Instantons

Lectures by Nikita Nekrasov notes by Bruno Le Floch

July 23–24, 2018

Nobody (even the typist) proof-read these notes, so there may be obvious mistakes: tell BLF.

Abstract

Instantons. These are lecture notes for the 2018 IHÉS summer school on *Supersymmetric localization and exact results.*

1 Lecture 1, July 23

We are interested in path integrals in Euclidean QFT. Space-time is a Riemannian manifold but the space of fields may be complexified.

1.1 Finite-dimensional analogy

In finite-dimensional integrals of analytic functions it is sometimes convenient to deform the contour in the complexified space. Let us say we want to compute

$$\int_X e^{-S/\hbar}\Omega \tag{1}$$

over some manifold X with volume form Ω, where $S\colon X \to \mathbb{R}$ is some function. The strategy is to embed X into a complexification $X^{\mathbb{C}}$ such that the volume form extends to a holomorphic volume form $\Omega_{\mathbb{C}}$ on $X^{\mathbb{C}}$, and S extends to a holomorphic function $S\colon X^{\mathbb{C}} \to \mathbb{C}$. Then X is a middle-dimensional contour inside $X^{\mathbb{C}}$ and we can deform it to a middle-dimensional contour Γ. The integral has better converge. Let

$$X_\infty = \left\{ x \mid \Re\left(\frac{S}{\hbar}\right)^{-1} > c \right\} \tag{2}$$

for some constant c very large. This defines some sectors in $X^{\mathbb{C}}$, and the contour must go to one of these sectors at infinty. What matters for the value of the integral is the homology class

$$[\Gamma] \in H_n(X^{\mathbb{C}}, X_\infty) \tag{3}$$

of Γ in the relative homology (abelian) group. Here $n = \dim X$.

Then

$$Z_\Gamma(\hbar) = \int_\Gamma e^{-S/\hbar}\Omega \tag{4}$$

obeys a fundamental system of Picard–Fuchs equations. The question then boils down to writing X in a nice basis of contours.

1.1.1 Basis of Lefschetz thimbles

In the $\hbar \to 0$ limit the integrals are dominated by the critical points p of S. For each such point p we can construct a contour L_p "Lefschetz thimble" as follows. These contours define a basis of $H_n(X^{\mathbb{C}}, X_\infty)$. Then the (relative) homology class of any contour can be written as

$$[\Gamma] = \sum_p n_p L_p. \tag{5}$$

How to build L_p?

Pick a generic Hermitian (i.e. compatible with complex structure) metric h on $X^{\mathbb{C}}$. Look at the gradient flow of $\Re(S/\hbar)$, namely

$$\dot{x} = \nabla^h \Re(S/\hbar). \tag{6}$$

Near p, expand $S \simeq S(p) + \sum_{i=1}^n \frac{1}{2} z_i^2$. Let $\hbar$ be real to make our lives simpler. Then

$$\Re(S/\hbar) = \Re(S(p)/\hbar) + \frac{1}{\hbar}\sum_{i=1}^n (x_i^2 - y_i^2) \tag{7}$$

so half of the eigenvalues are positive and the other half negative. The Hermitian metric is $h = \sum_i (\mathrm{d}x_i^2 + \mathrm{d}y_i^2)$. Then the gradient flow gives

$$\dot{x}_i = \frac{1}{\hbar} x_i, \qquad \dot{y}_i = -\frac{1}{\hbar} y_i. \tag{8}$$

We then define the contour L_p as the union of outgoing trajectories "x-lines".

Exercise: study the Airy function

$$A_\hbar(x) = \int \mathrm{d}t e^{(i/\hbar)(tx - t^3/3)}, \tag{9}$$

find its two critical points and its Lefschetz thimbles. The Airy function obeys a second-order differential equation.

Interestingly, $\Im(S/\hbar)$ is constant along the flow. Thus for generic (complex) $\hbar$, trajectories coming from a critical point do not intersect those coming from another critical point. There is an interesting story of when the trajectories do intersect, and a story of phase transition when $\hbar$ is varied.

1.2 Non-relativistic particle in a double-well potential

We consider the simplest infinite-dimensional case: path integral in quantum mechanics. Specifically we take a potential

$$U(x) = \frac{\lambda}{4}(x^2 - v^2)^2. \tag{10}$$

Classically we have motion around each minimum.

The Hamiltonian is of course

$$\widehat{H} = -\frac{\hbar^2}{2}\partial_x^2 + U(x). \tag{11}$$

It acts on $\mathcal{H} = L^2(\mathbb{R})$. Quantum mechanically we want

$$\widehat{H}\psi = E\psi. \tag{12}$$

We wish to analyse Low-energy states, namely those with "small" E.

Consider the Euclidean trace

$$Z_\hbar(T) = \mathrm{Tr}_{\mathcal{H}}\, e^{-\frac{T\widehat{H}}{\hbar}} \tag{13}$$

in the regime $\Re(T/\hbar) \gg 0$. Let us use a slightly unusual method of writing it as a phase-space path integral

$$Z_\hbar(T) = \int_{(p,x)\colon\, S^1\to\mathbb{R}^2} \mathcal{D}p\mathcal{D}x\, \exp\left[\frac{i}{\hbar}\int p\mathrm{d}x - \frac{T}{\hbar}\int H(p,x)\mathrm{d}t\right] \tag{14}$$

where $0 \leq t \leq 1$. We writ the exponent as $-S/\hbar$ with

$$S = -i\oint p\mathrm{d}x + T\oint H(p,x)\mathrm{d}t \tag{15}$$

complex.

Then we treat $\{(p,x)\colon S^1 \to \mathbb{R}^2\}$ as a middle-dimensional contour in the loop space $L\mathbb{C}^2$ of $\mathbb{C}^2$,

$$L\mathbb{C}^2 = \{(p,x)\colon S^1 \to \mathbb{C}^2\} = \Big\{(p,x)\;\Big|\; p(t), x(t) \in \mathbb{C},\ p(t{+}1) = p(t),\ x(t{+}1) = x(t)\Big\}. \tag{16}$$

We stress that we don't change the nature of time, t is not made complex (well, we have Wick rotated but that's a completely unrelated step). Critical points of the action obey

$$-i\dot{x} + Tp = 0 \tag{17}$$

$$+i\dot{p} + TU'(x) = 0. \tag{18}$$

In a usual treatment, p is already integrated out and we just find that x moves in an inverted potential. Here we see more: even for real x, p needs to be complex.

We need to find periodic solutions. Since $\partial_t H(p,x) = 0$, the trajectory (p,x) sits on a complex curve $H(p,x) = \mathcal{E}$ for some $\mathcal{E}$. For the given potential that curve is

$$\frac{p^2}{2} + \frac{\lambda}{4}(x^2 - v^2)^2 = \mathcal{E}, \tag{19}$$

which is an elliptic curve[1]. After a change of variables we get $y^2 = 4\widetilde{x}^3 - g_2\widetilde{x} - g_3$ for some parameters g_2 and g_3. In these coordinates the elliptic curve is a branched double-cover of the x complex plane (each sheet is one choice of sign for y) with two branch cuts. The two noncontractible cycles of the torus correspond to: a contour going around one branch cut, and a contour going through both branch cuts. The uniformizing coordinate $z = \int \mathrm{d}\widetilde{x}/y = \int \mathrm{d}x/p$ is such that $z \sim z + n_1\omega_1 + n_2\omega_2$ with ω_1 and ω_2 the periods of the torus.

In this coordinate, we want to solve $z(t) = z(0) + V_0(\mathcal{E})t$ for some constant $V_0(\mathcal{E})$ that should be $V_0 = n\omega_1(\mathcal{E}) + m\omega_2(\mathcal{E})$ for some $n, m \in \mathbb{Z}$. Solutions at a given energy will be characterized by the integers m, n. More precisely this depends on a choice of basis in the torus (what we call A-cycle and B-cycle), which can be made locally continuously in $\mathcal{E}$.

Then we have to tune the energy $\mathcal{E}$ so that

$$n \oint_{\text{A-cycle}} \frac{\mathrm{d}x}{p} + m \oint_{\text{B-cycle}} \frac{\mathrm{d}x}{p} = \frac{T}{i}. \tag{20}$$

We get infinitely many solutions depending on m, n. In 1969, Bender and Wu studied some other potential.

At low temperatures $T \to \infty$, one can safely expect that $\mathcal{E} \to 0$, which makes the elliptic curve nearly degenerate. Then

$$n + \frac{m}{\pi i}\log(\mathcal{E}/\mathcal{E}_0) \sim T/(iT_0) \tag{21}$$

with $\mathcal{E}_0 \sim \lambda v^4/4$ and $T_0 \sim 1/\sqrt{U''(v)}$. Then

$$\mathcal{E}_{n,m} \sim e^{-\pi T/(mT_0)} e^{2\pi i n/(2m)}. \tag{22}$$

This means that the trajectory passes many times through the pinched points in the nearly degenerate elliptic curve.

For most of the t-time the trajectory solves

$$\frac{i\dot{x}}{T} = p = \pm\sqrt{2U} \tag{23}$$

where the sign flips whenever the trajectory passes through the pinched point, so $2m$ times, where m is the number of times the B-cycle occurs.

As $\mathcal{E}$ goes around 0, the distance $\sqrt{\mathcal{E}}$ between branch points in the x plane picks up a sign, so the B-cycle is dressed by two A-cycles (two because there are two pairs of branch points).

[1] An elliptic curve is a complex one-dimensional manifold that is homeomorphic to the real two-torus. This page footer is too short to say everything about these beautiful objects.

The building blocks, solutions of $i\dot{x}/T = p = \sqrt{2U}$ are instantons, and $i\dot{x}/T = p = -\sqrt{2U}$ are anti-instantons. We are looking at m pairs of instantons and anti-instantons plus n perturbative fluctuations (to be glued on top of pairs of instanton and anti-instanton).

1.3 Many-body systems (algebraic integrable systems)

We shall look at the complexified phase-space of dimension $2n$. What coordinates do we have? The usual coordinates and momenta, but also the action-angle variables: the symplectic form is then

$$\sum_{a=1}^{n} \mathrm{d}p_a \wedge \mathrm{d}x^a = \sum_{a=1}^{n} \mathrm{d}a_a \wedge \mathrm{d}\varphi^a. \tag{24}$$

Here, $\varphi^a \sim \varphi^a + 2\pi(n^a + \tau^{ab}_{(\mathrm{a})} m_b)$ with $n^a \in \mathbb{Z}$ and $m_b \in \mathbb{Z}$.

The partitiono function is

$$Z_\hbar(T_k) = \int \mathcal{D}p\mathcal{D}x \, \exp\left[\frac{i}{\hbar}\int \sum_a p_a \mathrm{d}x^a - \sum_k \frac{T_k}{\hbar}\int H_k(p,x)\mathrm{d}t\right] \tag{25}$$

where

$$0 = \mathrm{d}\left(\sum_{a=1}^{n} n^a a_a + m_a a_D^a - \sum_{k=1}^{n} T_k H_k(a)\right). \tag{26}$$

1.4 Supersymmetric quantum mechanics

In previous stories we could not separate instantons and antiinstantons. Supersymmetry is when you can separate them.

On phase space we have coordinates (x,p) bosonic and $(\psi,\overline{\psi})$ fermionic. Then one supercharge does

$$\delta x = \psi, \quad \delta\overline{\psi} = p, \quad \delta\psi = 0, \quad \delta p = 0. \tag{27}$$

It obeys $\delta^2 = 0$. We ignore the other supercharge.

Consider V a Morse function on $\mathbb{R}$ where $V' = 0$ and $V'' \neq 0$ (at what point?). We consider a δ-exact action:

$$S = i\delta \int \overline{\psi}\left(\dot{x} + \frac{i}{2}p - V'(x)\right) \tag{28}$$

$$= i\int p\dot{x} - \int \frac{p^2}{2} - i\int pV' + \text{fermionic terms.} \tag{29}$$

On-shell, $p = i(\dot{x} - V')$. When $\delta = 0$ you get instantons, namely gradient trajectories $\dot{x} = V'$.

Now introduce a metric and write a new action

$$S = i\delta \int \overline{\psi}_a\left(\dot{x}^a + \frac{i}{2}g^{ab}(p_b - \gamma^c{}_{bd}\psi^d\overline{\psi}_c) - g^{ab}\partial_b v\right) \tag{30}$$

As in Zabzine's lecture, even though p looks like it transforms like a section it doesn't:

$$\delta\left(\overline{\psi}\frac{\partial x}{\partial \widetilde{x}}\right) = p\frac{\partial x}{\partial \widetilde{x}} + \overline{\psi}\frac{\partial^2 x}{\partial \widetilde{x}\partial \widetilde{x}}\psi\frac{\partial x}{\partial \widetilde{x}} \tag{31}$$

To make the action coordinate-invariant we need to add the extra term with ${\gamma^c}_{bd}$.

In the large-volume limit $g^{ab} \to 0$ and $V \to \infty$ such that $V^a = g^{ab}\partial_b V$ is finite. This retrieves Morse theory in the way Morse conceived it.

1.5 Infinite-dimensional case

We will be interested in 2d sigma models and 4d gauge theories, sometimes combined.

1.5.1 Sigma models

Let Σ be a Riemann surface and X be some Riemannian manifold with metric g and with a B-field. We write a sigma model with fields $\Sigma \to X$, with action

$$S = \frac{1}{2}\int_\Sigma g_{mn}\mathrm{d}x^m \wedge \star\mathrm{d}x^n + \frac{i}{2}\int_\Sigma x^* B \tag{32}$$

where x^*B is the pull-back of the two-form to Σ. This is the bosonic part of a supersymmetric system if

- X is Kähler and B is the symplectic form;
- or if X has an almost complex structure and the metric and B-field are related by this complex structure as $g_{mn} = B_{mk}J^k_n$.

In those cases, by the Bogomolny trick the action can be written as

$$S = \left\| {\left(\frac{1-iJ}{2}\right)^n}_m \overline{\partial} x^m \right\|^2 + i\int x^*\omega \tag{33}$$

If $\omega = B + ig\cdot J$ happens to be closed then this action is only sensitive to the topology of the space.

Instantons are pseudo-holomorphic maps, namely solutions to

$${(1-iJ)^n}_m \overline{\partial} x^m = 0. \tag{34}$$

1.5.2 Yang–Mills theory in 4d

We'll take $G = SU(N)$, the trace is taken in the fundamental representation and the Hodge star $\star^2 = +1$ on two-forms.

$$S_{\mathrm{YM}} = \frac{1}{4g^2}\int_{M^4} \operatorname{Tr} F_A \wedge \star F_A + \frac{i\theta}{8\pi^2}\int_{M^4} \operatorname{Tr} F_A \wedge F_A \tag{35}$$

$$= \|F_A^+\|^2 + 2\pi i\tau \int \frac{\operatorname{Tr} F_A \wedge F_A}{2(2\pi i)^2} \tag{36}$$

$$= \|F_A^-\|^2 + 2\pi i\bar{\tau} \int \frac{\operatorname{Tr} F_A \wedge F_A}{2(2\pi i)^2} \tag{37}$$

up to factors of 2 and similar. Here $F_A^+ = (F_A + \star F_A)/2$ and $\tau = \theta/(2\pi) + 4\pi i/g^2$.

Get instantons and antiinstantons in this way.

1.5.3 Combining

If we have an action of H on X we can make derivatives covariant in the sigma model:

$$S = \frac{1}{2}\int_\Sigma g_{mn} \nabla_A x^m \wedge \star \nabla_A x^n + \frac{i}{2}\int_\Sigma x^* B \tag{38}$$

where $\nabla_A x^m = \mathrm{d}x^m + A^a V_a^m$ and $V_a \in \mathrm{Vect}(X)$ is the vector field corresponding to the H action. The moment map is $\mathrm{d}\mu^a = i_{V_a} B$.

The Bogolmony trick gives

$$(1 - iJ)\nabla_A x = 0 \tag{39}$$

$$F_A + e^2 \mu \cdot \mathrm{vol}_\Sigma = 0 \tag{40}$$

As $e^2 \to \infty$, we find that $\mu = 0$ almost everywhere on Σ. Solutions are then basically

$$\phi\colon \Sigma \to \frac{\mu^{-1}(0)}{H} \tag{41}$$

which coincides with instantons in X/H. The slight difference is responsible for the difference between instantons in gauged linear sigma models and those in nonlinear sigma models. In mirror symmetry this shows up as a nontrivial mirror map.

2 Lecture 2

Today we'll consider instantons in gauge theory. Consider $M^3 \times \mathbb{R}^1_t$ with compact space M^3. Ansätze for $A = f(t)\Theta$ where Θ is a flat connection on M^3, namely

$$\mathrm{d}\Theta + \Theta \wedge \Theta = 0. \tag{42}$$

Then

$$F_A = \dot{f}\mathrm{d}t \wedge \Theta + (-f + f^2)\Theta \wedge \Theta. \tag{43}$$

The Yang–Mills action is

$$S_{\mathrm{YM}} = \int_{\mathbb{R}^1} \mathrm{d}t \Big[c_1 \dot{f}^2 + c_2 f^2 (f-1)^2 \Big]. \tag{44}$$

Note that Θ may be pure gauge but have non-zero

$$c_1 = \int_{M^3} \mathrm{Tr}(\Theta \wedge \star\Theta), \tag{45}$$

$$c_2 = \int_{M^3} \mathrm{Tr}\big(\Theta \wedge \Theta \wedge \star(\Theta \wedge \Theta)\big), \tag{46}$$

$$c_3 = \int_{M^3} \mathrm{Tr}\big(\Theta \wedge \Theta \wedge \Theta). \tag{47}$$

We are thus considering a particle in a quartic double-well potential, just like before. This is shifted slightly. The solution $f = 0$ corresponds to $A = 0$. The solution $f = 1$ corresponds to A flat. The solution with $\dot{f} \sim f(f-1)$ is an instanton in the quantum mechanics sense.

For instance for $M^3 = S^3$, it is a BPST instanton in radial coordinates (taking $G = SU(2)$). We mapped to $\mathbb{R}^4$ using

$$\mathrm{d}s_4^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\Omega_3^2 = r^2 \left(\frac{\mathrm{d}r^2}{r^2} + \mathrm{d}\Omega_3^2 \right) \tag{48}$$

with $r = e^t$ for $-\infty < t < \infty$. Unfortunately this method is difficult to generalize to higher instanton numbers.

2.1 All instanton solutions on $\mathbb{R}^4$

We embed $\mathbb{R}^4$ into its conformal compactification S^4. Fix $G = SU(N)$. The instanton charge

$$k = -\frac{1}{8\pi^2} \int \mathrm{Tr}\, F_A \wedge F_A \in \mathbb{Z} \tag{49}$$

is a topological number. We focus on $k \geq 0$. Solutions to $F_A^+ = 0$ are in one-to-one correspondence with solutions to algebraic equations on matrices: the ADHM construction. Their construction stemmed from the study of twistors.

2.1.1 Motivation

Fix a k-instanton configuration $F_A^+ = 0$ with finite action.

We can study the Dirac equation $\not{D}_A \psi = 0$ with $\psi \in L^2(S_\pm \otimes E)$, where E is a rank N complex vector bundle (on which $G = SU(N)$ acts) over $\mathbb{R}^4$ (we will always think of $\mathbb{R}^4$ as being compactified to S^4).

Identify $\mathbb{R}^4 \simeq \mathbb{C}^2$. Write the metric and symplectic form as

$$\mathrm{d}s^2 = \mathrm{d}z_1 \mathrm{d}\overline{z}_1 + \mathrm{d}z_2 \mathrm{d}\overline{z}_2, \qquad \omega = \frac{1}{2\sqrt{-1}} \big(\mathrm{d}z_1 \wedge \mathrm{d}\overline{z}_1 + \mathrm{d}z_2 \wedge \mathrm{d}\overline{z}_2 \big) \tag{50}$$

Then the spin bundles are $S_- \simeq \Omega^{0,1}$ (spanned by $\mathrm{d}\overline{z}_1$ and $\mathrm{d}\overline{z}_2$) and $S_+ \simeq \Omega^{0,0} \oplus \Omega^{0,2}$ (spanned by 1 and $\mathrm{d}\overline{z}_1 \wedge \mathrm{d}\overline{z}_2$). Then the Dirac operator is identified to $\not{D}_A \simeq \overline{\partial}_A \oplus \overline{\partial}_A^*$ where

$$\overline{\partial}_A \colon \Omega^{0,i} \to \Omega^{0,i+1}, \qquad \overline{\partial}_A^* \colon \Omega^{0,i} \to \Omega^{0,i-1}. \tag{51}$$

The equation $F_A^+ = 0$ is equivalent to $F_A^{0,2} = 0 = F_A^{1,1}\omega$. The relation $F_A^{0,2} = 0$ is equivalent to $\overline{\partial}_A^2 = 0$. The relation $F_A^{1,1}\omega = 0$ is equivalent to quotienting by complexified gauge transformations instead of real gauge transformations.

Altogether we are interested in solving $\overline{\partial}_A\widetilde{\psi} = 0$ for $\widetilde{\psi} \in L^2(\Omega^{0,i} \otimes E)$ up to $\widetilde{\psi} \sim \widetilde{\psi} + \overline{\partial}_A\chi$.

For S_+, $(\overline{\partial}_A \oplus \overline{\partial}_A^*)(\eta \oplus \chi) = 0 \in \Omega^{0,1} \otimes E$ with $\eta \in L^2(\Omega^{0,0} \otimes E)$ and $\chi \in L^2(\Omega^{0,2} \otimes E)$.

We reach the equation

$$\overline{\partial}_A\eta + \overline{\partial}_A^*\chi = 0 \tag{52}$$

Explilcitly

$$D_{\overline{1}}\eta + D_2\chi = 0 \tag{53}$$

$$D_{\overline{2}}\eta - D_1\chi = 0. \tag{54}$$

This impiles

$$\Big(\big\{D_1, D_{\overline{1}}\big\} + \big\{D_2, D_{\overline{2}}\big\}\Big)\eta = 0 = \Box_A\eta. \tag{55}$$

This implies $\eta = 0$. In other words there are no zero-modes of that chirality. Here we used $[D_1, D_2] = 0$.

Then an index theorem (see Maxim Zabzine's lectures)

$$\dim \underbrace{\ker_{L^2(S_-\oplus E)} \not{D}_A}_{\text{call it } K} - \dim \ker_{L^2(S_+\oplus E)} \not{D}_A = k. \tag{56}$$

Then $\psi \in K$ is equivalent to $\psi \in L^2(\Omega^{0,1} \otimes E)$ such that $\overline{\partial}_A\psi = 0$ and $\overline{\partial}_A^*\psi = 0$. Then multiply this by the coordinates. While the result is not a solution of the Dirac equation, we can project it onto that space.

More precisely, one can decompose orthogonally

$$\mathcal{H} = L^2(\Omega^{0,1} \otimes E) = \ker \not{D}_A \oplus \mathcal{H}^\perp \tag{57}$$

where $\not{D}_A^*\not{D}_A = \Delta_A > 0$ on $\mathcal{H}^\perp$. Then define a projector

$$\Pi = 1 - \not{D}_A^* \frac{1}{\Delta_A} \not{D}_A. \tag{58}$$

Then we can define four complex operators on K by

$$B_1\psi = \Pi(z_1\psi) \in K, \tag{59}$$

$$B_2\psi = \Pi(z_2\psi) \in K, \tag{60}$$

$$B_1^\dagger\psi = \Pi(\overline{z}_1\psi) \in K, \tag{61}$$

$$B_2^\dagger\psi = \Pi(\overline{z}_2\psi) \in K \tag{62}$$

so we have an action of four matrices on K.

Now consider $r^2 \to \infty$. There, A becomes pure gauge so ψ should approach the solutions of the flat Dirac equation. In coordinates $\psi = \psi_{\overline{\alpha}} \mathrm{d}\overline{z}^{\overline{\alpha}}$ we write the coordinates

$$D_{\overline{1}}\psi_{\overline{2}} - D_{\overline{2}}\psi_{\overline{1}} = 0 \tag{63}$$

$$D_1\psi_{\overline{1}} + D_2\psi_{\overline{2}} = 0. \tag{64}$$

Locally we can "solve" the first equation by $\psi_{\overline{\alpha}} = D_{\overline{\alpha}}\chi$ and the second equation says that χ should be harmonic at $r \to \infty$. The simplest nontrivial option is $1/r^2 = 1/(|z_1|^2 + |z_2|^2)$. Then

$$\psi_{\overline{\alpha}} r \overset{\sim}{\to} \infty D_{\overline{\alpha}}\left(\frac{1}{r^2} I^\dagger\right) - \epsilon_{\overline{\alpha}\overline{\beta}} g^{\gamma\overline{\beta}} D_\gamma \left(\frac{1}{r^2} J\right). \tag{65}$$

This is parametrized by $I\colon N \to K$ and $J\colon K \to N$, where $N = \mathbb{C}^N$ is the fiber of the vector bundle, while K is the space of solutions to the Dirac equation.

We thus have described ψ by the matrices $I\colon N \to K$, $J\colon K \to N$, $B_1, B_2\colon K \to K$.

While multiplication by coordinates z_1, z_2, $\overline{z}_1$, $\overline{z}_2$ commutes, after the projection by Π there is a noncommutativity. A tedious calculation shows

$$[B_1, B_2] + IJ = 0, \tag{66}$$

$$[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = 0. \tag{67}$$

Of course, I, J, B_1, B_2 are only defined up to a $U(K)$ change of basis.

We have shown how to go from an instanton to a collection of matrices. The ADHM construction consists of the opposite construction.

2.1.2 ADHM construction itself

Given (B_1, B_2, I, J), we want to construct A (and ψ).

Let $\mathcal{D}^\dagger\colon K \otimes \mathbb{C}^2 \oplus N \to K \otimes \mathbb{C}^2$ with

$$\mathcal{D}^\dagger = \begin{pmatrix} B_1 - z_1 & B_2 - z_2 & I \\ -B_2^\dagger + \overline{z}_2 & B_1^\dagger - \overline{z}_1 & -J^\dagger \end{pmatrix}, \tag{68}$$

which depends on $(z_1, z_2) \in \mathbb{C}^2$. Then an explicit calculation shows $\mathcal{D}^\dagger\mathcal{D}\colon K \otimes \mathbb{C}^2 \to K \otimes \mathbb{C}^2$ is

$$\mathcal{D}^\dagger\mathcal{D} = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}, \tag{69}$$

where $\Delta\colon K \to K$. For good (B_1, B_2, I, J) we have $\Delta > 0$.

Then let $E = \ker \mathcal{D}^\dagger$ and span the kernel using Ξ, namely $\mathcal{D}^\dagger\Xi = 0$ with $\Xi\colon N \to K \otimes \mathbb{C}^2 \oplus N$, where $\Xi = (\nu_+, \nu_-, \xi)$ with $\nu_\pm\colon N \to K$ and $\xi\colon N \to N$. Then $A = \Xi^\dagger \mathrm{d}\Xi$ solves $F_A^+ = 0$, and $\psi = (\nu_+^\dagger \Delta^{-1} \mathrm{d}\overline{z}_1 + \nu_-^\dagger \Delta^{-1} \mathrm{d}\overline{z}_2)$ obeys $\not{D}_A \psi = 0$.

The two constructions are inverses of each other.

By the way, the moduli space of instantons has real dimension $4Nk$.

2.2 Computing 4d $\mathcal{N}=2$ super Yang–Mills theory on $\mathbb{R}^4$

The field content is $(A_m, \sigma, \overline{\sigma})$ and the fermions $(\psi_m, \eta, \chi^+_{mn})$ are a one-form, scalar and self-dual two-form. We twisted the theory (since we are in flat space this is"just using a more mathematical language"). Supersymmetry is

$$\delta A_m = \psi_m, \qquad \delta\psi_m = D_m\sigma, \qquad \delta\sigma = 0, \tag{70}$$

$$\delta\overline{\sigma} = \eta, \qquad \delta\eta = [\sigma, \overline{\sigma}], \tag{71}$$

$$\delta\chi^+_{mn} = H^+_{mn}, \qquad \delta H^+_{mn} = [\sigma, \chi^+_{mn}]. \tag{72}$$

The usual super Yang–Mills action is

$$S_{\mathrm{SYM}} = \tau \int \operatorname{Tr} F_A \wedge F_A + \delta \int \operatorname{Tr}\Big(\chi^+(F^+ - g^2 H^+) + \psi\wedge\star D_A\overline{\sigma} + \eta\wedge\star[\sigma,\overline{\sigma}]\Big) \tag{73}$$

with the usual $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$.

Now expand the theory around a generic point in the Coulomb branch. The vev $\langle\sigma\rangle = \operatorname{diag}(a_1, \ldots, a_N)$ breaks $G = SU(N)$ to $U(1)^{N-1}$. The effective action, by supersymmetry, looks like (where i, j are indices of the unbroken $U(1)$ factors)

$$S_{\mathrm{eff}} = \int \tau_{ij}(a) F^{(i)} \wedge F^{(j)} + \frac{1}{3}(\partial_{a_k}\tau_{ij})\psi^{(k)}\wedge\psi^{(i)}\wedge F^{(j)} + \frac{1}{4}\partial_{a_k a_l}\tau_{ij}\psi^{(i)}\psi^{(j)}\psi^{(k)}\psi^{(l)} \tag{74}$$

and supersymmetry also forces $\tau_{ij} = \partial^2\mathcal{F}/\partial a_i \partial a_j$. Our goal is to find out $\mathcal{F}$.

We can split $\sigma = a + \sigma^{(0)}$ where $\sigma^{(0)}$ vanishes at infinity. Likewise we split gauge transformations into constant ones and those that vanish at infinity.

We use both these constant gauge transformations, and the $\mathrm{Spin}(4) = SU(2)\times SU(2)$ rotations of $\mathbb{R}^4$ to deform δ to

$$\delta_\epsilon A = \psi, \qquad \delta_\epsilon\psi = D_A\sigma + i_{V_\epsilon}F_A, \qquad \delta_\epsilon\sigma = i_{V_\epsilon}\psi. \tag{75}$$

Here

$$V_\epsilon = i\epsilon_1\left(z_1\frac{\partial}{\partial z_1} - \overline{z}_1\frac{\partial}{\partial\overline{z}_1}\right) + i\epsilon_2\left(z_2\frac{\partial}{\partial z_2} - \overline{z}_2\frac{\partial}{\partial\overline{z}_2}\right) \tag{76}$$

(Even though for actual rotations of space-time $\epsilon_1, \epsilon_2 \in \mathbb{R}$, the algebra allows complex ϵ_1 and ϵ_2.) Another comment is that rotations of $\mathbb{R}^4$ act on ADHM data by

$$(B_1, B_2, I, J) \to (e^{i\epsilon_1}B_1, e^{i\epsilon_2}B_2, I, e^{i(\epsilon_1+\epsilon_2)}J). \tag{77}$$

One checks that $\delta_\epsilon^2 = 0$ up to rotations and gauge and global symmetries. Note that the nonzero $\delta_\epsilon\sigma$ means that we can no longer insert $f(\sigma)$ everywhere in $\mathbb{R}^4$ so it seems the cohomology has shrunk. However, the position of insertion did not affect the cohomology classes so we could bring any insertion to the origin anyways. Now they are simply stuck there.

Beyond just writing the supersymmetry algebra, we would need to write down how the action should be modified to be δ_ϵ-invariant. The details are not so interesting for us now. In the limit $\epsilon_1, \epsilon_2 \to 0$ the deformation goes to zero.

Now we integrate out everything. V_ϵ is a Killing vector field on $\mathcal{M}_k^+$ (the moduli space of k-instantons). The manifold $\mathcal{M}_k^+$ is hyper-Kähler. Its metric g descends from the L^2 metric $\int_{\mathbb{R}^4} \operatorname{Tr} \partial A_m^2$.

We shall keep ϵ fixed and send $\bar{\epsilon}$ to infinity.

From this data we get a one-form $g(V_{\bar{\epsilon}}, \ )$ on $\mathcal{M}_k^+$. Then the contribution from k-instantons to the path integral is

$$Z_k = \int_{\mathcal{M}_k^+} \exp\Big[-\delta_\epsilon\Big(g(V_{\bar{\epsilon}}, \ )\Big)\Big] = \sum_{\substack{\text{fixed points } p \\ \text{of } SO(4)}} \frac{1}{\det A(\epsilon)}. \tag{78}$$

The first equality can be traced to the original Ω-deformed 4d $\mathcal{N}=2$ action. In the last expression, $A(\epsilon)^i_j = \partial_i V^j_\epsilon(p)$, namely we consider how rotations act near the fixed point $p \in \mathcal{M}_k^+$.

The space $\mathcal{M}_k^+$ has conical singularities. We resolve it by adding an FI term ζ in the ADHM equations:

$$[B_1, B_2] + IJ = 0, \tag{79}$$

$$[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = \zeta 1_k. \tag{80}$$

This blows up the conical singularities into smooth regions, on which rotations act with isolated fixed points.

To understand a bit better, let us consider $\zeta > 0$. It turns out that the ADHM equations are equivalent to $[B_1, B_2] + IJ = 0$ plus a stability condition $\mathbb{C}[B_1, B_2]I(N) = K$ (this is a mild genericity condition on (B_1, B_2, I, J)), taken modulo $GL(K)$ instead of just $U(K)$. Here $GL(K)$ acts by $h \cdot (B_1, B_2, I, J) = (h^{-1}B_1 h, h^{-1}B_2 h, h^{-1}I, Jh)$.

Consider now the fixed points.

Denoting $t_1 = e^{i\epsilon_1}$, $t_2 = e^{i\epsilon_2}$, $b = \operatorname{diag}(e^{ia_1}, \ldots, e^{ia_N})$, we have the action

$$(t_1, t_2, b) \cdot (B_1, B_2, I, J) = (t_1 B_1, t_2 B_2, I b^{-1}, t_1 t_2 b J). \tag{81}$$

We want this to be $GL(K)$-equivalent to the original (B_1, B_2, I, J), so we want some $h_{(t_1,t_2,b)} \in GL(K)$ such that

$$t_1 B_1 = h^{-1} B_1 h, \tag{82}$$

$$t_2 B_2 = h^{-1} B_2 h, \tag{83}$$

$$I b^{-1} = h^{-1} I, \tag{84}$$

$$t_1 t_2 b J = J h. \tag{85}$$

Write $N = \oplus_{\alpha=1}^N \mathbb{C}_\alpha$ where $\mathbb{C}_\alpha = \ker(b - e^{ia_\alpha})$. We have the vector $I_\alpha = I(1 \in \mathbb{C}_\alpha) \in K$ and the vectors

$$|i, j; \alpha\rangle = B_1^{i-1} B_2^{j-1} I_\alpha \in K. \tag{86}$$

We can easily work out

$$h|i, j; \alpha\rangle = b_\alpha t_1^{i-1} t_2^{j-1} |i, j; \alpha\rangle. \tag{87}$$

Using the constraints we find that the states $|i, j; \alpha\rangle$ are non-zero for (i, j) in some Young diagram Y_α (that depends on the fixed point. It is also possible to prove that $J|i, j; \alpha\rangle = 0$, so J vanishes on all of K.

Altogether, fixed points are labeled by N-tuples of Young diagrmas of total size K.

Then there is an algebraic calculation to find the contribution from each Young diagram. We need to know what happens when varying (B_1, B_2, I, J). Under the symmetry and the compensator h, we find

$$(\delta B_1, \delta B_2, \delta I, \delta J) \to (ht_1 \delta B_1 h^{-1}, ht_2 \delta B_2 h^{-1}, h\delta I b^{-1}, t_1 t_2 b \delta J h^{-1}). \qquad (88)$$

The eigenvalues can be worked out and reduce to the well-known product formula involving arm length and leg length of each box in the Young diagram.

Comment on rational ϵ_1/ϵ_2. When $\epsilon_1/\epsilon_2 = p/q \in \mathbb{Q}$, the fixed points are no longer isolated. The denominators in the localization formula end up vanishing so each contribution blow up. However, we can show that the set of fixed points is still **compact**, so the limit remains finite.

3 Lecture 3, July 25

Welcome to lecture 3 of a series of 2.

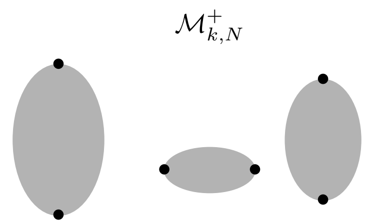
Yesterday we saw the ADHM equations, describing the moduli space of framed $U(N)$ instantons. Framed means we only identify instanton configurations up to gauge transformations that approach 1 at infinity.

The moduli space $\mathcal{M}^+_{k,N}$ has an $SU(N)$ symmetry coming from the possibility of performing a constant gauge transformation. We also have a Spin(4) symmetry acting by rotations of $\mathbb{R}^4$. Remember however that we deformed the moduli space by adding an FI parameter ζ. In order to single out one equation out of three, we have to break part of the symmetry: this deformation uses a specific complex structure $\mathbb{R}^4 \simeq \mathbb{C}^2$, so the Spin(4) rotation symmetry is broken to $U(2)$. Under this symmetry, (B_1, B_2) transforms in a doublet, I is invariant and J is in the determiant representation, namely $t \in U(2)$ maps $J \mapsto \det(t)J$. In applications we only use its maximal torus $U(1) \times U(1) = T_{\text{rotationos}}$.

In cases we considered yesterday (generic equivariant parameters) the fixed-points of $T_{\text{rotation}} \times T_{\text{gauge}} \subset U(2) \times SU(N)$ were isolated, and labeled by N-tuples of partitions $(\lambda^{(1)}, \dots, \lambda^{(N)})$ with total number of boxes $\sum_{\alpha=1}^N |\lambda^{(\alpha)}| = k$. A box in position (i, j) of $\lambda^{(\alpha)}$ corresponds to a vector $B_1^{i-1} B_2^{j-1} I(N_\alpha)$ in K.

Compacness theorem Yesterday we claimed that there is a compactness theorem, useful to study singularities (or lack thereof) of the instanton partition function.

When ϵ_1/ϵ_2 is rational, the one-parameter subgroup generated by $\operatorname{diag}(e^{i\epsilon_1}, e^{i\epsilon_2}) \in U(2)$ is not dense inside the two-torus $U(1)^2 \subset U(2)$; instead it is a one-dimensional subgroup. The set of fixed points of this $U(1) \subset U(1) \times U(1)$ can thus be bigger than fixed points of the full $U(1) \times U(1)$.



(89)

where each $\bullet$ is a fixed point of $U(1)^2$ and the shaded regions are fixed manifolds of the $U(1)$ subgroup. If the a's are also non-generic then the fixed point set can become non-compact, which is bad: the partition function can really blow up, it means that we are integrating over a flat direction. To avoid this we assume that

$$a_\alpha - a_\beta \notin \epsilon_1 \mathbb{Z}_{>0} + \epsilon_2 \mathbb{Z}_{>0} \qquad \text{for } \alpha \neq \beta. \tag{90}$$

Let $\epsilon_1 = p$ and $\epsilon_2 = q$, both integer, not both zero. Let $\phi \colon K \to K$ be the compensating transformation, namely $\phi \in \mathfrak{gl}(K)$. Then we want to solve the stability condition $\mathbb{C}[B_1, B_2]I(N) = K$ and the constraints

$$\epsilon_i B_i = [\phi, B_i], \qquad Ia = \phi I, \qquad (\epsilon_1 + \epsilon_2)J - aJ = -J\phi \tag{91}$$

modulo $GL(K)$. This defines the set of fixed points we want to study. Since $\mathbb{C}[B_1, B_2]I(N) = K$, we easily find that

$$\text{Spectrum}(\phi) = \bigl\{ a_\alpha + \epsilon_1(i-1) + \epsilon_2(j-1) \mid \text{for some } \alpha, i, j \bigr\}. \tag{92}$$

For $p, q > 0$ (or equivalently $p, q < 0$) we find that J must vanish because none of the $a_\alpha - \epsilon_1 - \epsilon_2$ appear in the spectrum. (For other signs of p and q we need to work harder.)

Given that the a_α are generic we can split

$$K = \bigoplus_{\alpha=1}^{N} K_\alpha, \qquad K_\alpha = \mathbb{C}[B_1, B_2]I(N_\alpha). \tag{93}$$

Then

$$\operatorname{Spec}\bigl(\phi|_{K_\alpha}\bigr) \subset a_\alpha + p\mathbb{Z}_{\geq 0} + q\mathbb{Z}_{\geq 0} \subset a_\alpha + \mathbb{Z}_{\geq 0}. \tag{94}$$

To prove compactness we want to find a bound on the norm squared $|B_1|^2 + |B_2|^2 + |I|^2 + |J|^2$ (note that we showed $J = 0$ so we can erase that term). We can split further to

$$K_\alpha = \bigoplus_{n \geq 0} K_{\alpha,n}, \qquad K_{\alpha,n} = \ker\bigl(\phi|_{K_\alpha} - a_\alpha - n\bigr). \tag{95}$$

There are finitely many non-zero summands of course; the danger is that matrix elements could become large.

Now let

$$\delta_{\alpha,n} = \frac{1}{\zeta}\,\mathrm{Tr}_{K_{\alpha,n}}\Big(B_1B_1^\dagger + B_2B_2^\dagger + II^\dagger\Big). \tag{96}$$

We want to bound the sum over n of this. Note that

$$B_1(K_{\alpha,n}) \subset K_{\alpha,n+p}, \qquad B_2(K_{\alpha,n}) \subset K_{\alpha,n+q}, \tag{97}$$

and the daggers act with the opposite sign. Rewrite the D-term equation as

$$[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger = \zeta 1_K \Longleftrightarrow B_1B_1^\dagger + B_2B_2^\dagger + II^\dagger = \zeta 1_K + B_1^\dagger B_1 + B_2^\dagger B_2. \tag{98}$$

We learn that

$$\begin{aligned}
\delta_{\alpha,n} &= \frac{1}{\zeta}\,\mathrm{Tr}_{K_{\alpha,n}}\Big(\zeta 1_K + B_1^\dagger B_1 + B_2^\dagger B_2\Big) && (99)\\
&= \dim K_{\alpha,n} + \frac{1}{\zeta}\,\mathrm{Tr}_{K_{\alpha,n+p}}\Big(B_1B_1^\dagger\Big) + \frac{1}{\zeta}\,\mathrm{Tr}_{K_{\alpha,n+q}}\Big(B_2B_2^\dagger\Big) && (100)\\
&\le \dim K_{\alpha,n} + \delta_{\alpha,n+p} + \delta_{\alpha,n+q}. && (101)
\end{aligned}$$

Of course this is a very conservative estimate. Now what makes it interesting is that we know that $\delta_{\alpha,n}$ vanishes for $n \ge k$ (again quite conservative when $N \ge 1$).

We can introduce generalized Fibonacci numbers

$$F_{<0} = 0, \qquad F_1 = 1, \qquad F_n = F_{n-p} + F_{n-q}. \tag{102}$$

This can be solved explicitly in terms of solutions to $\lambda^{-p} + \lambda^{-q} = 1$. Then one can in principle write a closed form formula as a sum over solutions to that equation:

$$F_n = \sum_\lambda f_\lambda \lambda^n, \qquad \text{so} \quad F_n < e^{c_1 n} \tag{103}$$

for some constant c_1. From our inequality (we write $k_n = \dim K_{\alpha,n}$ and drop the subscript α)

$$\delta_n \le k_n + \delta_{n+p} + \delta_{n+q} \tag{104}$$

it follows that

$$\delta_n \le \sum_{n'=0}^{\infty} k_{n+n'} F_{n'+1} \tag{105}$$

then

$$\sum_{n=0}^{\infty} \delta_n \le k e^{c_2 k} \tag{106}$$

has at most exponential growth.

For $p < 0$ and $q > 0$ (or equivalently the opposite signs) We first note that $J = 0$ nevertheless. Assume first $N = 1$. Then $JI = \mathrm{Tr}(IJ) = \mathrm{Tr}(IJ + [B_1, B_2]) = 0$ by the F-term equation.

For $(\alpha, \beta) \neq (0, 0)$ we have $[B_1, B_2] = [\alpha B_1 + \beta B_2, \gamma B_1 + \delta B_2]$ for some (γ, δ) (the choice doesn't matter) such that $\alpha\delta - \beta\gamma = 1$, from which we find that

$$J(\alpha B_1 + \beta B_2)^n I = \mathrm{Tr}\big((\alpha B_1 + \beta B_2)^n IJ\big) \tag{107}$$

$$= -\mathrm{Tr}\big((\alpha B_1 + \beta B_2)^n [B_1, B_2]\big) \tag{108}$$

$$= -\mathrm{Tr}\big((\alpha B_1 + \beta B_2)^n [\alpha B_1 + \beta B_2, \gamma B_1 + \delta B_2]\big) = 0. \tag{109}$$

By varying α and β we learn that J vanishes on $\mathbb{C}[B_1, B_2]I = K$.

Next we claim that for $pq < 0$, fixed points $(B_1, B_2, I, J = 0)$ are fixed by any ϵ_1 and ϵ_2. Consider $\mathcal{N} = B_1^q B_2^{-p}$. Then $\mathcal{N}$ is a nilpotent operator which commutes with the $U(1)$ generated by ϕ. Then decomposing into eigenspaces K_n of ϕ inside K, it follows from some theorem that there exists operators H and $\mathcal{N}^*$ such that we have an $\mathfrak{su}(2)$ algebra:

$$[H, \mathcal{N}] = \mathcal{N}, \qquad [H, \mathcal{N}^*] = -\mathcal{N}^*, \qquad [\mathcal{N}, \mathcal{N}^*] = 2^{(?)} H. \tag{110}$$

Then H provides a second grading, we have a $U(1) \times U(1)$ action, and we are back in business.

3.1 Adding matter

So far we only discussed pure 4d $\mathcal{N} = 2$ SYM. Let us add matter, but preserve asymptotic freedom (or conformality). This limits our choices. We can only work with quivers whose shape is of ADE type (either affine or non-affine). This in turn means that the theories can be obtained by certain orbifold truncations of 4d $\mathcal{N} = 4$ super Yang–Mills theory with bigger gauge group, or its mass deformation 4d $\mathcal{N} = 2^*$. So we can explain how to go from 4d $\mathcal{N} = 2$ SYM to 4d $\mathcal{N} = 4$ SYM, everything else being obtained by an orbifold.

Unfortunately there is no analogue of the reciprocity theorem relating ADHM data and configurations. We will have to proceed by analogy. More precisely, at a set-theoretic level everything reduces to usual instantons; the question is to work at a deeper level, keeping track of the measure etc.

In 4d $\mathcal{N} = 4$ theory we have the gauge field A and six scalar, which we collect into a self-dual two-form B^+ and a scalar C and σ and $\bar\sigma$ (alternatively just label fields using the flavour indices). We will single out one of the supercharges, and this relabelling (topological twist) is convenient. The B^+ splits further as

$$B^+ = B^{2,0} \oplus B^{1,1}_w \oplus B^{0,2} \tag{111}$$

and that gives what we later denote $B_4 = B^{1,1}_w + iC$ and what we call B_3.

The supersymmetric vacua are characterized by the equations $D_A\sigma = 0$ and $[\sigma, \overline{\sigma}] = 0$ and

$$F_A^+ + [B^+, C] + [B \times B]^+ = 0 \in \Omega^{2,+} \otimes \mathfrak{g} \tag{112}$$

$$D_A^* B^+ + D_A C = 0 \in \Omega^1 \otimes \mathfrak{g}. \tag{113}$$

These equations imply that $B, C = 0$ from which we deduce the instanton equations. However the measure of integration is different from the 4d $\mathcal{N} = 2$ case: instead of 1 we get the Euler class.

In contrast to 4d $\mathcal{N} = 2$ where we had a one-form Ω^1 and equations in $\Omega^{2,+} \oplus \Omega^0$, seemingly differnt, now our equations are in the same bundles as our fields. This is a "balanced" topological quantum field theory.

Now $W = \int \operatorname{Tr} B^+ \wedge \left(F_A^+ + \frac{1}{3}[B \times B]^+\right)$. Then $\delta W/\delta A$ and $\delta W/\delta B^+$ are gauge transformations of A and B^+ generated by C respectively.

3.1.1 ADHM equations for 4d $\mathcal{N} = 4$

The ADHM construction will have

$$[B_1, B_2] + IJ + [B_3, B_4]^\dagger = 0 \tag{114}$$

$$[B_1, B_3] + [B_4, B_2]^\dagger = 0 \tag{115}$$

$$[B_1, B_4] + [B_2, B_3]^\dagger = 0 \tag{116}$$

$$\sum_{a=1}^{4} [B_a, B_a^\dagger] + II^\dagger - J^\dagger J = \zeta \cdot 1_K \tag{117}$$

$$B_3 I + (JB_4)^\dagger = 0, \qquad B_4 I - (JB_3)^\dagger = 0. \tag{118}$$

Some of these come from a superpotential $W = \operatorname{Tr} B_3\big([B_1, B_2] + IJ\big)$ such that the equations read $(\delta W/\delta X)^\dagger = \delta^{\text{gauge}}_{B_4} X$.

The equations have more symmetry: the maximal torus $U(1)^3 \subset SU(4)$ plus some discrete symmetries. Namely we can multiply $B_a \to t_a B_a$ by any phases with $\prod_{a=1}^4 t_a = 1$. That's one more ϵ parameter than in the 4d $\mathcal{N} = 2$ case, namely we have $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ with $\sum_i \epsilon_i = 0$. Here ϵ_1 and ϵ_2 have to do with rotations of space-time while ϵ_3 is the mass of the adjoint hypermultiplet (there is a symmetry sending the mass of the adjoint to $-\epsilon_1 - \epsilon_2 - \epsilon_3 = \epsilon_4$).

After some work, the partition function ends up again being a sum over N-tuples of partitions:

$$Z_K = \sum_{\lambda = (\lambda^{(1)}, \dots)} \frac{\prod (\lambda + \epsilon_3)}{\prod\big(\lambda = (a_\alpha - a_\beta + \epsilon_1(\ ) + \epsilon_2(\ ))\big)} \tag{119}$$

where λ are eigenvalues of ϕ acting on $T\mathcal{M}^+_{k,N}$. There is a symmetry under $\epsilon_3 \leftrightarrow \epsilon_4$, which is not obvious in this sum expression.

Another symmetry the equations have is $SU(2)_{1,2} \times U(1) \times SU(2)_{3,4}$ where $SU(2)_{1,2}$ rotates (B_1, B_2) and $U(1)$ scales (B_1, B_2, B_3, B_4) by (t, t, t^{-1}, t^{-1}) and $SU(2)_{3,4}$ rotates (B_3, B_4).

From quotienting by discrete subgroups $\Gamma_1 \subset SU(2)_{1,2}$ and $\Gamma_2 \subset SU(2)_{3,4}$ we get quiver gauge theories on $\mathbb{R}^4/\Gamma_1$.

Note now that the equations are not symmetric in exchanging (B_1, B_2) with (B_3, B_4), so it is tempting to add $\widetilde{I}\colon \widetilde{N} \to K$ and $\widetilde{J}\colon K \to \widetilde{N}$. Then change a few equations:

$$[B_1, B_2] + IJ + \left([B_3, B_4] + \widetilde{I}\widetilde{J}\right)^\dagger = 0 \tag{120}$$

$$[B_1, B_3] + [B_4, B_2]^\dagger = 0 \tag{121}$$

$$[B_1, B_4] + [B_2, B_3]^\dagger = 0 \tag{122}$$

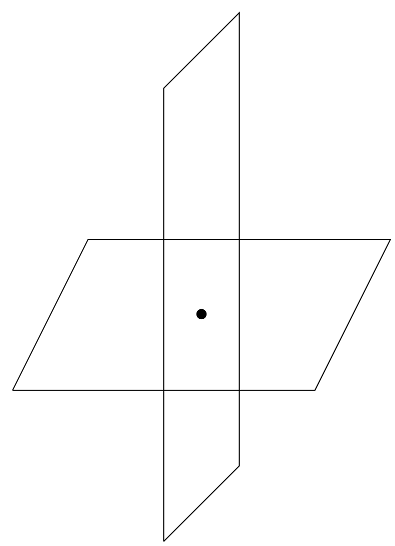
$$\sum_{a=1}^{4}[B_a, B_a^\dagger] + II^\dagger - J^\dagger J + \widetilde{I}\widetilde{I}^\dagger - \widetilde{J}^\dagger \widetilde{J} = \zeta \cdot 1_K \tag{123}$$

$$B_3 I + (JB_4)^\dagger = 0, \qquad B_4 I - (JB_3)^\dagger = 0 \tag{124}$$

$$B_1 \widetilde{I} + (\widetilde{J}B_2)^\dagger = 0, \qquad B_2 \widetilde{I} - (\widetilde{J}B_1)^\dagger = 0 \tag{125}$$

$$\widetilde{J}I - \left(J\widetilde{I}\right)^\dagger = 0. \tag{126}$$

What do these equations describe? Two $\mathbb{C}^2$ intersecting at a point. Let $K_{12} = \mathbb{C}[B_1, B_2]I(N)$ and $K_{34} = \mathbb{C}[B_3, B_4]\widetilde{I}(\widetilde{N})$ and $K_{12} + K_{34} = K$. There are $\dim K_{12}$ instantons on one plane, $\dim K_{34}$ instantons on the other, and $\dim(K_{12} \cap K_{34})$ instantons trapped at the intersection.



(127)

There are compactness theorems, which imply regularity of correlation functions. It is possible to use this to show that the partition function obeys some differential equation; then mapping the partition function to 2d CFT correlation functions we find some Ward identities of 2d CFT.

Nobody knows how to construct explicit gauge field configurations for cases where all B_i are non-zero.

To the observer who lives in one four-dimensional plane, it looks like an instanton can disappear. Of course it just goes off in the other four-dimensional plane.

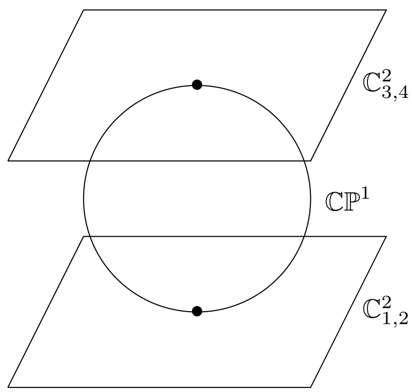
3.1.2 One instanton

Let us take $n = \widetilde{n} = k = 1$. Then all commutators vanish, everything is a number. We immediately find $J = \widetilde{J} = 0$, and

$$0 = B_3 I = B_4 I = B_1 \widetilde{I} = B_2 \widetilde{I}, \qquad \text{and} \qquad |I|^2 + |\widetilde{I}|^2 = \zeta \tag{128}$$

so we find

- a branch $\mathbb{C}^2_{3,4}$ parametrized by B_3 and B_4, which force $I = 0$ hence $\widetilde{I} = \sqrt{\zeta}$ (up to gauge transformations);
- a branch $\mathbb{C}^2_{1,2}$ parametrized by B_1 and B_2, which force $\widetilde{I} = 0$ hence $I = \sqrt{\zeta}$ (up to gauge transformations);
- a branch $\mathbb{CP}^1$ parametrized by I and $\widetilde{I}$, touching the other two branches at their origin.



(129)

Basically the instanton can probe one or the other $\mathbb{C}^2$, and at the intersection it has a whole moduli space where it hesitates whether to go to one or the other theory. The $U(1)$ symmetry has two fixd points, marked by $\bullet$ in the picture.

This has a II brane construction, basically

N D5	0	1	2	3	4	5				
$\widetilde{N}$ D5'	0	1					6	7	8	9