)|\Omega\rangle = 0$.

We are going to work in the cohomology of $\mathbb{Q}$. Acting on the vacuum with a Schur operator at 0 gives a $\mathbb{Q}$-closed state. What about moving it? We have $[\mathbb{Q}, P_1] \neq 0$, $[\mathbb{Q}, P_2] \neq 0$, but $[\mathbb{Q}, P_z] = 0$ so

$$\mathbb{Q} e^{zP_z} \mathcal{O}(0)|\Omega\rangle = 0. \tag{27}$$

What about moving in the $\overline{z}$ direction? Well, remember we contracted with u:

$$\partial_{\overline{z}}\Big(u_I(\overline{z})\mathcal{O}^I(z,\overline{z})\Big) = u_I(\overline{z})\big[P_{\overline{z}}, \mathcal{O}^I(z,\overline{z})\big] + \mathcal{O}^2(z,\overline{z}) = u_I(\overline{z})\big[P_{\overline{z}} + R^-, \mathcal{O}^I(z,\overline{z})\big] \tag{28}$$

and it turns out that $P_{\overline{z}} + R^- = \{\mathbb{Q}, \dots\}$. This implies not only that $P_{\overline{z}} + R^-$ acts on the cohomology (it is $\mathbb{Q}$-closed) but it actually acts trivially, in contrast with P_z which acts non-trivially. We learn that correlators of "twisted-translated" Schur operators only depend meromorphically on z (namely don't depend on $\overline{z}$).

We can construct a $\mathbb{Q}$-closed $\mathfrak{sl}(2)_z$ with generators

$$L_{-1} = P_z, \qquad L_1 = K^z, \qquad 2L_0 = D + M_+{}^+ + M^{\dot{+}}{}_{\dot{+}}. \tag{29}$$

We deduce that $[u_I(\overline{z})\mathcal{O}^I(z,\overline{z})]_\chi$ has conformal weight $h = (\Delta + j_1 + j_2)/2$. It is easy to check that this is consistent with the example of the free hypermultiplet.

We can construct a $\mathbb{Q}$-exact $\widehat{\mathfrak{sl}(2)}_{\overline{z}}$ (the hat means twisted) with generators

$$\widehat{L}_{-1} = P_{\overline{z}} + R^-, \qquad \widehat{L}_1 = K^{\overline{z}} - R^+, \qquad 2L_0 = D - M_+{}^+ - M^{\dot{+}}{}_{\dot{+}} - 2R. \tag{30}$$

In fact, the $\widehat{\mathfrak{sl}(2)}_{\overline{z}}$ algebra considered here is a diagonal subalgebra of $\mathfrak{sl}(2|2)$ that is inside $\mathfrak{sl}(2,2|2)$, so we should expect a similar chiral algebra story in other theories that are invariant under at least $\mathfrak{sl}(2|2)$, for instance

- 6d $(2,0)$ theories (the paper is already out);
- 4d $\mathcal{N} = 2$ theories (discussed in these notes);

- 2d $\mathcal{N} \geq (0,4)$ theories (upcoming paper).

Notice that our supercharge $\mathbb{Q}$ involves S, and not only Q, thus $\mathbb{Q}$ does not commute with P. This is why we could have some position dependence rather than none as in earlier cohomological stories.

1.6 Notable Schur operators

From $\widehat{B}_{1/2}$, $\widetilde{Q}^I$ and Q^J are Schur.

From $\widehat{B}_1$, $\mu^A_{(IJ)}$ is Schur, namely

$$j^A(z) = \left[u^I u^J \mu^A_{(IJ)}\right]_{\mathbb{Q}} \tag{31}$$

(we changed the subscript from χ to $\mathbb{Q}$ because the chiral algebra is just the $\mathbb{Q}$ cohomology). This operator j^A has $h = (\Delta + j_1 + j_2)/2 = 1$.

Recall the two-point function of conserved currents in 4d,

$$\left\langle J^A_\mu(z,\overline{z}) J^B_\nu(w,\overline{w})\right\rangle = 12 k_{4\mathrm{d}} \delta^{AB} \frac{x^2 g_{\mu\nu} - 2x_\mu x_\nu}{|x|^8} \tag{32}$$

where $k_{4\mathrm{d}} > 0$ (by unitarity) is the flavour central charge (a number independent of exactly marginal deformations). By supersymmetry this current two-point function gives the two-point function of $\mu^A_{(IJ)}$, from which we can deduce the singular OPE

$$j^A(z) j^B(w) \sim \frac{k_{2\mathrm{d}}\delta^{AB}}{(z-w)^2} + \frac{i f^{AB}{}_C j^C(w)}{z-w} \tag{33}$$

with $k_{2\mathrm{d}} = -k_{4\mathrm{d}}/2$. Altogether we learn that a 4d flavour symmetry gives rise to an affine Kač–Moody subalgebra of the 2d chiral algebra.

Question: the Sugawara construction starting from a Kač–Moody algebra gives a Virasoro subalgebra of the chiral algebra; does it have a physical meaning?

In $\widehat{C}_{0(0,0)}$ (stress-tensor multiplet) we have $j^{(IJ)}_{\alpha\dot{\alpha}}$ with $\Delta = 3$, $R = 1$, $j_1 = j_2 = 1/2$, giving rise to a (quasi)primary with $h = 2$ in the conformal algebra, neutral under all flavour symmetries. This is exactly right to produce a Virasoro algebra. Let

$$T(z) \coloneqq \left[u_I u_J j^{(IJ)}_{+\dot{+}}\right]_{\mathbb{Q}}. \tag{34}$$

From the two-point function $\langle T_{\mu\nu} T_{\rho\sigma}\rangle = c_{4\mathrm{d}}(\cdots)_{\mu\nu\rho\sigma}/|x|^8$ we find[3]

$$T(z)T(0) \sim \frac{c_{2\mathrm{d}}/2}{z^4} + \frac{2T(0)}{z^2} + \frac{\partial T(0)}{z}. \tag{35}$$

In all known examples this is the stress-tensor of the chiral algebra, namely its OPE with all chiral primaries is exactly right in order to generate the $\mathfrak{sl}(2)_z$ coming from 4d. This is not proven in general: the chiral algebra could have many Virasoro subalgebras.

[3] In our conventions the free hypermultiplet has $c_{4\mathrm{d}} = 1/12$. In any unitary theory, $c_{4\mathrm{d}} > 0$.

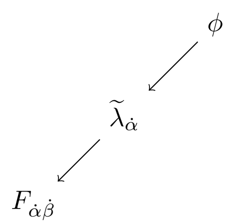
To come back to the question, the Sugawara construction is $:j_A j^A:(z)$; sometimes it coincides with the stress-tensor $T(z)$, but often it is a mixture of $T(z)$ with an operator that comes from some other Schur operator in the 4d theory. This leads to interesting consistency constraints on k_{4d} and c_{4d} for instance.

2 Lecture 3, July 27

Comment: the OPE in chiral algebras is single-valued so powers are integers.

Chiral algebras for gauge theories can be found using the fact that the Higgs branch data is independent of $(g_{\mathrm{YM}}, \theta)$ so we can work at zero coupling.

Gauge theories A free vector multiplet $\mathcal{E}_{1(0,0)}$, $\overline{\mathcal{E}}_{1(0,0)}$ looks like



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Here $\widetilde{\lambda}_{\dot\alpha}$ has $j_2 = 1/2$, $j_1 = 0$, $\Delta = 3/2$, $R = 1/2$, hence it is Schur, with $h = 1$. So if

$$b(z) = \left[u_I(\bar{z})\widetilde{\lambda}^I_+(z,\bar{z})\right]_{\mathbb{Q}} \tag{37}$$

$$\partial c(z) = \left[u_I(\bar{z})\widetilde{\lambda}^I_+(z,\bar{z})\right]_{\mathbb{Q}} \tag{38}$$

then the chiral algebra is generated by b and c after removing the zero mode of c. The OPE is

$$b(z)c(w) \sim \frac{1}{z-w}. \tag{39}$$

Consider now a gauge theory T_{gauge}, obtained by coupling vector multiplets in the adjoint representation of $\mathfrak{g}$, coupled to a matter theory T with global symmetry $\mathfrak{g}$ and with $k_{\mathrm{4d}} = 4h^\vee_{\mathfrak{g}}$. This relation between the flavour central charge of T and the dual Coxeter number of $\mathfrak{g}$ is needed for the result to be an SCFT.

How to get the chiral algebra $\chi[T_{\mathrm{gauge}}]$ of T_{gauge} from the chiral algebra $\chi[T]$ of T?

- Note that $\chi[T] \supset \mathfrak{g}$ contains an affine Kač–Moody algebra at level $k_{\mathrm{2d}} = -2h^\vee_{\mathfrak{g}}$; denote the currents by $j^A(z)$.

- Note that the chiral algebra $\chi[\text{f.v.}]$ of a free vector multiplet contains b_A and c^A minus the zero modes of c^A (we denote this by "\zero-modes").

Then define

$$Q_{\text{BRST}} = \oint \frac{\mathrm{d}z}{2\pi i} j_{\text{BRST}}(z), \tag{40}$$

$$j_{\text{BRST}} = c_A \left[j^A - \frac{i}{2} f^{ABC} c_B b_C \right]. \tag{41}$$

Then we claim that $\chi[T_{\text{gauge}}]$ is given by the cohomology of Q_{BRST}, namely

$$\chi[T_{\text{gauge}}] = H^{\bullet}_{\text{BRST}}\Big[\psi \in \chi[T] \otimes \chi[\text{f.v.}]\Big] \setminus \text{zero-modes}. \tag{42}$$

Note that Q_{BRST} is nilpotent: $[Q_{\text{BRST}}, [Q_{\text{BRST}}, \dots\}\} = 0$ only if $k_{4\text{d}} = 4h^{\vee}_{\mathfrak{g}}$.

Example: consider a charged operator $\mathcal{O}_i(z)$ in $\chi[T]$ (we would also need to work out the same for the gauginos). Then

$$j^A(z)\mathcal{O}_i(0) = \cdots + \frac{1}{z} T^A{}_i{}^j \mathcal{O}_j(0) + \cdots \tag{43}$$

while the OPE of c an b with $\mathcal{O}_i(0)$ is regular. Then using the definition of Q_{BRST},

$$[Q_{\text{BRST}}, \mathcal{O}_i(0)] = \oint \frac{\mathrm{d}z}{2\pi i} \left(c_A j^A - \frac{i}{2} f^{ABC} c_A c_B b_C \right) \mathcal{O}_i(0) = c_A(0) T^A{}_i{}^j \mathcal{O}_j(0) \tag{44}$$

and that's it. This only vanishes if $\mathcal{O}_i$ is actually neutral under the flavour symmetry that is being gauged, in other words if $\mathcal{O}_i$ is a genuine operator in the gauge theory.

$SU(N)$ **SQCD** Consider 4d $\mathcal{N} = 2$ $SU(N)$ with $2N$ fundamental hypermultiplets. Take $N \geq 3$ to avoid discussing flavour symmetry enhancement. Before gauging $SU(N)$ we have the following.

- Q^a_{Ii} and $\widetilde{Q}^i_{Ia}$ where I is still an $\mathfrak{su}(2)_R$ symmetry index, i is a flavour index and a a gauge index. These give rise to $q^a_i(z)$ and $\widetilde{q}^i_a(z)$ in the chiral algebra
- λ and λ giving rise to $b^A(z)$ and $c_A(z)$.

We want to build gauge-invariant, or Q_{BRST} closed, operators. They lie in various multiplets.

- $\widehat{B}_1$ multiplet: $Q_{(I|j}\widetilde{Q}^i_{|J)a}$, which gives rise to the $\mathfrak{u}(1) \times \mathfrak{su}(N_f)$ current $q^a_j \widetilde{q}^i_a = j^i_j$ where $\mathfrak{su}(N_f)$ is at level $-N_c$.
- $\widehat{C}^0_{(0,0)}$ multiplet: $j^{(IJ)}_\mu(z,\bar{z})$, which gives rise to $T(z) = \frac{1}{2}(q\partial_z\widetilde{q} - \widetilde{q}\partial_z q) - b\partial c$. Something funny happening here is that if we have a flavour symmetry current j^A we can define $T_{\text{Sugawara}} = :j^A j_A:$, which acts on currents like

a stress-tensor. In the original (ungauged) chiral algebra this is different from the $T(z)$, but in fact

$$T(z) - T_{\text{Sugawara}}(z) = \{Q_{\text{BRST}}, \cdots\} \tag{45}$$

so these operators are the same in the cohomology, hence the same in the chiral algebra of the gauge theory.

- We also have baryons $B_{i_1 \cdots i_N} = q_{i_1}^{a_1} \cdots q_{i_N}^{a_N} \epsilon_{a_1 \cdots a_N}$ and of course antibaryons $\widetilde{B}^{i_1 \cdots i_N}$.

Even for this theory, it is still conjectural whether there are other generators. This is a well-defined problem in cohomology.

One thing that can be done is to compute the Schur index (a certain limit of the $S^3 \times S^1$), which is almost the vacuum character of the chiral algebra: in fact there is a $(-1)^F$, so it is the graded character of the chiral algebra. The matching between Schur index and graded character of the chiral algebra gives some evidence that all the generators were found, but

- the $(-1)^F$ could lead to cancellations so some generators could be hiding;
- the comparison is only done level by level because the (graded) vacuum character is hard to compute for the given (negative) levels of the affine–Kač–Moody algebra.

Conjectures for chiral algebras of some theories The strategy is to compare the known Schur index of theories with the graded vacuum character.

- The chiral algebra of 4d $\mathcal{N} = 4$ super Yang–Mills has "small" 2d $\mathcal{N} = 4$ supersymmetry.
- Chiral algebras of 4d $\mathcal{N} = 3$ theories have 2d $\mathcal{N} = 2$ supersymmetry.
- The chiral algebra of $SU(N)$ with $2N$ fundamental hypermultiplets contains an $\mathfrak{su}(2N)_{-N} \oplus \mathfrak{u}(1)$ affine Kač–Moody algebra plus baryons.
- The chiral algebra of the Minahan–Nemeschansky E_6 theory must contain $(E_6)_{-3}$ affine Kač–Moody, and in fact that seems to be it.
- The chiral algebra of the T_N theory must contain $\mathfrak{su}(N)^3$ affine Kač–Moody at the critical level (because gauging together two $\mathfrak{su}(N)$ flavour symmetries should give an SCFT).

S-duality should not change the chiral algebra; it is very non-trivial to check or use such a bootstrap approach.

2.1 Unitarity bound

Consider a theory T with flavour symmetry $\mathfrak{g}$. Then in $\chi[T]$ we have $j^A(z)$, and the theory's R-symmetry current gives $T(z)$, with OPEs

$$j^A(z)j^B(0) \sim \frac{k_{2\mathrm{d}}\delta^{AB}}{z^2} + \frac{if^{AB}{}_c j^c(0)}{z}, \tag{46}$$

$$T(z)T(0) \sim \frac{c/2}{z^4} + \frac{2T}{z^2} + \frac{\partial T}{z}, \tag{47}$$

$$T(z)j^A(0) \sim \frac{j^A(0)}{z^2} + \frac{\partial j^A(0)}{z}. \tag{48}$$

This is enough to determine all correlators of T and j^A. For instance using the singularities at 0, 1, ∞ imposed by the jj OPE we find

$$\begin{aligned} &\big\langle j^A(0)j^B(z)j^C(1)j^D(\infty)\big\rangle \\ &= \frac{k_{2\mathrm{d}}^2}{z^2}\bigg(\delta^{AB}\delta^{CD} + z^2\delta^{AC}\delta^{BD} + \frac{z^2}{(1-z)^2}\delta^{AD}\delta^{BC} \\ &\quad - \frac{z}{k_{2\mathrm{d}}}f^{ACE}f^{BDE} - \frac{z}{k_{2\mathrm{d}}(z-1)}f^{ADE}f^{BCE}\bigg) \end{aligned} \tag{49}$$

Let $j^2(z) = :j^A j_A:(z) = \lim_{w\to z}\big[j^A(w)j_A(z) - k_{2\mathrm{d}}(\dim\mathfrak{g})/(w-z)^2\big]$. This can come from a $\widehat{C}_{0(0,0)}$ or $\widehat{B}_2$ supermultiplet in 4d, or a linear combination of the two. We are interested in the $\widehat{B}_2$ part. To get rid of the $\widehat{C}_{0(0,0)}$ piece we subtract a multiple of the stress-tensor so that the result is orthogonal to the stress tensor.

Compute $\langle T(z)T(0)\rangle = \frac{c/2}{z^4}$ and

$$\begin{aligned} \langle T(z)j^2(0)\rangle &= \lim_{w\to 0}\big\langle T(z)j^a(w)j_a(0)\big\rangle \\ &= \lim_{w\to 0}\left\langle j^a(w)\left(\frac{j_a(0)}{z^2} + \frac{\partial j^A}{z} + \cdots\right)\right\rangle + w \leftrightarrow 0 \\ &= \frac{k_{2\mathrm{d}}\dim\mathfrak{g}}{z^4} \end{aligned} \tag{50}$$

and

$$\langle j^2(z)j^2(0)\rangle = \cdots = 2k_{2\mathrm{d}}(\dim\mathfrak{g})\big(k_{2\mathrm{d}} + h^\vee\big). \tag{51}$$

Here $h^\vee$ comes as $f^{ABC}f_{ABC}$. We define

$$\widehat{j}(z) = j^2(z) - \frac{2k_{2\mathrm{d}}\dim\mathfrak{g}}{c}T(z). \tag{52}$$

This operator in the chiral algebra is a twisted $\widehat{B}_2$ operator and we can check

$$\langle\widehat{j}(z)T(0)\rangle = 0, \qquad \langle\widehat{j}(z)\widehat{j}(0)\rangle = \frac{2k_{2\mathrm{d}}\dim\mathfrak{g}}{z^4}\left((k_{2\mathrm{d}} + h^\vee) - \frac{k_{2\mathrm{d}}\dim\mathfrak{g}}{c_{2\mathrm{d}}}\right). \tag{53}$$

While two-point functions in the chiral algebra can have any sign, here it comes from twisting a $\widehat{B}_2$ operator in 4d:

$$\langle \widehat{j}(z)\widehat{j}(0)\rangle = \langle u_I u_J u_K u_L M^{(IJKL)}(z,\bar{z}) u_A u_B u_C u_D M^{(ABCD)}(0)\rangle = u\cdots u\frac{\epsilon^{IA}\epsilon^{JB}\epsilon^{KC}\epsilon^{LD}}{|x|^8}\mathcal{N} \tag{54}$$

Unitarity of the 4d theory implies $\mathcal{N}\geq 0$ (in some conventions), from which we learn that

$$\frac{\dim\mathfrak{g}}{c_{4\mathrm{d}}}\geq\frac{24h^\vee}{k_{4\mathrm{d}}}-12. \tag{55}$$

This inequality has been checked explicitly to hold in a large class of 4d theories. When it is saturated, this means that the Sugawara construction gives the true stress tensor; it means that the $\widehat{B}_2$ decouples from the theory.

Comment: the key benefit from using the chiral algebra here was to be able to deduce the four-point function $\langle jjjj\rangle$ (and $\langle Tjj\rangle$) from knowing the OPE of a pair of j.

There are similar inequalities

$$c_{4\mathrm{d}}\geq\frac{11}{30},\qquad k_{4\mathrm{d}}\geq\cdots \tag{56}$$