2 Exercise session 2, July 17

2.1 Exercises about Guido Festuccia's course

Exercise 2.1. Solve the old-minimal generalized Killing spinor equations on S^4.

$$\nabla_\mu \zeta_\alpha = \frac{i}{6} M \sigma_\mu \bar{\zeta} + \frac{i}{3} b_\mu \zeta + \frac{i}{3} b^\nu \sigma_{\mu\nu} \zeta \tag{1}$$

$$\nabla_\mu \bar{\zeta}_{\dot{\alpha}} = \frac{i}{6} \overline{M} \sigma_\mu \zeta - \frac{i}{3} b_\mu \bar{\zeta} - \frac{i}{3} b^\nu \bar{\sigma}_{\mu\nu} \bar{\zeta} \tag{2}$$

Hint: you'll find $b^\mu = 0$.

Answer. (Provided by Pieter Bomans.) We have

$$\frac{1}{6} \mathcal{R}_{\mu\nu} \zeta = [\nabla_\mu, \nabla_\nu] \zeta = \left(-\frac{1}{27} (b_\mu b_\nu - g_{\mu\nu} b_\rho b^\rho) + \frac{1}{18} g_{\mu\nu} M \bar{M} \right) \zeta \tag{3}$$

For the sphere we have

$$\mathcal{R}_{\mu\nu} = -\frac{3}{r^2} g_{\mu\nu} \tag{4}$$

and thus we find that b_μ indeed has to be zero. If we then put

$$M = \bar{M} = -\frac{3i}{r} \tag{5}$$

we find the conformal killing spinor equation on the four-sphere

$$\nabla_\mu \zeta = \frac{1}{2r} \sigma_\mu \bar{\zeta} \tag{6}$$

$$\nabla_\mu \bar{\zeta} = \frac{1}{2r} \sigma_\mu \zeta \tag{7}$$

which has been calculated in excercise 1.11. □

2.2 Exercises about Francesco Benini's course

Exercise 2.2. Check that the Wilson line W_R in some representation R of some gauge group G is gauge-invariant. Show that this operator is equivalent to a 1d defect operator with: a 1d gauge field $\widetilde{A}$, the pull-back A_τ of the bulk gauge field, some 1d fermions ψ in representation R of the bulk gauge group G and charge 1 under $\widetilde{A}$, namely the 1d Lagrangian is $\mathcal{L}_D = \overline{\psi}(\partial_\tau - iA_\tau - i\widetilde{A}_\tau)\psi + i\widetilde{A}_\tau$.

Answer. (Provided by Francesco Benini.) Wilson loop operators are defined by

$$W_\mathcal{R}[\gamma] = \mathrm{Tr}_\mathcal{R}\, \mathrm{Pexp} \oint_\gamma A\ . \tag{8}$$

We would like to find a defect theory description of these operators.

First, consider a 1d theory along γ, given by a free complex spinor ψ in representation $\mathcal{R}$ of the bulk gauge group G, minimally coupled to the bulk. Its Lagrangian is

$$\mathcal{L}_D = \overline{\psi} \not{D} \psi = \overline{\psi}(\partial_\tau - iA_\tau)\psi = \sum_{\rho \in \mathcal{R}} \overline{\psi}_\rho \big(\partial_\tau - i\rho(A_\tau)\big)\psi_\rho\ . \tag{9}$$

Here τ is a coordinate along γ, A is the bulk connection pulled back to γ, ρ are the weights of $\mathcal{R}$, and the 1d gamma matrix $\gamma_\tau = 1$ in einbein basis. For simplicity, we will take τ such that the pulled back metric is 1. Let γ be a circle of length β and let us choose antiperiodic (thermal) boundary conditions for the fermions. Then the path-integral is easily evaluated, since ψ is free. Let us choose a gauge where A_τ is constant. Then

$$Z_D = \int \mathcal{D}\overline{\psi}\,\mathcal{D}\psi\; e^{-\int d\tau\, \overline{\psi}(\partial_\tau - iA_\tau)\psi} = \prod_{\rho \in \mathcal{R}} \prod_{k \in \mathbb{Z}} \left(\tfrac{2\pi i}{\beta}\big(k + \tfrac{1}{2}\big) - i\rho(A_\tau) \right)\ . \tag{10}$$

That is because the modes of ψ are $e^{2\pi i\left(k+\frac{1}{2}\right)\tau/\beta}$. The regularization has some ambiguity, as the function should have zeros at $\beta\rho(A_\tau) = 2\pi\big(k+\frac{1}{2}\big)$, but we can choose

$$Z_D = \prod_{\rho\in\mathcal{R}} \left(1 + e^{i\beta\rho(A_\tau)}\right) \;\equiv\; \prod_{\rho\in\mathcal{R}} (1 + x_\rho) \;. \tag{11}$$

This is just the partition function of the fermionic Fock space, where the excited levels have energies $-i\rho(A_\tau)$. Notice that x_ρ are the eigenvalues of the holonomy $\mathrm{Pexp}\oint_\gamma A$ in representation $\mathcal{R}$, therefore the gauge-invariant expression for Z_D is

$$Z_D = \det\nolimits_{\mathcal{R}} \Big(1 + \mathrm{Pexp}\oint_\gamma A\Big) \;. \tag{12}$$

This is not yet the Wilson line operator in representation $\mathcal{R}$. However notice that if we decompose $\prod_\rho(1+x_\rho)$ into characters, we find all antisymmetric products of $\mathcal{R}$, which can be further decomposed into irreducible representations:

$$\prod_\rho (1 + x_\rho) \;\sim\; \sum_{\ell=0}^{\dim\mathcal{R}} \mathcal{R}^{\otimes_A \ell} \;.$$

Each level ℓ is the partition function restricted to fermion number ℓ. To select a specific fermion number, we gauge it – which corresponds to imposing Gauss law – and include a Chern-Simons coupling which includes $-\ell$ units of electric charge so that gauge-invariant states have fermion number ℓ. Thus, we consider the action

$$\widetilde{\mathcal{L}}_D = \overline{\psi}\big(\partial_\tau - iA_\tau - i\widetilde{A}_\tau\big)\psi + i\ell\widetilde{A}_\tau \;, \tag{13}$$

where $\widetilde{A}$ is a 1d gauge field. The path-integral over $\tilde{A}$ gives a delta function on $\psi\overline{\psi} = \ell$, which projects the partition function to the sector with fermion number ℓ. Alternatively, we perform the path-integral over ψ first and introduce a fugacity $y = e^{i\beta\widetilde{A}_\tau}$ for the 1d $U(1)$ symmetry; then the CS term gives a classical contribution $y^{-\ell}$ and finally the path-integral over $\widetilde{A}$ – imposing Gauss law – reduces to a contour integral along $|y| = 1$:

$$\widetilde{Z}_D = \oint_{|y|=1} \frac{dy}{2\pi i\, y}\, y^{-\ell} \prod_{\rho\in\mathcal{R}} \big(1 + x_\rho y\big) = \sum_{\rho_1<\ldots<\rho_\ell} x_\rho \;, \tag{14}$$

where, with some abuse of notation, we have assumed an ordering of the weights.

If now we consider the special case $\ell = 1$, we precisely produce the trace of the holonomy in representation $\mathcal{R}$:

$$\widetilde{Z}_D(\ell=1) = \sum_\rho x_\rho = \mathrm{Tr}_{\mathcal{R}}\, \mathrm{Pexp}\oint_\gamma A \;. \tag{15}$$

Representations $\mathcal{R}$ which are the antisymmetric product of some representation $\mathcal{R}'$ can be obtained either by choosing higher ℓ, or by choosing $\mathcal{R}$ directly.

□

2.3 Exercises about Wolfger Peelaers' course

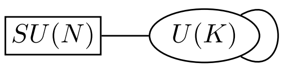
Exercise 2.3.

Exercise 2.4.

Exercise 2.5.

Exercise 2.6. The goal is to derive the ADHM constraints as describing the Higgs branch of the worldvolume theory of instantons in 4d $\mathcal{N} = 4$ SYM. A reference is Tong's lectures `http://www.damtp.cam.ac.uk/user/tong/tasi/instanton.pdf` around equation (1.37).

1. Instantons preserve half of the supersymmetry, namely their world-volume theory is a 0d theory (matrix model) with 8 supercharges. From the brane picture described by a stack of N D3 branes in the presence of a stack of k D(-1) branes argue that the worldvolume theory on the D(-1) branes is the dimensional reduction to 0d of a 4d $\mathcal{N}=2$ theory with gauge group $U(k)$ with an adjoint hypermultiplet and a collection of N fundamental hypermultiplets, described by the following quiver. Write down its bosonic action explicitly.



2. Perform the Gaussian integral over the auxiliary fields D_{IJ}.
3. Write down the vacuum equations.
4. These equations admit in particular a Higgs branch of solutions where scalar fields originating from the $\mathcal{N}=2$ vector multiplet vanish. Recover in this way the ADHM equations of `https://en.wikipedia.org/wiki/ADHM_construction`
5. Compute the one-instanton partition function for $SU(N)$ instantons explicitly using the integral representation provided in the lecture.

Answer. (Provided by Tom Bourton.)

1. Quantisation of open D(-1)–D(-1) strings gives rise to the reduction to zero dimensions of 10d $\mathcal{N}=1$ SYM with gauge group $U(k)$. D(-1)–D3 strings gives rise to the dimensional reduction of 4d $\mathcal{N}=2$ hypermultiplets in the $\mathbf{K}\times\overline{\mathbf{N}}$ representation of $U(K)\otimes(S)U(N)$. The coupling of these bifundamental hypers to the maximally supersymmetric $U(k)$ theory is fixed by demanding $\mathcal{N}=2$ supersymmetry. The number of supercharges is $32/(2\cdot 2)=8$ as required for 1/2-BPS instantons of $\mathcal{N}=4$ SYM. The superpotential is

$$W = q\phi\widetilde{q} + \mathrm{tr}_k[\phi,z]\widetilde{z} + \mathrm{tr}_k W_\alpha W^\alpha\,. \tag{16}$$

The bosonic part of the action reduced to zero dimensions is

$$\begin{aligned}\mathcal{L}_{bos} =& \frac{1}{2}\mathrm{tr}_k\left([X_\mu,X_\nu]^2 + |[X_\mu,\phi]|^2 + |[X_\mu,z]|^2 + |[X_\mu,\widetilde{z}]|^2\right) + q^\dagger X^\mu X_\mu q \\ &+ \widetilde{q}^\dagger X^\mu X_\mu \widetilde{q} + \left(\sum_{f\in\{q,\widetilde{q},z,\widetilde{z},\phi\}} \mathrm{tr}_k\left(\frac{\partial W}{\partial f}F_f + F_f^2\right) + h.c.\right) \\ &+ \frac{1}{2}\mathrm{tr}_k D^2 + \mathrm{tr}_k D\left([\phi,\phi^\dagger] + [\widetilde{z},\widetilde{z}^\dagger] + [z,z^\dagger] + qq^\dagger - \widetilde{q}^\dagger\widetilde{q} + \zeta\right)\,,\end{aligned} \tag{17}$$

where $\mu = 1,2,3,4$.

2. Since D appears only up to quadratic order it can be integrated out, this is equivalent to solving its equations of motion (20)

$$\begin{aligned}\mathcal{L}_{bos} =& \frac{1}{2}\mathrm{tr}_k\left([X_\mu,X_\nu]^2 + |[X_\mu,\phi]|^2 + |[X_\mu,z]|^2 + |[X_\mu,\widetilde{z}]|^2\right) + q^\dagger X^\mu X_\mu q \\ &+ \widetilde{q}^\dagger X^\mu X_\mu \widetilde{q} + \left(\sum_{f\in\{q,\widetilde{q},z,\widetilde{z},\phi\}} \mathrm{tr}_k\left(\frac{\partial W}{\partial f}F_f + F_f^2\right) + h.c.\right) \\ &- \frac{1}{2}\mathrm{tr}_k\left([\phi,\phi^\dagger] + [\widetilde{z},\widetilde{z}^\dagger] + [z,z^\dagger] + qq^\dagger - \widetilde{q}^\dagger\widetilde{q} + \zeta\right)^2\end{aligned} \tag{18}$$

3. The F terms are

$$-F_\phi = q\widetilde{q} + [z,\widetilde{z}]\,,\quad -F_q = \phi\widetilde{q}\,,\quad -F_{\widetilde{q}} = q\phi\,,\quad -F_z = [\widetilde{z},\phi]\,,\quad -F_{\widetilde{z}} = [z,\phi] \tag{19}$$

and the D-term is

$$-D = qq^\dagger - \widetilde{q}^\dagger\widetilde{q} + [z,z^\dagger] + [\widetilde{z},\widetilde{z}^\dagger] + [\phi,\phi^\dagger] + \zeta\mathbb{I}_k = 0\,. \tag{20}$$

4. The Higgs branch is reached by setting $\phi = 0$. In this limit, we have

$$\mathcal{E}_{\mathbb{C}} := q\widetilde{q} + [z, \widetilde{z}] = 0\,, \quad \mathcal{E}_{\mathbb{R}} := qq^\dagger - \widetilde{q}^\dagger\widetilde{q} + [z, z^\dagger] + [\widetilde{z}, \widetilde{z}^\dagger] + \zeta\mathbb{I}_k = 0 \tag{21}$$

These are precisely the ADHM equations with $z = B_1$, $\widetilde{z} = B_2$, $q = I$ and $\widetilde{q} = J$. The Higgs branch is

$$\mathcal{M}^{\mathrm{D}(-1)}_{\mathrm{Higgs}} = \{q, \widetilde{q}, z, \widetilde{z} | \mathcal{E}_{\mathbb{C}} = \mathcal{E}_{\mathbb{R}} = 0\,, \phi = X_\mu = 0\}\,/U(k) \tag{22}$$

and we have

$$\mathcal{M}^{\mathrm{D}(-1)}_{\mathrm{Higgs}} \cong \mathcal{M}^{\mathcal{N}=4\ \mathrm{SYM}}_{\mathrm{ADHM}}\,. \tag{23}$$

5. The k instanton partition function for 4d $G = U(N)$ $\mathcal{N} = 2^*$ has a contour integral expression

$$\begin{aligned} Z_k =& \frac{1}{k!}\int_{JK}\prod_{I=1}^{k}\frac{d\phi_I}{2\pi i}\prod_{I,J=1}^{k}\frac{\phi'_{IJ}(\phi_{IJ}-2\varepsilon_+)}{(\phi_{IJ}-\varepsilon_1)(\phi_{IJ}-\varepsilon_2)}\frac{(\phi_{IJ}+m+\varepsilon_-)(\phi_{IJ}+m-\varepsilon_-)}{(\phi_{IJ}+m+\varepsilon_+)(\phi_{IJ}+m-\varepsilon_+)} \\ &\times\prod_{I=1}^{k}\prod_{i=1}^{N}\frac{(\phi_I-a_i+m)(\phi_I-a_i-m)}{(\phi_I-a_i-\varepsilon_+)(\phi_I-a_i+\varepsilon_+)} \end{aligned} \tag{24}$$

where m is the adjoint hypermultiplet mass, $2\varepsilon_\pm = \varepsilon_1 \pm \varepsilon_2$ are the Ω-background parameters, $\phi_{IJ} = \phi_I - \phi_J$ and the prime means that the $I = J$ terms should be omitted from the product. The $k = 1$ instanton result can be easily computed

$$Z_1 = \frac{(-2\varepsilon_+)(m+\varepsilon_-)(m-\varepsilon_-)}{\varepsilon_1\varepsilon_2(m+\varepsilon_+)(m-\varepsilon_+)}\int\frac{d\phi}{2\pi i}\prod_{i=1}^{N}\frac{(\phi-a_i+m)(\phi-a_i-m)}{(\phi-a_i-\varepsilon_+)(\phi-a_i+\varepsilon_+)} \tag{25}$$

We can close the contour to pick up the poles at, say, $\phi = a_i + \varepsilon_+$

$$Z_1 = \frac{(m+\varepsilon_-)(m-\varepsilon_-)}{\varepsilon_1\varepsilon_2}\sum_{j=1}^{N}\prod_{\substack{i=1\\ i\neq j}}^{N}\frac{(a_j-a_i+\varepsilon_++m)(a_j-a_i+\varepsilon_+-m)}{(a_j-a_i)(a_j-a_i+2\varepsilon_+)}\,. \tag{26}$$

□