Index theorems and 5d localization

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Nobody (even the typist) proof-read these notes, so there may be obvious mistakes: tell BLF.

Abstract

Mathematical aspects (elliptic, transverse elliptic operators) and 5d gauge theories. These are lecture notes for the 2018 IHÉS summer school on *Supersymmetric localization and exact results.*

1 Lecture 1, July 23

See Pestun `https://arxiv.org/abs/1608.02954`.

1.1 Equivariant cohomology

Consider a smooth manifold M with a $\mathfrak{u}(1)$ action $\mathfrak{u} \times M \to M$. We want to talk about the cohomology of $M/\mathfrak{u}(1)$, but of course that space may be singular.

We construct a supermanifold $T[1]M$, modeled on TM but with the Grassmann degree of tangent vectors being shifted by 1 (this is the meaning of [1]). The bosonic coordinates are x^μ (like those of M) and Grassmann odd coordinates are ψ^μ, that transform like $\mathrm{d}x^\mu$, namely

$$\widetilde{\psi}^\mu = \frac{\partial f^\mu}{x^\nu}\psi^\nu \quad \text{under} \quad \widetilde{x}^\mu = f^\mu(x). \tag{1}$$

Exercise: the measure is invariant under coordinate changes, namely $\int \mathrm{d}^n x \mathrm{d}^n \psi (\cdots) = \int \mathrm{d}^n \widetilde{x} \mathrm{d}^n \widetilde{\psi}(\cdots)$. Remember that the ψ anticommute.

(Hint: note that in Grassmann integrals $\int \mathrm{d}\psi = 0$ and $\int \mathrm{d}\psi\psi = 1$ so under a change of variables $\widetilde{\psi} = a\psi$, the relation $\int \mathrm{d}\widetilde{\psi}\widetilde{\psi} = \int \mathrm{d}\psi\psi$ forces $\mathrm{d}\widetilde{\psi} = \frac{1}{a}\mathrm{d}\psi$, the inverse transformation compared to bosonic variables.)

Note that smooth functions on $T[1]M$ can be expanded as

$$\alpha(x,\psi) = \sum_k \frac{1}{k!}\alpha_{\mu_1\cdots\mu_k}\psi^{\mu_1}\cdots\psi^{\mu_k} \tag{2}$$

into forms so $C^\infty(T[1]M) = \Omega^\bullet(M)$.

Using the $\mathfrak{u}(1)$ action we can define

$$\delta x^\mu = \psi^\mu, \qquad \delta\psi^\mu = V^\mu(x) \tag{3}$$

where V generates the action. This is a kind of supersymmetry. Acting on a polyform (a function on $T[1]M$) we find

$$\delta\alpha(x,\psi) = \sum_k \left(\frac{1}{k!}\partial_\rho \alpha_{\mu_1\cdots\mu_k}\psi^\rho\psi^{\mu_1}\cdots\psi^{\mu_k} + \frac{1}{k!}\alpha_{\mu_1\cdots\mu_k}V^{\mu_1}\cdots\psi^{\mu_k} \right) \tag{4}$$

so δ acts on polyforms as $\mathrm{d}_V = \mathrm{d} + i_V$. Last week we checked that $\mathrm{d}_V^2 = \mathcal{L}_V$.

In Francesco Benini's class we had the opposite sign in d_V.

Since d and i_V change degrees in different ways, it is useful to introduce a formal parameter ξ of degree 2 and use $\mathrm{d} + \xi i_V$. This now acts with degree 1 on $\Omega^\bullet(M)[\xi]$ (polynomials in ξ) or $\Omega^\bullet(M)[[\xi]]$ (formal series in ξ).

The Cartan model of equivariant cohomology consists of the cohomology of the restriction of d_V to $\mathfrak{u}(1)$-invariant forms

$$\Omega^\bullet_{\mathrm{inv}}(M)[\xi] = \{\mathcal{L}_V\alpha(\xi) = 0\}. \tag{5}$$

(We define in the obvious way equivariantly closed forms and equivariantly exact forms.) Then $H^\bullet_{\mathfrak{u}(1)}(M) = H^\bullet(M/\mathfrak{u}(1))$ if the latter exists.

While ξ is important to get the complete grading of the cohomology, we will work as physicists and just set $\xi = 1$.

1.2 Atiyah–Bott formula

Our goal is to compute

$$\int \mathrm{d}^n x \mathrm{d}^n\psi \alpha(x,\psi) = Z[0] \tag{6}$$

for an equivariantly closed polyform α. Then we note that for any W with $\delta^2 W = 0$,

$$Z[t] = \int \mathrm{d}^n x \mathrm{d}^n\psi\, \alpha(x,\psi) e^{-t\delta W(x,\psi)} \tag{7}$$

is independent of t. Then to compute the integral we shall send $t\to\infty$.

Consider $W = v^\mu g_{\mu\nu}\psi^\nu$. Exercise: $\delta^2 W = 0 \Leftrightarrow \mathcal{L}_V g = 0$, namely V is a Killing vector.

Let us now assume we have a metric g that is $\mathfrak{u}(1)$-invariant (if our $\mathfrak{u}(1)$ action is a $U(1)$ action then averaging an arbitrary metric works). Then compute

$$\delta W = |V|^2 + \psi^\mu\psi^\nu\partial_\nu(g_{\mu\rho}V^\rho). \tag{8}$$

Then we want to compute

$$\lim_{t\to\infty}\int \mathrm{d}^n x \mathrm{d}^n\psi\, \alpha(x,\psi) e^{-t|V|^2 - t\psi^\mu\psi^\rho(\mathrm{d}v)_{\mu\rho}}. \tag{9}$$

Any point with $|V|^2 > 0$ is exponentially suppressed so only fixed points ($|V|^2 = 0$) contribute.

We look at a fixed point $V(P) = 0$ and choose coordinates with P at zero. Expand

$$\delta W = H_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + O(x^3) + S_{\mu\nu}\psi^\mu\psi^\nu + O(\psi^2 x) + O(\psi^4). \tag{10}$$

Rescale $\widetilde{x} = \sqrt{t}x$ and $\widetilde{\psi} = \sqrt{t}\psi$. This introduces no Jacobian factor (because the measure on $T[1]M$ is invariant under coordinate changes, see an exercise above). Then

$$Z[0] = \lim_{t\to\infty} \int \mathrm{d}^n\widetilde{x}\mathrm{d}^n\widetilde{\psi}\,\alpha\left(\frac{\widetilde{x}}{\sqrt{t}}, \frac{\widetilde{\psi}}{\sqrt{t}}\right) e^{-H_{\mu\nu}\widetilde{x}^\mu\widetilde{x}^\nu - S_{\mu\nu}\widetilde{\psi}^\mu\widetilde{\psi}^\nu - O(1/\sqrt{t})} \tag{11}$$

$$= \alpha_0(0) \int \mathrm{d}^n\widetilde{x}\mathrm{d}^n\widetilde{\psi} e^{-H_{\mu\nu}\widetilde{x}^\mu\widetilde{x}^\nu - S_{\mu\nu}\widetilde{\psi}^\mu\widetilde{\psi}^\nu} \sim \alpha_0(0)\frac{\mathrm{Pf}(S)}{\sqrt{\det H}} \tag{12}$$

up to some factors of π, where the Pfaffian of an antisymmetric matrix S is essentially $\mathrm{Pf}(S) = \sqrt{\det S}$. The last expression is only defined if $H_{\mu\nu}$ is non-singular (no quartic point, no manifold of fixed points).

Recall the supersymmetry $\delta x^\mu = \psi^\mu$ and $\delta\psi^\mu = V^\mu(x)$. We can linearize it to $\delta_l \widetilde{x}^\mu = \widetilde{\psi}^\mu$ and $\delta\widetilde{\psi}^\mu = \partial_\rho V^\mu(0)\widetilde{x}^\rho$. Then

$$- H_{\mu\nu}\widetilde{x}^\mu\widetilde{x}^\nu - S_{\mu\nu}\widetilde{\psi}^\mu\widetilde{\psi}^\nu = -(\delta W)_2 \tag{13}$$

where the subscript means we only keep quadratic terms. Then

$$\delta_l(\delta W)_2 = 0 \tag{14}$$

implies $H_{\mu\nu} = \partial_\mu V^\rho(0) S_{\rho\nu}$. We deduce the simplification

$$\alpha_0(0)\frac{\mathrm{Pf}(S)}{\sqrt{\det H}} = \alpha_0(0)\frac{1}{\sqrt{\det \partial_\mu V^\rho(0)}} \tag{15}$$

Theorem 1 (Atiyah–Bott formula)**.** *Given a $U(1)$ action with no isolated fixed points, for any equivariantly closed form α,*

$$\int \mathrm{d}^n x \mathrm{d}^n\psi\,\alpha(x,\psi) = \sum_{\textit{fixed points } x} \frac{\pi^{\dim M/2}\alpha_0(x)}{\sqrt{\det(\partial_\mu V^\rho(x))}}. \tag{16}$$

Always two sets of calculations: either take quadratic piece and compute Gaussian integrals, or your term has more symmetries and we have use the simplifications giving just the determinant in the denominator.

There is a more general version of Atiyah–Bott when the fixed points are not isolated:

$$\int_M \alpha = \int_{M_{\mathfrak{u}(1)}} \frac{\alpha|_M}{e(NM_{\mathfrak{u}(1)})} \tag{17}$$

Exercise (hard): understand this very carefully (e.g., using Riemann normal coordinates). You have to split coordinates along the fixed locus and transverse to the locus, take curvature into account etc.

1.3 Bundles

Consider now a vector bundle E on M. Define $E[1]$ to be the supermanifold whose coordinates χ^a along the fiber (expressed in a basis of sections of E) are Grassmann odd, and whose coordinates x^μ along M are even.

Then we consider the vector bundle $T[1]E[1]$ over the manifold $E[1]$. It has coordinates $(x^\mu, \psi^\mu, \chi^a, H^a)$, where ψ^μ transforms as $\mathrm{d}x^\mu$ and H^a transforms as $\mathrm{d}\chi^a$, where d is the exterior derivative on the supermanifold $E[1]$. Here x^μ and H^a are even and ψ^μ and χ^a are even.

A vector bundle over a vector bundle is not a vector bundle!

Consider a change of coordinates $\widetilde{\chi}^a = g^a{}_b(x)\chi^b$ on the fibers of $E[1]$. Then

$$\widetilde{H}^a = \mathrm{d}\widetilde{\chi}^a = g^a{}_b(x)\mathrm{d}\chi^b + \partial_\rho(g^a{}_b)\mathrm{d}x^\rho\chi^b = g^a{}_b(x)H^b + \partial_\rho(g^a{}_b)\mathrm{d}x^\rho\chi^b. \tag{18}$$

There is an extra term compared to how a section (such as χ^a) would transform. For instance $H^a h_{ab} H^b$ is BAD.

Now consider a $\mathfrak{u}(1)$-equivariant bundle on M, namely we have an action of $\mathfrak{u}(1)$ on E that acts compatibly with the action $\mathfrak{u}(1) \times M \to M$. Then

$$\delta x^\mu = \psi^\mu, \quad \delta\psi^\mu = v^\mu(x), \quad \delta\chi^a = H^a, \quad \delta H^a = \mathcal{L}_V\chi^a \tag{19}$$

following the previous definitions. It is sometimes convenient to change H (in a non-canonical way) to get a covariantized version

$$\delta_\nabla\chi^a = H^a - A^a{}_{b\mu}\psi^\mu\chi^b, \qquad \delta_\nabla H^a = \mathcal{L}_V\chi^a + \text{extra terms}. \tag{20}$$

This is useful because H can be made to transform as a section.

We can do all of Atiyah–Bott.

The relevant thing here is the Mathai–Quillen formalism. There are different ways of expressing it: a non-equivariant version, an equivariant version, and a version with $P \times V/G$ that should be used to deal with gauge theories properly (see Atiyah–Jeffrey).

1.4 Atiyah–Bott with bundles

Let us see some aspects of Atiyah–Bott.

We consider W such that

$$\delta W = \delta\big(\chi^a h_{ab}(H^b - S^b(x))\big) + \delta(\psi^\mu g_{\mu\nu}V^\nu). \tag{21}$$

Here h_{ab} is a metric on E, $g_{\mu\nu}$ is a metric on M, s is a section on E, and $\delta^2 W = 0$ and $\mathcal{L}_V g = 0$ and s is a $\mathfrak{u}(1)$-equivariant section. For later purposes we call D the matrix with components $\partial_\rho s^b$ at the fixed point.

Linearize this near a fixed point. First we linearize the supersymmetry transformations to[1]

$$\delta X = \psi, \qquad \delta\psi = R_0 X, \qquad \delta\chi = H, \qquad \delta H = R_1\chi \tag{22}$$

[1]It may be more covariant to redefine H such that $\delta\chi = H - iDX$ and $\delta H = R_1\chi + iD\psi$?

where R_0 and R_1 are linear operators. Then we write W itself:

$$W = \langle \psi, R_0 X\rangle + \langle \chi, H - iDX\rangle. \tag{23}$$

Exercise: check that $\delta^2 W = 0$ using the nice property that $R_1 D = D R_0$.

$$\begin{tikzcd} 0 \arrow[r] & \overset{X}{E_0} \arrow[r, "-iD"] & \overset{\chi}{E_{-1}} \arrow[r] & 0 \\ 0 \arrow[r] & \underset{\psi}{E_0} \arrow[r, "iD"] \arrow[u, "R_0"] & \underset{H}{E_{-1}} \arrow[r] \arrow[u, "R_1"'] & 0 \end{tikzcd} \tag{24}$$

We can work out that R_0 and R_1 are antiHermitian like $\mathcal{L}_V$.

Then compute (recall that Grassmann-odd quantities, including δ, pick up signs when we exchange them; this shows up in the Leibniz rule)

$$\delta W = \langle R_0 X, R_0 X\rangle - \langle \psi, R_0\psi\rangle + \langle H - iDX, H\rangle - \langle \chi, R_1\chi + iD\psi\rangle \tag{25}$$

and integrate H to get

$$\delta W|_{H \text{ integrated}} = \underbrace{\langle X, (-R_0^2 + \frac{1}{4}D^\dagger D)X\rangle}_{\text{analogue of } H_{\mu}\nu x^\mu x^\nu} - \langle \psi, R_0\psi\rangle - \langle \chi, R_1\chi\rangle - \langle \chi, \frac{i}{2}D\psi\rangle + \langle \psi, \frac{i}{2}D^\dagger\chi\rangle. \tag{26}$$

The last four terms are the analogue of $S_{\mu\nu}\psi^\mu\psi^\nu$. We get

$$Z_{\text{1-loop}} \sim \frac{\det^{1/2}\begin{pmatrix} R_0 & \frac{i}{2}D^\dagger \\ -\frac{i}{2}D & R_1 \end{pmatrix}}{\det^{1/2}(-R_0^2 + \frac{i}{4}D^\dagger D)}. \tag{27}$$

Note that the formula only makes sense if the denominator has no kernel (or if the numerator has corresponding zero modes), so $-R_0^2 + \frac{i}{4}D^\dagger D$ needs to be a second order elliptic operator. For the numerator we compute

$$\begin{pmatrix} R_0 & \frac{i}{2}D^\dagger \\ -\frac{i}{2}D & R_1 \end{pmatrix}\begin{pmatrix} R_0 & \frac{i}{2}D^\dagger \\ -\frac{i}{2}D & R_1 \end{pmatrix}^\dagger = \begin{pmatrix} -R_0^2 + \frac{1}{4}D^\dagger D & \frac{i}{2}(R_0 D^\dagger - D^\dagger R_1) \\ \frac{i}{2}(DR_0 - R_1 D) & -R_i^2 + \frac{1}{4}DD^\dagger \end{pmatrix}. \tag{28}$$

The off-diagonal blocks vanish and we get partial cancellation:

$$Z_{\text{1-loop}} \sim \frac{\det^{1/4}(-R_1^2 + \frac{1}{4}DD^\dagger)}{\det^{1/4}(-R_0^2 + \frac{1}{4}D^\dagger D)}. \tag{29}$$

This is the analogue of $\sqrt{\det S}/\sqrt{\det H}$.

Outside the kernels of $DD^\dagger$ and $D^\dagger D$ there are huge cancellations. The kernel contributes

$$Z_{\text{1-loop}} \sim \frac{\det^{1/4}_{DD^\dagger}(-R_1^2)}{\det^{1/4}_{D^\dagger D}(-R_0^2)} \sim \frac{\det^{1/2} R_1}{\det^{1/2} R_0} \tag{30}$$

(we ignore phases). This final formula is an analogue of $1/\sqrt{\det \partial v(0)}$.

Next time we will see elliptic and transversely elliptic operators.

2 Lecture 2, July 24

In infinite-dimensional settings, R_0, R_1, D are differential operators.

2.1 Elliptic operators

The symbol $\sigma(D)$ of an operator D with

$$Du = \sum_{|\alpha|\le m} a_\alpha(x)\partial^\alpha u = \sum_{|\alpha|\le m} a_\alpha(x)\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n} u \tag{31}$$

is defined in terms of the highest-order part ($|\alpha| = m$) as

$$\sigma(D) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha = \sum_{|\alpha|=m} a_\alpha(x)\xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n} \in C^\infty(M)[\xi_1,\ldots,\xi_n]. \tag{32}$$

The operator is called elliptic if the symbol is non-degenerate away from $\xi = 0$.

Example: the symbol of the Laplacian is $\sigma(\Delta) = \sum_{i=1}^N \xi_i^2$ so this operator is elliptic.

Example: the operator $\Delta_p = \mathrm{d}\mathrm{d}^\dagger + \mathrm{d}^\dagger\mathrm{d}$ on p-forms is elliptic.

Exercise: the Dirac operator is $\not{D}$ elliptic.

Definition: D is a Fredholm operator if both $\ker D$ and $\operatorname{coker} D$ are finite-dimensional.

Theorem: on a compact manifold M, any elliptic operator D is Fredholm.

Consider now the action for a scalar field

$$S = \int_M \mathrm{d}\phi \wedge \star\mathrm{d}\phi = \langle \phi, \Delta\phi\rangle. \tag{33}$$

Then the Gaussian integral gives us $1/\sqrt{\det\Delta}$, so this operator has better have no kernel or cokernel, or at least a finite-dimensional one. This is guaranteed by ellipticity.

For the gauge field,

$$\int F\wedge\star F = \int \mathrm{d}A\wedge\star\mathrm{d}A = \langle A, \mathrm{d}^\dagger\mathrm{d}A\rangle \tag{34}$$

is not elliptic. This is not surprising because of gauge-invariance. After gauge-fixing though the operator is elliptic.

$$\int F\wedge\star F + \mathrm{d}^\dagger A\wedge\star\mathrm{d}^\dagger A = \int \mathrm{d}A\wedge\star\mathrm{d}A = \langle A, \mathrm{d}^\dagger\mathrm{d}A\rangle \tag{35}$$

2.2 Transversally elliptic operators

Example: Consider the elliptic operator ∂ on S^2.

Lift the problem to $S^1 \times S^2$ with coordinates $(t, z, \bar{z})$. (The Laplace operator is $\Delta = -\partial_t^2 + \partial_z \partial_{\bar{z}}$.) A horizontal form is a form that has no component in the t direction. We denote these by $\Omega_H^{p,q}$ where p, q are the holomorphic/antiholomorphic grading. Then the holomorphic exterior derivative[2] maps $\partial_H \colon \Omega_H^{p,q} \to \Omega_H^{p+1,q}$.

This has no way of being elliptic because the t-dependence is not controlled at all. On the other hand we can expand in Fourier modes $\mathcal{L}_t \omega_n = in\omega_n$. Then $\partial \colon (\Omega_H^{0,0})_n \to (\Omega_H^{1,0})_n$ and we get an elliptic operator for each n. The original operator ∂_H is then called transversally elliptic.

Example: S^3 embedded as $|z_1|^2 + |z_2|^2 = 1$ in $\mathbb{C}^2$. The Hopf fibration is an S^1 fibration of S^3 with base S^2. This defines v and κ with $i_v \kappa = 1$ and $i_v \mathrm{d}\kappa = 0$. Then forms can be decomposed into vertical and horizontal forms:

$$\begin{cases} \Omega & = \Omega_V \oplus \Omega_H \\ \omega & \mapsto (\kappa \wedge i_v \omega, \omega - \kappa \wedge i_v \omega). \end{cases} \tag{36}$$

Now on to the general definition. Consider a manifold with an action of G, then

$$T_G^* M = \Big\{ \xi \in T^* M, \langle \xi, v \rangle = 0 \forall v \text{ associated to the action of } G \Big\}. \tag{37}$$

Then a differential operator D is transversally elliptic if for each $\xi \in T_G^* M$, $\sigma(D)$ is invertible.

Typically, when writing $\Delta = -R_0^2 + DD^\dagger$ with Δ a second-order elliptic operator and R_0 roughly a Lie derivative, the residual operator $DD^\dagger$ is transversely elliptic.

Typically $\dim(\ker D) = \infty$ but the space $\ker D$ decomposes into representations of the symmetry G and the pieces are finite-dimensional. (Sometimes this is also useful in non-compact spaces because elliptic does not imply Fredholm.)

Example: on $\mathbb{C}$, $\overline{\partial}$ is elliptic but not Fredholm since $\{\ker \overline{\partial}\} = \mathrm{Span}\{1, z, z^2, \dots\}$. However, using the rotation action of $U(1)$ on $\mathbb{C}$ we find that the index

$$\operatorname{ind} \overline{\partial} = 1 + t + t^2 + \cdots = \sum_{k=0}^{\infty} t^k = \frac{1}{1-t}. \tag{38}$$

2.3 Serious examples

2.3.1 Two dimensions version 1

We consider maps $S^2 \to \mathbb{C}^n$ (chiral fields).

Then

$$\delta X^\mu = \psi^\mu, \qquad \delta \psi^\mu = 0, \qquad \delta \chi_{\bar{z}}^i = H_{\bar{z}}^i - \overline{\partial} X^i, \qquad \delta H_{\bar{z}}^i = i \overline{\partial} \psi^i. \tag{39}$$

Now $R_0 = 0$ and $R_1 = 0$. Here, $\chi_{\bar{z}}^i \in \Omega^{(0,1)}\big(S^2, X^*(T^{1,0}\mathbb{C}^n)\big)$ to be pedantic.

[2] The exterior derivative splits as $\mathrm{d} = \mathrm{d}_V + \mathrm{d}_H$, then $\mathrm{d}_H = \partial_H + \overline{\partial}_H$ as in any almost-complex manifold.

In our earlier formalism $M = \mathrm{Maps}(S^2 \to \mathbb{C}^n)$ and we are working with $T[1]M$.

We have holomorphic map that is isolated. ∂ is elliptic.

$$\delta W = \delta\big(\chi^i_{\bar{z}} g_{i\bar{j}} (H^{\bar{j}}_z - \partial X^{\bar{j}})\big) = \overline{\partial} X^i g_{i\bar{j}} \partial X^{\bar{j}} + \chi^i_z g_{i\bar{j}} \partial_z \psi^{\bar{j}}. \tag{40}$$

Fixed points $\overline{\partial} X^i = 0$ We find determinants $\det^{1/2}(\partial\overline{\partial})/\det^{1/2}(\partial\overline{\partial}) = 1$. Studying these maps gives rise to Gromov–Witten invariants.

2.3.2 Two dimensions version 2 (equivariant)

Using $D = \partial$ and $R_0 = \mathcal{L}_v$ on $\Omega^{0,0}$ and $R_1 = \mathcal{L}_v$ on $\Omega^{(1,0)}$ (or $\Omega^{(0,1)}$). (Importantly D and R commute.) Then

$$\delta X^\mu = \psi^\mu, \qquad \delta\psi^\mu = \mathcal{L}_v X^\mu, \qquad \delta\chi^i_{\bar{z}} = H^i_{\bar{z}} - \overline{\partial} X^i, \qquad \delta H^i_{\bar{z}} = \mathcal{L}_v \chi^i_{\bar{z}} + \overline{\partial}\psi^i. \tag{41}$$

Then $\delta^2 = \mathcal{L}_v$. We compute

$$\delta W = \delta\big(\psi^\mu g_{\mu\nu} \mathcal{L}_v X^\nu + \chi^i_{\bar{z}} g_{i\bar{j}} (H^{\bar{j}}_z - \partial X^{\bar{j}})\big) \tag{42}$$

Now we are working with $-\mathcal{L}_v^2 + \partial\overline{\partial}$. The determinant is non-trivial

$$\frac{\det_{\Omega^{(1,0)}(T^{(0,1)})}(\mathcal{L}_v)}{\det^{1/2}_{\Omega^{(0,0)}}(\mathcal{L}_v)}. \tag{43}$$

Now this is a non-trivial calculation.

2.3.3 Three-dimensions

We take the $U(1)$ translation along the Hopf fiber of S^3, and we decompose into vertical and horizontal forms and introduce $\Omega^{p,q}_H$ and ∂_H as before.

$$\delta X^\mu = \psi^\mu, \qquad \delta\psi^\mu = \mathcal{L}_v X^\mu, \qquad \delta\chi^i_{\bar{z}} = H^i_{\bar{z}} - \overline{\partial}_H X^i, \qquad \delta H^i_{\bar{z}} = \mathcal{L}_v \chi^i_{\bar{z}} + \overline{\partial}_H \psi^i. \tag{44}$$

Now $\chi^i_{\bar{z}}, H^i_{\bar{z}} \in \Omega^{0,1}_H\big(X^*(T^{1,0}M)\big)$. We write

$$\delta W = \delta\big(\psi^\mu g_{\mu\nu} \mathcal{L}_v X^\nu + \chi^i_{\bar{z}} g_{i\bar{j}} (H^{\bar{j}}_z - \partial X^{\bar{j}}). \tag{45}$$

Then we see in the first term the operator $-\mathcal{L}_v^2 + \partial_H \overline{\partial}_H$ appear. The operator $\partial_H \overline{\partial}_H$ is transversely elliptic. The determinant is

$$\frac{\det_{\Omega^{(1,0)}(T^{(0,1)})}(\mathcal{L}_v)}{\det_{\Omega^{(0,0)}}(\mathcal{L}_v)} \tag{46}$$

In all three situations we just saw, the operator $-R_0^2 + D^\dagger D$ has to be elliptic.

2.4 Index theorems

(Index theorems and some examples on $\mathbb{CP}^2$ and $\mathbb{CP}^3$.)

Take a compact M and elliptic differential operator D (D is Fredholm). Then we define the analytical index[3]

$$\operatorname{ind} D = \dim\ker D - \dim\operatorname{coker} D \in \mathbb{Z}. \tag{47}$$

Computing it analytically is tough, but there are ways to compute it using topology. A slight generalization of this index is when we have a complex instead of just a single operator:

$$\cdots \to \Omega^p \xrightarrow{\mathrm{d}} \Omega^{p+1} \xrightarrow{\mathrm{d}} \Omega^{p+1} \xrightarrow{\mathrm{d}} \cdots \tag{48}$$

with $\mathrm{d}^2 = 0$ then we define the index as an alternating sum. For instance for the Dolbeault operator $\overline{\partial}$ twisted by a vector bundle E (we mean that we add a connection term),

$$\operatorname{ind}(\overline{\partial}, E) = \sum_k (-1)^k \dim H^k(M, E). \tag{49}$$

3 Lecture 3, July 25

Theorem 2 (Atiyah–Singer index theorem)**.** *The index can be computed as an integral of characteristic classes:*

$$\operatorname{ind}(\overline{\partial}, E) = \frac{1}{(-2\pi i)^n} \int_M \operatorname{td}(T_M^{1,0}) \operatorname{ch}(E). \tag{50}$$

There are many variants of this index theorem.

Note that the characteristic classes can be written as invariant polynomials in field strengths, but the full integral is an integer independent of the choice of connection on the bundle. In fact, it is a priori surprising that the integral gives an integer. In a sense, the Atiyah–Singer theorem is a generalization of the Gauss–Bonnet theorem.

Consider an element $p \in \mathbb{R}[\mathfrak{g}]^G$ namely a G-invariant polynomial. Then choose a connection and let F be its field strength. Then $p(F) \in H^\bullet(M, \mathbb{R})$ is independent of the choice of connection.

Elliptic complex We have seen the notion of elliptic operator $D\colon E \to F$ mapping sections to sections. Now we can define an elliptic complex as

$$0 \to E_1 \xrightarrow{D_1} E_2 \xrightarrow{D_2} \cdots \xrightarrow{D_{n-1}} E_n \to 0 \tag{51}$$

such that at the level of the symbols the complex is exact, namely $\sigma(D_{i+1})\sigma(D_i) = 0$ for all i. The usual case is retrieved by looking at the complex

$$0 \to E \to F \to 0. \tag{52}$$

[3] Note that $\dim\operatorname{coker} D = \dim\ker D^\dagger$.

Indeed, such a complex is exact if and only if the map in the middle is invertible.

Example: we saw the elliptic PDE $F^+ = 0$ and $\mathrm{d}^\dagger A = 0$. A similar problem is to show that the following complex is elliptic

$$0 \to \Omega^0(M_4) \xrightarrow{\mathrm{d}} \Omega^1(M_4) \xrightarrow{\mathrm{d}^+} \Omega^{2+}(M_4) \to 0 \tag{53}$$

where $\mathrm{d}^+ = \frac{1}{2}(1+\star)\mathrm{d}$.

3.1 Equivariant version of index theorem

Consider a G-equivariant bundle on a manifold M with G action. Consider a G-equivariant connection $D_{A,G} = D_A + \epsilon^a i_{v^a}$ and

$$F_{A,G} = D_{A,G}^2 - \epsilon^a \mathcal{L}_{v^a}. \tag{54}$$

This is a 0-form plus a 2-form.[4] Then for p any invariant polynomial, $p(F_{A,G}) \in H_G^\bullet(M)$ is an equivariantly closed form. Then the integral can be computed using the Atiyah–Bott localization theorem

$$\int_M p(F_{A,G}) = \sum_{\text{fixed points}} \frac{p_0(F_{A,G})}{\sqrt{\cdots}}. \tag{55}$$

The big difference compared to the non-equivariant setting is that now we know how to compute the integral coming out of the Atiyah–Singer theorem.

$$\operatorname{ind}(\overline{\partial}, E)(e^{\varphi_a T^a}) = \frac{1}{(2\pi i)^k} \int \operatorname{td}_G(T_M)\operatorname{ch}_G(E) \tag{56}$$

$$= \sum_{\text{fixed points } x} \frac{\operatorname{Tr}_{E_x}(g)}{\det_{T_x^{1,0}}(1-g^{-1})} \tag{57}$$

where E_x and $T_x^{1,0}$ are fibers at x. The end slogan is: when we want to compute the index of some G-equivariant differential operator, we only need to find fixed points of the G action and do a linear algebra problem near that point.

Example: $\mathbb{CP}^1 = S^2$ Define $\mathbb{CP}^1 = \big\{(z_1, z_2) \sim (\lambda z_1, \lambda z_2) \mid \lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}\big\}$. Cover it by

- $U_1 = \{z_1 \neq 0\}$ with coordinate $\xi = z_2/z_1$;
- $U_2 = \{z_2 \neq 0\}$ with coordinate $\lambda = z_1/z_2$.

Of course $\xi = 1/\lambda$ on the overlap. The $U(1)$ action on $\mathbb{CP}^1$ is given by $\xi \to t\xi$ and $\lambda \to t^{-1}\lambda$ for $t \in U(1) = \big\{|t| = 1 \mid t \in \mathbb{C}\big\}$.

[4] Side-note: in localization in any dimension we find a BPS locus with $0 = i_v F + \mathrm{d}_A \varphi$, namely $F + \varphi$ is equivariantly-closed.

Now consider $E = \Omega^{0,0}(\mathbb{CP}^1)$ and $F = \Omega^{0,1}(\mathbb{CP}^1)$ and $\overline{\partial}\colon E \to F$. Let us try to see slowly how to apply the index theorem. First check

$$\overline{\xi} \to t^{-1}\overline{\xi}, \qquad \mathrm{d}\overline{\xi} \to t^{-1}\mathrm{d}\overline{\xi}, \qquad \mathrm{d}\xi \to t\mathrm{d}\xi, \tag{58}$$

$$\overline{\lambda} \to t\overline{\lambda}, \qquad \mathrm{d}\overline{\lambda} \to t\mathrm{d}\overline{\lambda}, \qquad \mathrm{d}\lambda \to t^{-1}\mathrm{d}\lambda. \tag{59}$$

The index theorem (57) states

$$\operatorname{ind}\overline{\partial} = \sum_{\text{fixed points } x} \frac{\operatorname{Tr}_{E_x} t - \operatorname{Tr}_{F_x} t}{\det_{T_x}(1-\rho(t))} \tag{60}$$

$$= \frac{1-t^{-1}}{(1-t)(1-t^{-1})} + \frac{1-t}{(1-t)(1-t^{-1})} = 1. \tag{61}$$

Here E_x and F_x are fibers of the bundles defined above, while T_x is the tangent space of M at x, while $\rho(t)$ is the representation in which t acts on the bundle.

Each of the two terms comes from one fixed point. Each of these is equal to the index we saw on $\mathbb{C}$ yesterday, namely $1 + t + \cdots = 1/(1-t)$, and the analogue with $t \to t^{-1}$.

Example: $\mathcal{O}(n)$ bundle on $\mathbb{CP}^1$ The total space of the line bundle is defined by identifying points of $\big(\mathbb{C}^2 \setminus \{(0,0)\}\big) \times \mathbb{C}$ as $(z_1, z_2, z_3) \sim (\lambda z_1, \lambda z_2, \lambda^n z_3)$. Again we have to open sets U_1 and U_2 with coordinate ξ_1 and λ_1 on the base of the line bundle, and with coordinates

$$\xi_2 = \frac{z_3}{z_1^n} \qquad \text{and} \qquad \lambda_2 = \frac{z_3}{z_2^n} \tag{62}$$

on the fibers, respectively. On the overlap,

$$\xi_1 = \frac{1}{\lambda_1}, \qquad \text{and} \qquad \xi_2 = \frac{\lambda_2}{\lambda_1^n}. \tag{63}$$

Consider the $U(1)$ action defined by $\xi_1 \to t\xi_1$ and $\xi_2 \to \xi_2$, which implies $\lambda_1 \to t^{-1}\lambda_1$ and $\lambda_2 \to t^{-n}\lambda_2$.

Then consider $E = \Omega^{0,0} \otimes \mathcal{O}(n)$ and $F = \Omega^{0,1} \otimes \mathcal{O}(n)$ and

$$\overline{\partial}\colon E \to F. \tag{64}$$

Then the equivariant index theorem implies

$$\operatorname{ind}\overline{\partial} = \frac{1-t^{-1}}{(1-t)(1-t^{-1})} + \frac{t^{-n}-t^{1-n}}{(1-t)(1-t^{-1})} = 1-n. \tag{65}$$

Example: S^3 Consider $|z_1|^2 + |z_2|^2 = 1$ in $\mathbb{C}^2$. Consider the T^2 action $z_1 \to e^{i\alpha}z_1$ and $z_2 \to e^{i\beta}z_2$. Consider the diagonal $U(1)$ action $U(1) \times S^3 \to S^3$. Orbits define the Hopf fibration $U(1) \to S^3 \to S^2$.

There are nice coordinates

$$z_1 = \frac{e^{i\theta}}{\sqrt{1+\xi\bar{\xi}}}; \qquad z_2 = \frac{e^{i\theta}\xi}{\sqrt{1+\xi\bar{\xi}}} \tag{66}$$

mapping to $\mathbb{C}\times S^1$. Exercise: find another $\mathbb{C}\times S^1$ coordinate patch with $\lambda = 1/\xi$ and some $\widetilde{\theta} = (?)$.

Given a horizontal form on $S^1 \times M$ we can do a Fourier expansion: $\Omega^p_H(S^1 \times S^2) = \oplus_n \Omega^p_{H,n}(S^2)$. Similarly, when you have a horizontal form in $\Omega^p_H(S^3)$ you can Fourier expand it in one of the $\mathbb{C}\times S^1$ coordinate patches and (exercise) doing carefully the change of patch we get

$$\Omega^p_H(S^3) = \oplus_n \Omega^p_H(S^2, \mathcal{O}(n)). \tag{67}$$

Then we will use the index result (65) above to compute one-loop determinants on S^3.

Question from the audience: will we compute the index on S^3, what will this be used for?

Answer: remember we had $\det_{\ker D} R_1 / \det_{\ker D^\dagger} R_0$, so we will work out how the Lie derivative gets expanded into modes through (67), then find that the ratio of determinants includes an exponent equal to the index $1-n$ computed above.

3.2 Chern–Simons, unlike Witten did

The goal is to compute the S^3 partition function. All fields are Lie-algebra valued, we'll ignore this in the notations.

Consider a gauge bundle $P \to S^3$; the Chern–Simons action can be written

$$S_{\mathrm{CS}}(A) = \frac{k}{4\pi}\int \mathrm{Tr}\left(A\mathrm{d}A + \frac{2}{3}A^3\right) \tag{68}$$

for A in the space $\mathcal{A}$ of connections on the gauge bundle. We have a $U(1)$ symmetry on S^3. Then in analogy to $\delta x^\mu = \psi^\mu$, $\delta\psi^\mu = \phi^a v^\mu_a(x)$, $\delta\phi^a = 0$, we have

$$\delta A = \Psi, \qquad \delta\Psi = \mathcal{L}_v A + \mathrm{d}_A \Phi, \qquad \delta\Phi = 0 \tag{69}$$

where ψ is a Grassmann-odd one-form. A variant, using the field strength F is

$$\delta A = \Psi, \qquad \delta\Psi = i_v F + i\mathrm{d}_A\sigma, \qquad i\delta\sigma = i_v\Psi. \tag{70}$$

The map back is $\Phi = i\sigma - i_v A$, and $\sigma \in \Omega^0$ is a (Lie algebra valued) zero-form.

Note that (A, Ψ) lives in $T[1]\mathcal{A}$, on which there is a canonically defined measure (important for supersymmetric localization, as stressed earlier).

Supersymmetric Chern–Simons is

$$S_{\mathrm{SCS}} = S_{\mathrm{CS}}(A - i\kappa\sigma) - \frac{k}{4\pi}\int \kappa\wedge\psi\wedge\psi \tag{71}$$

and is invariant under δ. Exercise: show this using that $\kappa = g(v,\ ) \in \Omega^1(S^3)$ obeys $i_v\kappa = 1$ and $i_v \mathrm{d}\kappa = 0$.

We need more fields to ensure that the operators are transversally elliptic where appropriate. Then we add an odd zero-form χ and an even zero-form H and write

$$\delta A = \Psi, \qquad \delta\Psi = i_v F + i\mathrm{d}_A\sigma, \qquad \delta\sigma = -i i_v\Psi, \qquad \delta\chi = H, \qquad \delta H = \mathcal{L}_v^A\chi - i[\sigma,\chi] \tag{72}$$

with

$$\mathcal{L}_v^A = \mathrm{d}_A i_v + i_v \mathrm{d}_A = \mathcal{L}_v + [i_v A,\ ]. \tag{73}$$

Incidentally this is a 3d $\mathcal{N}=2$ vector multiplet; counting odd components we see enough to build a 4d $\mathcal{N}=1$ Dirac spinor, consistent with 3d $\mathcal{N}=2$ supersymmetry.

We compute

$$\delta W = \delta\Big(\Psi\wedge\star\overline{\delta\Psi} + \chi\wedge\star(H - F_H)\Big) = F\wedge\star F + \mathrm{d}_A\sigma\wedge\star\mathrm{d}_A\sigma + \cdots \tag{74}$$

Recall $i_v\kappa = 1$ and the vertical/horizontal forms $\Omega^\bullet = \Omega^\bullet_V \oplus \Omega^\bullet_H$ where the first factor is obtained by $\kappa\wedge i_v$. In particular, Ω^2_V and Ω^2_H have ranks 2 and 1 respectively.

Exercise: write the fixed-point equations and find $F=0$ and σ constant. Then using our supersymmetric localization machinery, get an incomplete result

$$\int \mathrm{d}\sigma e^{-\#\operatorname{Tr}(\sigma^2)} \frac{\sqrt{\det_{\Omega^0(S^3)}(\mathcal{L}_v + \mathrm{ad}_\sigma)}}{\sqrt{\det_{\Omega^1(S^3)}(\mathcal{L}_v + \mathrm{ad}_\sigma)}}. \tag{75}$$

We forgot to gauge fix! We need to add c, $\bar{c}$, b. This gives an extra determinant

$$\int \mathrm{d}\sigma e^{-\#\operatorname{Tr}(\sigma^2)} \frac{\sqrt{\det_{\Omega^0(S^3)}(\mathcal{L}_v + \mathrm{ad}_\sigma)}\det_{\Omega^0(S^3)}(\mathcal{L}_v + \mathrm{ad}_\sigma)}{\sqrt{\det_{\Omega^1(S^3)}(\mathcal{L}_v + \mathrm{ad}_\sigma)}}. \tag{76}$$

Since we know a lot about the harmonics we can compute these determinants very explicitly. But instead we can use the identity we saw in the first lecture,

$$\frac{\sqrt{\det S}}{\sqrt{\det H}} = \frac{1}{\sqrt{\det \partial v}}. \tag{77}$$

We can decompose one-forms to

$$\Omega^1(S^3) = \Omega^1_V + \Omega^{1,0}_H + \Omega^{0,1}_H. \tag{78}$$

Then Ω_V is basically zero-forms, and $\Omega^{1,0}_H$ and $\Omega^{0,1}_H$ are the same up to the phase. Ignoring the phase we find a cancellation:

$$\int \mathrm{d}\sigma e^{-\#\operatorname{Tr}(\sigma^2)} \frac{\det_{\Omega^0}(\mathcal{L}_v + \mathrm{ad}_\sigma)}{\det_{\Omega^{1,0}_H}(\mathcal{L}_v + \mathrm{ad}_\sigma)}. \tag{79}$$

Next we shall use that ∂_H is transversally elliptic, and decompose into modes then uses $\mathcal{L}_v \omega_n = 2\pi i n \omega_n$ with $\omega_n \in \Omega^{\bullet}_H(S^2, \mathcal{O}(n))$. Then we need to compute the mismatch between kernel and cokernel and the one-loop determinant is

$$\prod_n (2\pi i n + \mathrm{ad}_\sigma)^{n-1} \tag{80}$$

where the $n-1$ comes from the index theorem we worked with earlier.

4 Lecture 4: Chern–Simons theory

The reference is Källén in 2012.

$$\begin{array}{ccccccc} \Omega^0 & & \Omega^1 & & \Omega^2_H & & \Omega_0 \\ [c] & \longrightarrow & [A] & \longrightarrow & [\chi] & \otimes & [\bar{c}] \\ \uparrow & & \uparrow & & \circlearrowleft & & \circlearrowright \\ [\sigma] & \longrightarrow & [\psi] & \longrightarrow & [H] & \otimes & [b] \\ & & & \xrightarrow[D]{} & & & \end{array} \qquad R \simeq \mathcal{L}_v + \mathrm{ad}_\sigma \tag{81}$$

The problem with $[c]$, $[A]$, $[\chi]$ is a transversally elliptic problem.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & F_0 & \longrightarrow & F_1 & \longrightarrow & F_2 & \longrightarrow & 0 \\ & & & & \xrightarrow[D]{} & & & & \end{array} \qquad R \tag{82}$$

Of course all of this is made less clear by the supersymmetric formulation.

The correct way to deal with all of this is to consider the space $\mathcal{A}$ of connections on some bundle, etc, and to take into account constant gauge transformations. The BPS equations are $F=0$ and σ constant. The end result is

$$Z_{S^3} = \int_{\mathfrak{g}} \mathrm{d}\sigma \, e^{-\# \operatorname{Tr} \sigma^2} \operatorname{sdet}_{\Omega^{0,0}_H(S^3,\mathfrak{g})} \big(\mathcal{L}_v + \mathrm{ad}_\sigma\big) = \int_{\mathfrak{t}} \mathrm{d}\sigma \, e^{-\# \operatorname{Tr}(\sigma^2)} \prod_{\text{root } \beta \neq 0} \sin\big(i\langle \beta, \sigma\rangle\big). \tag{83}$$

In even dimensions the problem is elliptic and lifting it one dimension up gives a transversally elliptic problem.

2d	3d	4d	5d	6d	7d
$\mathcal{N}=2$	$\mathcal{N}=2$	$\mathcal{N}=2$	$\mathcal{N}=1$		
$F=0$	$F_H=0$	$F^+=0$	$F^+_H=0$		

4.1 Contact geometry

Original idea from 1997 (B., Losev, Nekrasov). The toy model for our situation is as follows. Given a M_4 with the equation $F^+=0$ on it, we can consider $S^1 \times M_4$ with $v = \partial_t$ and the equations $i\partial_t F = 0$ and $F^+_H = 0$.

Crash course on contact geometry A $2n-1$ dimensional manifold M is a contact manifold if there exists a one-form κ such that $\kappa \wedge (\mathrm{d}\kappa)^{n-1}$ is nonzero. Then there exists a unique Reeb vector field v (also denoted R) such that $i_v \kappa = 1$ and $i_v \mathrm{d}\kappa = 0$. There also exists some metric g such that $g(v,\ ) = \kappa$. [Then on the contact plane (orthogonal to v) such that $\mathrm{d}\kappa$ is a symplectic form? and there exists an almost complex structure on that plane?]

Every orientable 3d manifold is a contact manifold. In 5d most examples are contact manifolds.

Given a contact structure we can split forms into vertical and horizontal, with Ω^p_V obtained by projecting using $\kappa \wedge i_v$ and Ω^p_H using $1 - \kappa \wedge i_v$.

Exercise: the two spaces are orthogonal. (Hint: $i_v \star \sim \star(\kappa \wedge\ )$.)

In 5d we can split

$$\Omega^2(M_5) = \Omega^2_V + \Omega^2_H = \Omega^2_V + \Omega^{2+}_H + \Omega^{2-}_H \,. \tag{84}$$

Exercise: show that these spaces are orthogonal (the projectors are $\frac{1}{2}(1 + i_V \star)$ and $\frac{1}{2}(1 - i_V \star)$). The problem we are interested in is not at all elliptic because there are not the right number of equations (but it can be embedded in an elliptic problem):

$$F^+_H = 0 \quad \text{and} \quad F_V = 0 \quad \Longleftrightarrow \quad \star F = -\kappa \wedge F \quad \text{(contact instanton)}. \tag{85}$$

The choice of sign in this last equation is important; supersymmetry chooses that sign; with that sign the Yang–Mills equation is automatically satisfied.

Symplectization The cone $M_{2n-1} \times \mathbb{R}_+$ where we denote the coordinate on $\mathbb{R}_+$ as r is the symplectization. The symplectic form used is $\omega = \mathrm{d}(r^2 k)$; the metric is $g_{\text{cone}} = \mathrm{d}r^2 + r^2 g_{M_{2n-1}}$.

- If the cone is Kähler then M_{2n-1} is Sasaki.
- If the cone is Calabi–Yau then M_{2n-1} is Sasaki–Einstein.

Of course these also have intrinsic definitions. Killing spinors on the base of the cone is equivalent to covariantly constant spinors on the cone. On Calabi–Yau manifolds there are such covariantly constant spinors, so all SYM is defined on all Sasaki–Einstein manifolds. Thus we have millions of (toric) examples.

Example: the cone over S^5 is $\mathbb{C}^3$; there are many covariantly constant spinors.

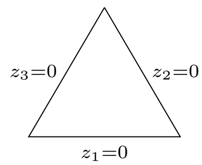
Example: the conifold $\mathbb{C}^4//(1,1,-1,-1)$ is a cone with base called $T^{1,1}$, topologically $S^2 \times S^3$.

Example: $Y^{p,q}$ (again topologically $S^2 \times S^3$) is the base of $\mathbb{C}^4//(p-q, p+q, -p, -p)$ for coprime $p, q > 0$.

Round sphere Consider $S^5\colon \{|z_1|^2 + |z_2|^2 + |z_3|^2 = 1\}$ and the T^3 action $z_i \to e^{i\alpha_i} z_i$. Denote e_i the action on z_i.

- $v = e_1 + e_2 + e_3$ gives the Hopf fibration. Exercise: write κ.

- For generic $\omega_i \in \mathbb{R}_+$ choose $v = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3$. Exercise: compute κ. We can picture S^5 as follows:



(86)

  where each corner is an S^1, each point on the sides is T^2 and each point in the bulk is T^3.

  Conjecture: $\star F = -\kappa F$ has no smooth solution and only has singular solutions on the closed orbits of v, which are three S^1 at the three corners of the triangle.

Weinstein conjecture: prove that every contact manifold has at least one closed orbit.

4.2 Super Yang–Mills on Sasaki–Einstein manifolds

On a Sasaki–Einstein manifold consider

$$\begin{gathered} \delta A = \Psi, \qquad \delta\Psi = i_v F + i\mathrm{d}_A\sigma, \qquad \delta\sigma = -i\, i_v\Psi, \\ \delta\chi_H^+ = H_H^+, \qquad \delta H_H^+ = \mathcal{L}_v^A \chi_H^+ - i[\sigma, \chi_H^+] \end{gathered} \tag{87}$$

where A is a connection, Ψ is an odd 1-form, σ an even 0-form, χ_H^+ an odd horizontal self-dual 2-form, H_H^+ an even horizontal self-dual 2-form, all in the adjoint representation of the gauge group. This is in fact a $\mathcal{N} = 1$ vector multiplet expressed in a cohomological field theory form.

There are exactly the right number of fields, for instance counting odd fields we find Ψ has 5 components and χ_H^+ being self-dual has 3 components, and $5 + 3 = 8$ is the number of odd components in a 5d $\mathcal{N} = 1$ vector multiplet. Supersymmetry is more clever than us: if we had tried to put more fields by hand we would get badly behaved problems.

Of course we are missing the ghosts $(c, \bar{c}, b)$ namely a ghost, anti-ghost, and Lagrangian multiplier. Then

$$\begin{tikzcd} \Omega^0 & \Omega^1 & \Omega^2_H & \Omega_0 \\ {[c]} \arrow[r, "\mathrm{d}"'] & {[A]} \arrow[r, "\mathrm{d}_H^+ \oplus \mathrm{d}^\dagger i_v"'] & {[\chi_H^+]} \otimes & {[\bar{c}]} \\ {[\sigma]} \arrow[r] \arrow[u] & {[\psi]} \arrow[r] \arrow[u] & {[H]} \otimes \arrow[u, bend left] & {[b]} \arrow[u, bend right] \end{tikzcd} \tag{88}$$

Then we find $A = 0$ and the determinant we want is

$$\frac{\det^{1/2}_{\Omega_H^{2+}}(\mathcal{L}_v + \mathrm{ad}_\sigma)\,\det_{\Omega^0}(\mathcal{L}_v + \mathrm{ad}_\sigma)}{\det^{1/2}_{\Omega^1}(\mathcal{L}_v + \mathrm{ad}_\sigma)} \tag{89}$$

4.2.1 Localization

Write

$$\delta W = \delta\big(\Psi \wedge \star\overline{\delta\Psi} + \chi_H^+ \wedge \star(H_H^+ - F_H^+)\big) = F_V \wedge \star F_V + F_H^+ \wedge \star F_H^+ + \mathrm{d}_A\sigma \wedge \star \mathrm{d}_A\sigma. \tag{90}$$

Note that $F_V \wedge \star F_V + F_H^+ \wedge \star F_H^+ = F \wedge \star F + \kappa \wedge F \wedge F$. The equations are written either as

$$F_V = 0, \qquad F_H^+ = 0, \qquad \mathrm{d}_A\sigma = 0 \tag{91}$$

or equivalently as

$$\star F = -\kappa \wedge F, \qquad \mathrm{d}_A\sigma = 0. \tag{92}$$

The solutions are $F = 0$ (hence $A = 0$ up to gauge transformation) and σ constant, except for singular solutions of $\star F = -\kappa \wedge F$.

We decompose using the Kähler form ω and using v:

$$\Omega_H^{2+} = \Omega_H^{2,0} + \Omega_H^{0,2} + \Omega^0\omega \tag{93}$$

$$\Omega^1 = \Omega_H^{1,0} + \Omega_H^{0,1} + \Omega^0 v \tag{94}$$

Split determinants according to this decomposition, and ignore phases (somehow in 5d they don't matter) so that $\Omega_H^{2,0}$ and $\Omega_H^{0,2}$ contribute the same. The determinant is then

$$\frac{\det_{\Omega_H^{0,2}}(\mathcal{L}_v + \mathrm{ad}_\sigma)\det_{\Omega_0}(\mathcal{L}_v + \mathrm{ad}_\sigma)}{\det_{\Omega_H^{0,1}}(\mathcal{L}_v + \mathrm{ad}_\sigma)} = \mathrm{sdet}_{\Omega_H^{0,\bullet}}(\mathcal{L}_v + \mathrm{ad}_\sigma). \tag{95}$$

We have the complex

$$\Omega_H^{0,0} \xrightarrow{\overline{\partial}_H} \Omega_H^{0,1} \xrightarrow{\overline{\partial}_H} \Omega_H^{0,2}. \tag{96}$$

Just like in 3d,

$$\Omega_H^{0,p}(S^5) = \oplus_n \Omega^{0,p}\big(\mathbb{CP}^2, \mathcal{O}(n)\big) \tag{97}$$

and we can derive that $\mathrm{ind}\big(\overline{\partial}, \mathcal{O}(n)\big) = 1 + \frac{3}{2}n + \frac{1}{2}n^2$. Thus the super determinant reduces to

$$\prod_{n\neq 0}\big(2\pi i n + \mathrm{ad}_\sigma\big)^{1+\frac{3}{2}n+\frac{1}{2}n^2} \tag{98}$$

for the round sphere. This is a triple-sine function at special values.

4.2.2 Final result

Consider the squashed sphere, obtained using generic $v = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3$. Then

$$Z^{\text{perturbative}}_{S^5_{\omega_1,\omega_2,\omega_3}} = \int_{\mathfrak{t}} \mathrm{d}\sigma\, e^{-\#\,\mathrm{Tr}(\sigma^2)} \prod_{\beta\neq 0} S_3\big(i\langle\sigma,\beta\rangle \bigm| \omega_1,\omega_2,\omega_3\big) \tag{99}$$

for some coefficient # (see the original references).

There is a nice pattern:

- S^1 gives sin functions, which is periodic;
- S^3 gives double-sine function $S_2(x|\omega_1, \omega_2)$, which is periodic up to sines;
- S^5 gives triple-sine functions $S_3(x|\omega_1, \omega_2, \omega_3)$, which is periodic up to double-sines.

Explicitly,

$$S_3(x|\vec{\omega}) = \prod_{n_1,n_2,n_3\geq 0} (x + \vec{n}\cdot\vec{\omega}) \prod_{n_1,n_2,n_3\geq 1} (-x + \vec{n}\cdot\vec{\omega}) \qquad \text{regularized} \tag{100}$$

$$\begin{aligned} &= e^{\cdots}\left(e^{2\pi i x/\omega_1} \,\middle|\, e^{2\pi i\omega_2/\omega_1}, e^{2\pi i\omega_3/\omega_1}\right)_\infty \\ &\left(e^{2\pi i x/\omega_2} \,\middle|\, e^{2\pi i\omega_3/\omega_2}, e^{2\pi i\omega_1/\omega_2}\right)_\infty \left(e^{2\pi i x/\omega_3} \,\middle|\, e^{2\pi i\omega_1/\omega_3}, e^{2\pi i\omega_2/\omega_3}\right)_\infty \end{aligned} \tag{101}$$

where

$$(z|q_1, q_2)_\infty = \prod_{n,m\geq 0} \left(1 - z q_1^n q_2^m\right) \tag{102}$$

for $|q_1| < 1$ and $|q_2| < 1$ and otherwise the function is defined by analytic continuation.

Each of the three factors in (101) is the perturbative part of the 5d $\mathbb{R}^4 \times S^1$ Nekrasov partition function. In the first factor for instance, the S^1 has radius $1/\omega_1$ and epsilon parameters are ω_2/ω_1 and ω_3/ω_1. This leads to a conjecture:

$$\begin{aligned} Z^{\text{perturbative}}_{S^5_{\omega_1,\omega_2,\omega_3}} = \int_{\mathfrak{t}} \mathrm{d}\sigma\, e^{\cdots} Z^{\text{Nekrasov}}_{\mathbb{R}^4\times S^1}\left(\frac{\omega_2}{\omega_1}, \frac{\omega_3}{\omega_1}, \frac{1}{\omega_1}\right) \\ Z^{\text{Nekrasov}}_{\mathbb{R}^4\times S^1}\left(\frac{\omega_3}{\omega_2}, \frac{\omega_1}{\omega_2}, \frac{1}{\omega_2}\right) Z^{\text{Nekrasov}}_{\mathbb{R}^4\times S^1}\left(\frac{\omega_1}{\omega_3}, \frac{\omega_2}{\omega_3}, \frac{1}{\omega_3}\right). \end{aligned} \tag{103}$$

This is a *conjecture*, but it has passed many tests.

Actually the problem is also there in Pestun's S^4 result: what does it even mean to say that we get localization to singular solution? This is very different from the non-compact setting (Omega background) where the moduli space of instantons is under control, and the non-commutative deformation is well-defined.

Question: why do we stop at 7d? The main problem is that we cannot put SYM on S^8 while preserving supersymmetry and isometry. Indeed, the bosonic group would have $\mathfrak{so}(9)$ isometry, and that is the same as asking for a 7d SCFT.