Topics on 4d $\mathcal{N} = 2$ theories

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Nobody (even the typist) proof-read these notes, so there may be obvious mistakes: tell BLF.

Abstract

We discuss extremal correlators in 4d $\mathcal{N} = 2$ theories and related aspects. These are lecture notes for the 2018 IHÉS summer school on *Supersymmetric localization and exact results.*

These lecture notes assume familiarity with supersymmetry at the level of the first few chapters of the book by Wess and Bagger.

1 Lecture 1, July 25

Around 1998, people introduced 4d $\mathcal{N} = 4$ maximally-supersymmetric Yang–Mills theory. We will start with the story of 4d $\mathcal{N} = 4$, then explain how it is generalized to 4d $\mathcal{N} = 2$.

1.1 Aspects of zero-coupling 4d $\mathcal{N} = 4$ SYM

$SU(N)$ gauge field A_μ, six real scalars ϕ^i, $1 \leq i \leq 6$, and some fermions ψ^A, all in the adjoint representation of $SU(N)$. People have considered

$$\mathcal{O}^I = C^I_{i_1 \cdots i_k} \operatorname{Tr}(\phi^{i_1} \dots \phi^{i_k}) \tag{1}$$

where for each I, $C^I_{i_1 \cdots i_k}$ is a tensor. These tensors should be symmetric (other parts don't contribute) and traceless to be in an irreducible representation of the R-symmetry $\mathrm{Spin}(6) = SU(4)$. These operators are half-BPS and played an important role in the early days.

We normalize the tensors such that

$$C^I_{i_1 \cdots i_k} C^{J, i_1 \cdots i_k} = \delta^{IJ}. \tag{2}$$

The action is normalized as

$$S = \frac{1}{2g^2_{\mathrm{YM}}} \operatorname{Tr}(F^2) + \cdots \tag{3}$$

As a consequence, the propagator is normalized as

$$\langle \phi^i_a(x)\phi^j_b(y)\rangle = \frac{g^2_{\rm YM}}{(2\pi)^2}\delta^{ij}\delta_{ab}\frac{1}{|x-y|^2}. \tag{4}$$

Note that by "zero-coupling" we mean that all loop diagrams are set to zero, namely we only consider the leading zero-loop order using Wick contractions. We also restrict ourselves to the planar limit.

As an exercise show that

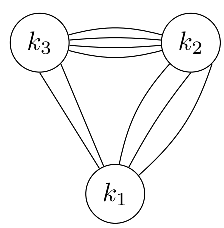
$$g(x,y) = \left\langle \mathrm{Tr}\left(\phi^{i_1}\dots\phi^{i_k}\right)(x)\,\mathrm{Tr}\left(\phi^{j_1}\dots\phi^{j_k}\right)\right\rangle = \frac{N^k g^{2k}_{\rm YM}}{(2\pi)^{2k}}\left(\delta^{i_1j_1}\delta^{i_2j_2}\dots\delta^{i_kj_k}+\text{cyclic}\right)\frac{1}{|x-y|^{2k}} \tag{5}$$

in the planar limit. We deduce, setting $\lambda = Ng^2_{\rm YM}$, that

$$\langle \mathcal{O}^I(x)\mathcal{O}^J(y)\rangle = \lambda^k\frac{k}{(2\pi)^{2k}|x-y|^{2k}}\delta^{IJ}. \tag{6}$$

The two-point function just tells us that the dimension is k (from the $|x-y|$ power).

Three-point function How can we draw a planar diagram for $\langle\mathcal{O}^{I_1}\mathcal{O}^{I_2}\mathcal{O}^{I_3}\rangle$? The only option is



We immediately see the triangle inequality $k_i \le k_j + k_k$ for $\{i,j,k\} = \{1,2,3\}$ has to be obeyed (otherwise there are no diagrams). Let $\Sigma = k_1+k_2+k_3$ and $\alpha_i = \Sigma/2 - k_i$. We find

$$\langle\mathcal{O}^{I_1}\mathcal{O}^{I_2}\mathcal{O}^{I_3}\rangle \overset{\text{planar}}{=} \frac{\lambda^{\Sigma/2}}{N(2\pi)^\Sigma}\frac{k_1k_2k_3\langle C^{I_1}C^{I_2}C^{I_3}\rangle}{|x-y|^{2\alpha_3}|x-z|^{2\alpha_2}|y-z|^{2\alpha_1}} \tag{7}$$

where $\langle C^{I_1}C^{I_2}C^{I_3}\rangle$ are some combinatorical numbers.

Observations from 1998–2000:

- $\Delta(\mathcal{O}^I) = k$, independent of the coupling constant; this is unsurprising since these operators are half-BPS and their dimension is set by their R-charge;
- surprisingly, $\langle\mathcal{O}\mathcal{O}\mathcal{O}\rangle$ has exactly the same λ dependence as given by AdS/CFT for infinite λ (now known for finite N too);

Note that while each operator $\mathcal{O}^I$ preserves half of the supersymmetry, in general they preserve different halves of the supersymmetry.

Higher correlators Let us label operators by their dimension $\Delta = k$ (number of fields used when constructing the operator).

Extremal correlators are those for which one of the Δ is equal to the sum of the others. For a three-point function this means $\Delta_3 = \Delta_1 + \Delta_2$, so all tree-level contractions are from one of the operators to all others.

Then for extremal correlators $\Delta = \Delta_1 + \cdots + \Delta_N$,

$$\left\langle \mathcal{O}^{\Delta_1}(x_1) \cdots \mathcal{O}^{\Delta_N}(x_N) \mathcal{O}^{\Delta}(y) \right\rangle = \lambda^{\#} \mathcal{A}(\Delta, N) \prod_{i=1}^{N} \frac{1}{|y - x_i|^{2\Delta_i}} \tag{8}$$

for some power $\#$ (sum of dimensions). The position-dependence is fixed and the coupling dependence is trivial.

The best source of information on this is the TASI lectures by D'Hoker et al. There it is explained how AdS/CFT relates this to Witten diagrams.

Question from the audience: the OPE of $\mathcal{O}^{\Delta_i}$ and $\mathcal{O}^{\Delta_j}$ is singular, so how can adding a heavy operator in the Andromeda galaxy affect this?? Zohar's answer. The operator in the Andromeda galaxy is hungry and wants to have all the lines. If even a single line goes between $\mathcal{O}^{\Delta_i}$ and $\mathcal{O}^{\Delta_j}$ then the resulting operator (whose prefactor is a singular function of $x_i - x_j$) has zero overlap with the heavy operator. Despite the distance, taking the correlator with a heavy operator projects out all these singular terms.

1.2 Broad overview of what's true in 4d $\mathcal{N} = 2$ SCFT

We will define here chiral ring operators and define extremal correlators thereof. The coordinate dependence will still be $\prod_{i=1}^{N} |y - x_i|^{-2\Delta_i}$. Unlike $\mathcal{N} = 4$, there is a non-trivial dependence on the coupling constant. It is *exactly computable*, and *non-holomorphic* in τ and $\overline{\tau}$.

(Incidentally, there hasn't been a lot of work on theories which are not superconformal; the questions we may want to ask are a little bit different.)

1.2.1 Coordinate dependence

Let us first understand the coordinate dependence

$$\left\langle \mathcal{O}^{\Delta_1}(x_1) \cdots \mathcal{O}^{\Delta_N}(x_N) \mathcal{O}^{\Delta}(y) \right\rangle = (\cdots) \prod_{i=1}^{N} \frac{1}{|y - x_i|^{2\Delta_i}}. \tag{9}$$

Back to general CFT. A two-point function of primary operators goes like $\langle \mathcal{O}_1 \mathcal{O}_2 \rangle \sim |x - y|^{-2\Delta}$, while a three-point function goes like $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle \sim |x - y|^{-(\Delta_1 + \Delta_2 - \Delta_3)} |x - z|^{-(\Delta_1 + \Delta_3 - \Delta_2)} |y - z|^{-(\Delta_2 + \Delta_3 - \Delta_1)}$. Something we learn in 2d CFT is that the conformal group can be used to formally put an operator at infinity. But we need to avoid making the formulas non-sensical. So an operator at infinity is defined as $\mathcal{O}^{\Delta}(\infty) := \lim_{y \to \infty} y^{2\Delta} \mathcal{O}^{\Delta}(y)$. Then the two and three-point functions are

$$\langle \mathcal{O}_1(0) \mathcal{O}_2(\infty) \rangle \sim 1, \qquad \langle \mathcal{O}_1(x) \mathcal{O}_2(y) \mathcal{O}_3(\infty) \rangle \sim |x - y|^{-(\Delta_1 + \Delta_2 - \Delta_3)}. \tag{10}$$

All correlators behave nicely. Of course sending a point to infinity is a reversible process.

Let's apply this procedure to the extremal correlator. Putting $y \to \infty$ we get no dependence on x_i positions:

$$\left\langle \mathcal{O}^{\Delta_1}(x_1) \cdots \mathcal{O}^{\Delta_N}(x_N) \mathcal{O}^{\Delta}(\infty) \right\rangle = (\cdots). \tag{11}$$

This looks very similar to things happening in supersymmetric theories. However a crucial difference is that the lack of position-dependence only happens once we put a specific one of the operators at infinity. Since the result is independent of distances we can just OPE these $\mathcal{O}^{\Delta_i}(x_i)$ operators and keep only the regular piece, until we only have a single operator $\widetilde{\mathcal{O}}(0)$ left.

1.2.2 Review of 4d $\mathcal{N} = 2$ SCFT

The supercharges are Q^i_α, $\overline{Q}^i_{\dot\alpha}$, S^i_α and $\overline{S}^i_{\dot\alpha}$ with $\alpha = 1, 2$, $\dot\alpha = 1, 2$ spinor indices, and $i = 1, 2$ R-symmetry indices. The conformal group is $SO(5, 1)$ and the R-symmetry is $\mathfrak{su}(2) \times \mathfrak{u}(1)$.

Superconformal primaries are those annihilated by the supercharges S and $\overline{S}$. The quantum numbers of these operators are:

- Δ conformal dimension;
- j_L and j_R spins under $SU(2) \times SU(2)$ space-time symmetries;
- S the $SU(2)$ R-symmetry quantum number;
- R the $U(1)$ R-symmetry charge.

Knowing this classification is important in the conformal bootstrap program. There is a huge amount of work on special classes of these operators. The superconformal index, the chiral ring, the chiral algebra, the Schur index, … all of these are names for some classes of these operators.

Today we are interested in half-BPS operators. This means they are annihilated by half of the Q and $\overline{Q}$ in addition to all of the S and $\overline{S}$. That's 12 supercharges.

Chiral operators We choose $\overline{Q}$.

Chiral operators, also called chiral ring operators, also called Coulomb branch operators, are those that obey $[\overline{Q}^i_\alpha, \mathcal{O}] = 0$ for all α and i.

Very schematically, from the $\mathcal{N} = 2$ superconformal algebra, (see appendix of Minwalla et al on Superconformal Index)

$$\{\overline{Q}, \overline{S}\} = \epsilon\epsilon(\Delta + R_{U(1)}) + \text{Lorentz} + R_{SU(2)} \tag{12}$$

An operator annihilated by both $\overline{Q}$ and $\overline{S}$ must be such that all these charges vanish. In particular, $j_r = 0$ and $S = 0$ and $\Delta = |R/2|$. In all known Lagrangian constructions, $j_l = 0$, but this is not a consequence of the algebra.[1] Note the absolute value. More precisely, $\Delta = R/2$ for chiral operators and $\Delta = -R/2$ for anti-chiral operators. These operators saturate a unitarity bound

$$\Delta \geq |R|/2. \tag{13}$$

The relation between R-charge and dimension implies that the OPE

$$\mathcal{O}_I(x)\mathcal{O}_J(y) = \sum_K C^k_{IJ}(x,y)\mathcal{O}_k(x) \tag{14}$$

where $\mathcal{O}_k$ is not necessarily chiral. The R-charge of $\mathcal{O}_k$ is $R_I + R_J$, so its dimension is at least $(R_I + R_J)/2 = \Delta_I + \Delta_J$. Given the $x - y$ dependence of C^k_{IJ} we learn that the OPE has no singular term. In addition, the term of order 1 has dimension exactly equal to half of its R-charge, in other words it is a chiral ring operator. Altogether,

$$\mathcal{O}_I(x)\mathcal{O}_J(y) = \sum_K C^K_{IJ}\mathcal{O}_K(z) + \text{regular, non-chiral.} \tag{15}$$

Here the positions x, y, z are arbitrary and C^K_{IJ} is independent of them. The residual terms go to zero as $|x-y| \to 0$. We will show that these non-chiral operators contribute nothing to extremal correlators. If we knew the C^K_{IJ}, which dependend non-holomorphically on coupling constants, then we would know the extremal correlator by repeatedly taking the OPE.

By the way, the OPE (15) defines a multiplication, hence a ring structure on the set of chiral operators.

Special case $R = 4$. Then $\Delta = 2$ and we can add $\int \mathrm{d}^4\theta\mathcal{O}$ to the action (here we are in $\mathcal{N} = 2$ superspace) and this deformation is exactly marginal (we are not going to prove it has no anomalous dimension). In particular, we will learn in the next lectures that $\langle \mathcal{O}\mathcal{O}^\dagger \rangle$ is the Zamolodchikov metric in theory space. It is easy to show that all exactly marginal deformations correspond to some $\Delta = 2$ chiral ring operator. It is reasonable to assume that all such deformations can be seen (locally in the conformal manifold) as g_{YM} and ϑ of some simple gauge group.

In all Lagrangian SCFTs, Coulomb branch operators have no spin, and the chiral ring is freely generated.[2] A (false) conjecture is that this holds for non-Lagrangian theories too. There are some non-Lagrangian constructions of models where the chiral ring is not freely generated, but in all cases where the construction is on solid ground, the same chiral ring can be obtained by gauging a discrete symmetry in a theory where the chiral ring is freely generated.

[1] In Lagrangian theories one can show that this holds, but in non-Lagrangian theories it is an open question to know whether there exists any spinning Coulomb branch operator. It is reasonable to conjecture that this is universally true.

[2] A ring is freely generated if there is a set of generators $(A_i)_{i\in I}$ such that the ring consists of all polynomials in the A_i, with no relations between them.

Higgs branch sector Here the condition is

$$[Q^1_\alpha, \mathcal{H}] = [\overline{Q}^1_{\dot\alpha}, \mathcal{H}] = 0. \tag{16}$$

(An imprecise terminology is “Schur sector”.) Then we find $\Delta = 2S$ while $j_l = j_r = R = 0$.

While the chiral ring coefficients had perturbative and non-perturbative contributions to C^K_{IJ}, the corresponding C^K_{IJ} for the Higgs branch are tree-level exact.[3] This is therefore very boring for Lagrangian theories. In non-Lagrangian theories this is still interesting and hard to compute.

... Typically an SCFT has coupling constants such as g_{YM} and ϑ, but also a space of vacua, namely different superselection sectors in infinite volume. For instance in $\mathcal{N} = 4$ we have a huge space of vacua $\mathbb{R}^6/\Gamma$. Among these vacua there are some vacua where $U(1)_R$ is spontaneously broken, while there are some where $SU(2)_R$ is spontaneously broken. These branches of vacua are essentially orthogonal.

- The expectation value of Coulomb branch operators is a $U(1)$ R-symmetry order parameter parametrizing whether $U(1)$ R-symmetry is broken.
- The expectation value of Higgs branch operators is a $SU(2)$ R-symmetry order parameter parametrizing whether $SU(2)$ R-symmetry is broken.

It is not clear in full generality what the C^K_{IJ} mean geometrically.

Tomorrow we will compute the C^K_{IJ} of the chiral ring (Coulomb branch) and extremal correlators using supersymmetric localization.

Question: why are vacua superselection sectors? Answer by example:

- in quantum mechanics with a double-well potential there is a single vacuum $\psi_+ + \psi_-$, while the state $\psi_+ - \psi_-$ has energy difference proportional to the instanton action;
- in $d > 2$ QFT with a double-well potential there are two vacua, because the tunneling rate has a factor of volume (in other words the instanton action which controls the (log of the) tunneling rate is proportional to the volume).

This is related to how there are no Goldstone boson in 2d. If the potential is exactly flat, there is a complication due to strong correlations in 2d; the volume factor in $e^{-\mathrm{Vol} S_{\mathrm{QM}}}$ can get erased.

[3] Intuitively, the reason is as follows: the coupling constant dependence comes from deforming the theory by an exactly marginal operator, hence by a Coulomb branch operator. Then the Higgs branch operators don’t talk to the Coulomb branch operators. Since the Yang–Mills coupling (and other couplings) are coefficients of Coulomb branch deformations, the Higgs branch correlators are independent of it.

2 Lecture 2, July 26

Recall the half-BPS sectors

- Coulomb branch operators $[\overline{Q}^i_\alpha, \mathcal{O}] = 0$ with $\Delta = R/2$ (the $U(1)$ R-charge), while (assuming the conjecture above) $j_l = j_r = S = 0$.
- Higgs branch operators $[\overline{Q}^i_\alpha, \mathcal{O}] = [Q^i_\alpha, \mathcal{O}] = 0$ with $\Delta = 2S$ (this is the $SU(2)$ R-symmetry isospin) and $R = j_l = j_r = 0$.

Since $(g_{\mathrm{YM}}, \theta)$ are parameters for a Coulomb branch operator, Higgs branch ring data is perturbatively (tree-level?) exact.

Extremal correlators Extremal correlators are

$$\big\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\mathcal{O}^\dagger(y)\big\rangle \tag{17}$$

where the total R-charge must vanish. Given the relation of dimension and R-charge for the chiral operators and the single antichiral operator $\mathcal{O}^\dagger$, we must take the latter to have

$$\Delta = \sum_{i=1}^n \Delta_i. \tag{18}$$

We shall now prove that the coordinate-dependence is very simple and not renormalized:

$$\big\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\mathcal{O}^\dagger(y)\big\rangle = \mathcal{A}\big(\{g_{\mathrm{YM}},\theta\}\big)\prod_{i=1}^n |y-x_i|^{-2\Delta_i}. \tag{19}$$

We will use $\overline{Q}\mathcal{O} = 0$, $Q\mathcal{O}^\dagger = 0$, and $\partial \sim \{Q, \overline{Q}\}$, and finally a trick: mapping $y \to \infty$ by conformal symmetry through $\mathcal{O}^\dagger(\infty) = \lim_{y\to\infty} y^{2\Delta_{\mathcal{O}}}\mathcal{O}^\dagger(y)$. Then we want to prove that $\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\mathcal{O}^\dagger(\infty)\rangle$ is independent of the x_i. Of course that's equivalent to the original statement, but it's a bit nicer to work with. We care about $\partial/\partial x_1$ acting on the correlator. This is equivalent to acting with $\{Q, \overline{Q}\}$. Now,

$$[\{Q,\overline{Q}\}, \mathcal{O}_1] = \{Q, [\overline{Q}, \mathcal{O}_1]\} + \{\overline{Q}, [Q, \mathcal{O}_1]\} = \{\overline{Q}, [Q, \mathcal{O}_1]\} \tag{20}$$

by chirality. Then we can integrate by parts $\overline{Q}$ and note that it does not act on any $\mathcal{O}_i$:

$$\frac{\partial}{\partial x_1}\Big\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\mathcal{O}^\dagger(\infty)\Big\rangle = \Big\langle [Q,\mathcal{O}_1](x_1)\dots\mathcal{O}_n(x_n)[\overline{Q},\mathcal{O}^\dagger](\infty)\Big\rangle \tag{21}$$

Recall that $[\overline{Q}, \mathcal{O}^\dagger](\infty)$ is defined as the limit of $y^{2\Delta_{\mathcal{O}}}[\overline{Q}, \mathcal{O}^\dagger](y)$. The key is that this is not the right power of y: the correlator at finite y behaves as

$$\Big\langle [Q,\mathcal{O}_1](x_1)\dots\mathcal{O}_n(x_n)[\overline{Q},\mathcal{O}^\dagger](y)\Big\rangle \sim y^{-2(\Delta_{\mathcal{O}}+1/2)} \quad \text{as } y\to\infty \tag{22}$$

because $[\overline{Q}, \mathcal{O}^\dagger]$ has dimension $\Delta_{\mathcal{O}} + 1/2$ (the details of the other operators are unimportant). Multiplying by $y^{2\Delta_{\mathcal{O}}}$ and sending $y \to \infty$ gives a y^{-1} scaling, which goes to zero. This concludes the proof.

Exactly marginal operators Consider the special case $n = 1$ and $\Delta = 2$, such that we are considering a two-point function of operators that can be used to construct exactly marginal deformations of the Lagrangian. Now the prefactor $\mathcal{A}$ has a physical meaning:

$$\left\langle \mathcal{O}_i(x)\mathcal{O}_j^\dagger(y)\right\rangle = G_{i\bar{j}}(g_{\mathrm{YM}}, \theta)\frac{1}{|x-y|^4}. \tag{23}$$

Here $G_{i\bar{j}}$ are (components of) the Zamolodchikov metric in theory space.

Don't get confused: the Coulomb branch operators we consider here have dimension 2 but the marginal operator actually added to the Lagrangian is $\int \mathrm{d}^4\theta\mathcal{O}$ which has dimension 4 hence is marginal (and exactly marginal it turns out).

2.1 Forget about susy

Suppose we have a CFT in dimension d and suppose it has real (Hermitian) operators $\mathcal{O}_i$ whose conformal dimension is d. It is natural to deform the action by

$$\delta S = \sum_i \lambda^i \int \mathcal{O}_i(x)\mathrm{d}^d x. \tag{24}$$

Does the theory remain conformal.

Examples:

- 4-dimensional free-field theory $\int(\partial\phi)^2\mathrm{d}^4x$. An interesting deformation is to add $\lambda\int\phi^4\mathrm{d}^4x$. This does not remain a conformal field theory: the beta function is non-zero and in fact the deformation is marginally irrelevant. In this case we go back to the original free theory. No conformal manifold.
- 4d $\mathcal{N} = 4$ super Yang–Mills has an exactly marginal deformation corresponding to g_{YM} and θ.
- 4d $\mathcal{N} = 2$ super QCD with gauge group $SU(N)$ and $N_f = 2N$ fundamental hypermultiplet. Again the exactly marginal deformation is g_{YM} and θ.
- 4d $\mathcal{N} = 1$ beta-deformation of 4d $\mathcal{N} = 4$ super–Yang–Mills theory.
- (not good): 2d WZW model, actually the parameter k is discrete, not continuous.
- (not good): Liouville, where we can change the linear dilaton coupling, which is very strange because it changes the central charge.
- Ashkin–Teller model (has $c = 1$), consisting of a compact boson of radius R, where the radius R is an exactly marginal deformation. This is equivalent to a fermionic model with quartic fermion interaction (Thirring model).

Let β^i be the beta function of the coupling λ^i. Then

$$\beta^i = 0\lambda^a + C^i_{ab}\lambda^a\lambda^b + C^i_{abc}\lambda^a\lambda^b\lambda^c + \cdots \tag{25}$$

For the operator to be exactly marginal we need to have $\beta^i = 0$ so $0 = C^i_{ab} = C^i_{abc} = \cdots$ This gives infinitely many constraints on the CFT data at $\lambda = 0$. For instance $C^i_{ab} = 0$ states

$$\mathcal{O}_a(x)\mathcal{O}_b(y) \text{ OPE } \not\supset \mathcal{O}_i(x). \tag{26}$$

The next condition $C^i_{abc} = 0$ states that a specific integral of a four-point function must vanish, etc.

This is reviewed in recent work by Bashmakov et al, and by Connor Behan et al. The hope would be to show that these constraints could only be satisfied (so that there exists a conformal manifold) if we have supersymmetry (or are in $d = 2$). By the way, we say "conformal manifold" but it could have singularities.

Assume we have a conformal manifold What structure does the conformal manifold $M_{\text{conformal}}$ have? Coordinates are coupling constants. Observation by Zamolodchikov (well, this is a bit misattributed, he didn't really discuss this question but related questions): the two-point function

$$\langle \mathcal{O}_i(0)\mathcal{O}_j(\infty)\rangle_\lambda = G_{ij}(\lambda) \tag{27}$$

defines a positive matrix $G(\lambda)$ at each point $\lambda \in M_{\text{conformal}}$.

Recall from differential geometry that the metric at a point does not make sense on its own.

- One can choose an orthonormal basis for the $\mathcal{O}_i$ so that $G_{ij} = 0$ at one point.
- Then (harder) one can choose Riemann normal coordinates, such that $\partial_k G_{ij} = 0$. This corresponds to a subtle choice of contact terms, which sets $C^i_{ab} = 0$ in the notations above.
- The first invariant quantity is R_{ijkl}, which is an integrated four-point function with appropriate antisymmetrizations. For instance in two dimensions,

$$R_{ijkl} = \int \mathrm{d}^2\eta \big\langle \mathcal{O}_i(0)\mathcal{O}_j(\eta)\mathcal{O}_k(1)\mathcal{O}_l(\infty)\big\rangle \log|\eta|^2. \tag{28}$$

2.2 Anomalies on the conformal manifold

Let us re-introduce the coordinates x and y. The two-point function is

$$\big\langle \mathcal{O}_i(x)\mathcal{O}_j(y)\big\rangle = \frac{G_{ij}(\lambda)}{|x-y|^{2d}}. \tag{29}$$

Use

$$\int e^{ipx}x^{-2d}\mathrm{d}^d x = \begin{cases} p^d \log(p^2) & \text{if } d \text{ is even;} \\ p^d & \text{if } d \text{ is odd} \end{cases} \tag{30}$$

to switch to momentum space.

Consider d even, which is a more interesting case. Note that the log must be there because in even dimensions p^d is a polynomial so its Fourier transform would be ultralocal $\Box^d\delta(x)$. We learn that

$$\big\langle \widetilde{\mathcal{O}}_i(p)\widetilde{\mathcal{O}}_j(-p)\big\rangle = G_{ij}(\lambda)p^d \log\left(\frac{p^2}{\Delta^2}\right). \tag{31}$$

At first this looks like the scale Δ violates conformal invariance. Under scaling we get an extra contribution p^d in addition to the usual conformal transformations.

This is very similar to the anomaly that gives rise to the central charges a and c in 4d.

The mathematical way to handle this is to let couplings depend on space: $\lambda^i \to \lambda^i(x)$. The partition funciton becomes a functional:

$$Z[\lambda(x)] = \int \mathcal{D}X\, e^{S_{\text{CFT}}+\int \lambda^i(x)\mathcal{O}_i(x)}. \tag{32}$$

Then under a Weyl transformation $g \to e^{\pm 2\sigma(x)}g$ (not sure about the sign) we compute

$$\delta_{\sigma(x)} \log Z[\lambda, g] = \int \mathrm{d}^d x\, \mathcal{L}\big(\lambda^i(x), g_{\mu\nu}(x)\big) \tag{33}$$

In 4d, we get

$$\delta_{\sigma(x)} \log Z[\lambda, g] = \frac{1}{192\pi^2}\int \mathrm{d}^4 x\,\sqrt{g}\,\delta\sigma\left[G_{ij}\widehat{\Box}\lambda^i\widehat{\Box}\lambda^j + \cdots\right] \tag{34}$$

where $\widehat{\Box}\lambda^I = \Box\lambda^I + \Gamma^I{}_{JK}\lambda^J\lambda^K$ is the Paneitz–Fradkin–Tseytlin operator. The WZ consistency conditions imply that $\Gamma^I{}_{JK}$ is the Levi–Civita connection of the Zamolodchikov metric.

Supersymmetry Supersymmetry (4d $\mathcal{N}=2$) implies various restrictions on the anomaly. A particularly interesting term that pops out is

$$\delta_\sigma \log Z \supset \frac{1}{2}\int \mathrm{d}^4 x\, K(\lambda^i, \overline{\lambda}^i)\Box^2(\delta\sigma). \tag{35}$$

Here the λ^i are $\frac{4\pi}{g^2_{\text{YM}}} + \frac{i\theta}{2\pi}$ and appear in front of the $\int \mathrm{d}^4\theta\mathcal{O}$ complex operators built from chiral ring operators, and $\overline{\lambda}^i$ are their complex conjugates. The conformal manifold is automatically an even-dimensional manifold, and in fact the metric is complex, namely only the $G_{i\bar{j}}$ components corresponding to λ^i and $\overline{\lambda^j}$ are non-zero. Moreover,

$$G_{i\bar{j}} = \partial_i\overline{\partial}_{\bar{j}}K(\lambda, \overline{\lambda}) \tag{36}$$

Question: can the anomaly polynomial here be found by anomaly inflow? Answer: people have not been successful in getting such conformal anomalies with anomaly inflow.

Question: what geometric structures do conformal manifolds M have? Answer: without supersymmetry, M has a Riemannian structure (a metric).

- In 4d $\mathcal{N}=4$: M is a Kähler manifold, which in known Lagrangian examples (SYM) has constant curvature, essentially upper half plane mod $SL(2,\mathbb{Z})$.
- In 4d $\mathcal{N}=2$: M is a Kähler manifold.
- In 4d $\mathcal{N}=1$: M is a Kähler manifold.
- In 3d $\mathcal{N}=4$: there are no exactly marginal deformations so M is a point.
- In 3d $\mathcal{N}=2$: M is a Kähler manifold.
- In 3d $\mathcal{N}=1$: M is just Riemannian, like in the absence of supersymmetry.

There are interesting questions about these spaces. For instance a conjecture consistent with what we know is that the volume may be finite.

Question: don't conformal anomalies break conformal symmetry. Answer: there are two types of anomalies in physics.

- An ABJ anomaly implies an explicit violation of the symmetry. It is as bad as breaking a symmetry by turning on a mass term etc. In 4d it is tested by triangle diagrams with two dynamical fields and one background one.
- A 't Hooft anomaly has to do purely with background fields, and does not point to any breaking of symmetry (in a trivial background). It has to match between UV and IR for instance (while ABJ anomalies don't need to match since the symmetry is not existent at all). In 4d it is tested by triangle diagrams with three background fields.
- Another interesting case in 4d is to have to background and one dynamical fields.

2.3 Meaning of sphere partition function

The stereographic map from S^4 to $\mathbb{R}^4$ involves a (finite) Weyl transformation, which can be probed by integrating small $\delta\sigma$ transformations. Integrating the anomaly polynomial we find (the actual computation is quite messy and needs all terms in the anomaly polynomial)

$$Z_{S^4} = e^{\frac{1}{12}K(\lambda^i,\overline{\lambda}^i)}. \tag{37}$$

This is the physical meaning of the four-sphere partition function. Therefore, since the metric is a derivative of the Kähler potential,

$$G_{i\bar{j}} = \left\langle \mathcal{O}_i(0)\mathcal{O}^\dagger_{\bar{j}}(\infty)\right\rangle_{\mathbb{R}^4} = 16\frac{\partial}{\partial\lambda^i}\frac{\partial}{\partial\overline{\lambda}^{\bar{j}}}\log Z_{S^4} = \frac{1}{\left(Z_{S^4}\right)^2}\det\begin{pmatrix} Z_{S^4} & \frac{\partial}{\partial\lambda^i}Z_{S^4} \\ \frac{\partial}{\partial\overline{\lambda}^{\bar{j}}}Z_{S^4} & \frac{\partial}{\partial\lambda^i}\frac{\partial}{\partial\overline{\lambda}^{\bar{j}}}Z_{S^4}\end{pmatrix}. \tag{38}$$

Question: K is not well-defined, it can be shifted $K \to K + F(\lambda) + \overline{F}(\overline{\lambda})$ so shouldn't this mean that the four-sphere partition function is ambiguous?

Answer: yes, Pestun's S^4 partition function was computed in one "choice of Kähler frame", physically meaning one choice of regularization scheme.

Question: is there a similar formula for other four-manifolds. Answer. As a side-comment, for 2d $\mathcal{N} = (2,2)$ theories, $Z_{S^2} = e^{-K}$ exactly, which lets you compute the Zamolodchikov metric etc. What about 4d $\mathcal{N} = 2$ on $S^3 \times S^1$? Speculation $Z_{S^3 \times S^1}$ could be related to the exponential of a line-bundle introduced by Neitzke. For 3d, $Z_{S^3} = e^{-F}$ but F is independent of the coupling constants; it is monotonic in RG flow.

An open question in 3d is to compute the Zamolodchikov metric. Nobody knows.

Question: the sphere partition function can be split into conformal blocks (hemisphere partition function), so you can cut and glue. Answer: there is a paper by Bachas et al about the Calabi diastasis. The idea is that the hemisphere partition function teaches us about the Calabi diastasis.

Next time we will consider $SU(2)$ and $SU(N)$.

3 Lecture 3, July 27

Yesterday we discussed the two half-BPS sectors of 4d $\mathcal{N} = 2$ SCFTs:

- Higgs branch operators, whose structure constants are independent of g_{YM} and θ hence are boring for Lagrangian theories, while for non-Lagrangian theories they can be studied using the chiral algebras discussed in Balt van Rees' class;
- Coulomb branch operators, whose structure constants depend non-trivially (and non-holomorphically) on g_{YM} and θ, so that they are intersting even in Lagrangian theories; in non-Lagrangian theories it is not clear how to start studying them but there has been some progress using bootstrap (Hamburg group).

Yesterday we saw that $Z_{S^4} = e^K$ namely the four-sphere partition function captures the Kähler potential, hence the Zamolodchikov metric. Consider $\mathcal{O}_i$ some chiral operators with $\Delta(\mathcal{O}_i) = 2$. Then what we are saying is

$$\left\langle \mathcal{O}_i(0)\mathcal{O}^\dagger_{\bar{j}}(\infty)\right\rangle = G_{i\bar{j}}(g_{\text{YM}},\theta) = \partial_i\overline{\partial}_{\bar{j}} \log Z_{S^4}(g_{\text{YM}},\theta). \tag{39}$$

3.1 $SU(2)$ gauge theory

There are two 4d $\mathcal{N} = 2$ SCFTs with gauge group $SU(2)$:

- with an adjoint hypermultiplet we get 4d $\mathcal{N} = 4$ $SU(2)$ SYM;
- with four fundamental hypermultiplets it is called SQCD; this is a genuine $\mathcal{N} = 2$ theory.

Exercise: $\phi_2 \coloneqq -4\pi i \operatorname{Tr}(\varphi^2)$ is a chiral ring (Coulomb branch) operator, where φ is the scalar in the 4d $\mathcal{N}=2$ vector multiplet. Note that $\operatorname{Tr}(\varphi)=0$ so we have to take a square; the factor of $-4\pi i$ is for later convenience. It turns out that the chiral ring (in both SCFTs we study here) is freely generated by ϕ_2, so it is spanned as a vector space by

$$\mathcal{O}_n = (\phi_2)^n \tag{40}$$

where $\Delta(\mathcal{O}_n)=2n$. (Here, $\mathcal{O}_0=1$.) In fact, $\mathcal{O}_1$ is the exactly marginal operator: the action has

$$\tau \int \mathcal{O}_1 \mathrm{d}^4\theta \mathrm{d}^4 x + \text{c.c.} \tag{41}$$

with coupling constant $\tau = \frac{4\pi i}{g_{\text{YM}}^2} + \frac{\theta}{2\pi}$.

We chose to normalize the operators such that the chiral ring multiplication is simple:

$$\mathcal{O}_n \mathcal{O}_m = \mathcal{O}_{n+m}, \tag{42}$$

but two-point functions

$$\big\langle \mathcal{O}_n(0) \mathcal{O}_m^\dagger(\infty) \big\rangle = \delta_{mn} G_{2n}(\tau,\overline{\tau}) \tag{43}$$

are complicated. We could have normalized operators canonically to get rid of the G_{2n} factor, but this would complicate the multiplication. From the G_{2n} we can deduce all extremal correlators:

$$\big\langle \mathcal{O}_{n_1}(x_1)\cdots\mathcal{O}_{n_k}(x_k)\mathcal{O}^\dagger_{n_1+\cdots+n_k}(y)\big\rangle = G_{2(n_1+\cdots+n_k)}(\tau,\overline{\tau}) \prod_{i=1}^{k} \frac{1}{|y-x_i|^{4n_i}}. \tag{44}$$

Since G_2 is the Zamolodchikov metric on the space of theories, we learned yesterday that

$$G_2(\tau,\overline{\tau}) = 16\frac{1}{(Z_{S^4})^2} \det\begin{pmatrix} Z_{S^4} & \partial_\tau Z_{S^4} \\ \partial_{\overline{\tau}} Z_{S^4} & \partial_\tau\partial_{\overline{\tau}} Z_{S^4}\end{pmatrix} = 16\partial_\tau\partial_{\overline{\tau}} \log Z_{S^4}. \tag{45}$$

For 4d $\mathcal{N}=4$ $SU(2)$ SYM,

$$Z_{S^4} = \int_{-\infty}^{+\infty} \mathrm{d}a\, e^{-4\pi\Im(\tau)a^2}(2a)^2, \tag{46}$$

while for 4d $\mathcal{N}=2$ $SU(2)$ with 4 fundamental hypermultiplets,

$$Z_{S^4} = \int_{-\infty}^{\infty} \mathrm{d}a\, e^{-4\pi\Im(\tau)a^2}(2a)^2 \frac{H(2ia)H(-2ia)}{\big(H(ia)H(-ia)\big)^4} \big|Z_{\text{instanton}}(ia,\tau)\big|^2. \tag{47}$$

A word of warning: these partition functions are not quite invariant under $SL(2,\mathbb{Z})$ S-duality: instead it changes precisely by a Kähler transformation. In fact, no one checked this explicitly for the SQCD case but it should be possible to get it from the AGT corespondence. A (seemingly open) question is whether the Kähler transformations that come up in 4d $\mathcal{N}=4$ and in SQCD are the same.

For 4d $\mathcal{N}=4$ $SU(2)$ SYM we find exactly

$$G_2 = 6\frac{g_{\rm YM}^2}{4\pi}. \tag{48}$$

This is saying that only tree-level diagrams contribute to the Zamolodchikov metric.

For 4d $\mathcal{N}=2$ $SU(2)$ with 4 fundamental hypermultiplets,

$$\begin{aligned} G_2 = 6\frac{g_{\rm YM}^2}{4\pi} & \underbrace{- \frac{135\zeta(3)}{2\pi^2}\frac{1}{(\Im\tau)^4} + \frac{1575\zeta(5)}{4\pi^3}\frac{1}{(\Im\tau)^5} + \cdots}_{\text{perturbative}} \\ & + \cos\theta e^{-8\pi^2/g^2}\left(\frac{6}{(\Im\tau)^2} + \frac{3}{\pi}\frac{1}{(\Im\tau)^3} + \cdots\right) + \cdots \end{aligned} \tag{49}$$

where we see a perturbative series, a one instanton contribution that comes with a perturbative series, etc. Both SCFTs have a conformal manifold with coordinate $(g_{\rm YM}, \theta)$ in the upper half plane modulo $SL(2,\mathbb{Z})$. The two SCFTs give rise to two different metrics on the same manifold. The 4d $\mathcal{N}=4$ theory gives the standard metric with constant negative curvature on the Poincaré disk:

$$\mathrm{d}s^2 = \frac{6}{(\Im\tau)^2}\mathrm{d}\tau\mathrm{d}\overline{\tau}. \tag{50}$$

The SQCD theory gives

$$\mathrm{d}s^2 = \frac{6}{(\Im\tau)^2}\mathrm{d}\tau\mathrm{d}\overline{\tau} + \cdots \tag{51}$$

This deformed metric shares many features with the Poincaré disk metric:

- the volume of a fundamental domain of the action of $SL(2,\mathbb{Z})$ is finite;
- the weak coupling point is logarithmically far away.

In the AGT correspondence, we can think of SQCD as arising from twisted-compactifying the 6d $\mathcal{N}=(2,0)$ theory on a sphere with four punctures. Then the metric we get now is an interesting metric on Teichmüller space, which is different from the standard Weyl–Peterson metric. On the other hand 4d $\mathcal{N}=4$ comes in AGT from the torus with one puncture, and we would say that the same Teichmüller space is equipped with the standard metric.

3.2 Relation to resurgence

Dyson argued that the coefficients of a perturbative series

$$a_1\lambda + a_2\lambda^2 + \cdots + e^{-1/\lambda}(\#\lambda + \#\lambda^2 + \cdots) + \cdots \tag{52}$$

should obey $|a_{n+1}/a_n| \sim n$. We can check that this holds for the perturbative series we find now.

Brodsky, Karliner, etc. proposed a more stringent criterion (which in 4d can only be checked in the present situation of extremal correlators). Suppose we have computed n loops of a perturbative series (n even). Then we can make a $(n/2, n/2)$ Padé approximation by fitting

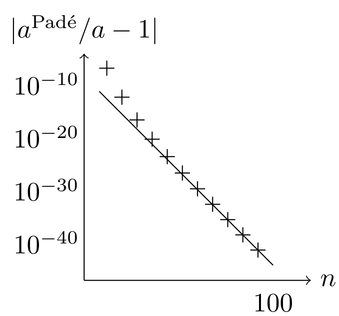
$$\frac{\sum_{i=0}^{n/2-1} c_i \lambda^i}{\sum_{i=0}^{n/2-1} d_i \lambda^i} \tag{53}$$

to the first n terms computed so far. This makes a prediction for a_{n+1}. We can ask how well this works. The conjecture is that in any QFT,

$$\left| \frac{a_{n+1}^{\text{Padé}}}{a_{n+1}} - 1 \right| \leq C e^{-\sigma n} \quad \text{as } n \to \infty, \tag{54}$$

for some constants C and σ.

We can plot[4]



and the slope is aroud $\sigma \simeq 0.7$, which matches with QCD calculations suggesting that many observables have $\sigma \simeq \log 2$.

The series we get are Borel summables, namely the Borel transform has no poles on the positive axis. In fact, the Borel transform only has poles on the negative real axis, whose interpretation is not completely clear. These poles should have to do with instantons that we need to add to repair the perturbative series. This suggests that there could be some way to retrieve the Nekrasov instanton contributions from the perturbative series, maybe some recursion relations?

3.3 Some references

- `arXiv:1405.7271` Derivation of $Z = e^{-K}$ from an approach well-suited to supersymmetric localization.

[4] The figure in the present notes is built from fake data; for the real data, see the original paper `arXiv:1602.05971`.

- arXiv:1509.08511 Derivation of $Z = e^{-K}$ from the trace anomaly.
- arXiv:1602.05971 Extremal correlators.
- arXiv:1803.07366 and arXiv:1805.04202 and Donagi–Morrison on global properties.
- arXiv:1603.06207 and subsequent papers on resurgence.
- arXiv:1710.07336 and arXiv:1804.01535 and arXiv:1803.00580 on limits of heavy operators.
- arXiv:0910.4963 and arXiv:1409.4212 and arXiv:1409.4217 and arXiv:1508.03077 and arXiv:1602.05871 etc. on tt^* geometry, Toda, integrability.
- arXiv:1712.02551 and arXiv:1712.01164 on the 2d story, there is probably a lot more to be explored there.

3.4 tt^* geometry

Let M be the infinite matrix with components $M_{mn} = \frac{1}{Z_{S^4}}\partial_\tau^n \partial_{\overline{\tau}}^m Z_{S^4}$, namely

$$M = \begin{pmatrix} 1 & \partial_\tau Z/Z & \cdots \\ \partial_{\overline{\tau}} Z/Z & \partial_\tau\partial_{\overline{\tau}} Z/Z & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}. \tag{55}$$

Then it turns out that $G_{2n} = \left\langle \mathcal{O}_n(0)\mathcal{O}_n^\dagger(\infty)\right\rangle$ is given by

$$G_{2n} = (16)^n \frac{D_n}{D_{n-1}}, \qquad \text{with} \quad D_n = \det_{2n\times 2n} (M_{ij})_{0\leq i,j<2n}. \tag{56}$$

Nontrivial exercise: check that this is invariant under $Z \to e^{F+\overline{F}} Z$.

The G_{2n} obey recursion relations. This is true for more general 4d $\mathcal{N} = 2$ theories, but we don't know in general what the integrable system is. It is not terribly hard to show that

$$\partial_\tau \partial_{\overline{\tau}} \log D_n = \frac{D_{n+1}D_{n-1}}{D_n^2} - (n+1)D_1. \tag{57}$$

Now change variables to $D_n = 16^n e^{q_n - \log Z_{S^4}}$. Then the differential equation becomes

$$\partial_\tau \partial_{\overline{\tau}} q_n = e^{q_{n+1}-q_n} - e^{q_n - q_{n-1}} \tag{58}$$

This is called the half-infinite Toda chain, and it is familiar from the integrability literature. It basically describes a bunch of masses along springs. This Toda equation lives on the space of chiral operators; it is totally different from the one that lives on the Seiberg–Witten curve; it is totally different from the Toda system that lives on the other side of the AGT correspondence.

Given a boundary condition this system is exactly solvable. The typical technique involves taking ratios of determinants. Unsurprisingly our result takes the form of a ratio of determinants. In fact, whenever you see a ratio of determinants you should think integrability.

Actually 4d $\mathcal{N}=4$ $SU(2)$ SYM and 4d $\mathcal{N}=2$ $SU(2)$ SQCD both obey the same equations, but the initial conditions are different, trivial in the first case but not the second.

$SU(N)$ **SQCD** We now consider 4d $\mathcal{N}=2$ $SU(N)$ theory with $2N$ hypermultiplets in the fundamental representation. The chiral ring is generated by $\mathrm{Tr}(\varphi^2)$, $\mathrm{Tr}(\varphi^3)$, …, $\mathrm{Tr}(\varphi^N)$, where again φ is the scalar in the vector multiplet. Then extremal correlators boil down to two-point functions

$$\Big\langle \big(\mathrm{Tr}\,\varphi^2\,\mathrm{Tr}\,\varphi^3\big)\big(\mathrm{Tr}\,\overline{\varphi}^2\,\mathrm{Tr}\,\overline{\varphi}^3\big)\Big\rangle \tag{59}$$

and these are again expressed in a form that generalizes this ratio of determinants. The operator $\mathrm{Tr}(\varphi^2)$ has $\Delta=2$ (exactly marginal) and its correlators obey a generalization of the Toda chain found for $SU(2)$. It is not clear how this integrable system is called.

There is a problem: in the $SU(2)$ case we needed to know $\partial_\tau^n\partial_{\overline{\tau}}^m Z$, which were just derivatives of Z_{S^4} with respect to the gauge coupling. On the other hand in $SU(N)$ we need to access $\mathrm{Tr}(\varphi^k)$ for $k>2$. This requires adding

$$\lambda\int \mathrm{d}^4\theta\,\mathcal{O}_n \tag{60}$$

to the Lagrangian, at least infinitesimally, to get all orders in perturbation theory. This is an irrelevant deformation. Then we want derivatives like

$$\frac{\partial^{\cdots} Z_{S^4}}{\partial^{\cdots}\lambda\,\partial^{\cdots}\overline{\lambda}}\bigg|_{\lambda=0}. \tag{61}$$

Unfortunately, the partition function has not been computed using localization. More precisely it takes the form

$$\int \mathrm{d}a\, e^{\lambda a^{2n}}\, Z^{\text{standard}}_{\text{perturbative}}(a)\,|Z_{\text{instanton}}(\tau,\lambda,\dots)|^2. \tag{62}$$

Here the perturbative part is the same as the standard one (except for the classical contribution λa^{2n}), while the instanton partition function is affected.

Question by Maxim Zabzine: why is the instanton partition function affected? Answer: […] for $U(N)$ gauge groups, see Fucito–Morales, it's known exactly. The really tough thing is to extract the $U(1)$ factor in a principled way.