Extended operators and localization

Lectures by Takuya Okuda   notes by Bruno Le Floch

July 23–24, 2018

Nobody (even the typist) proof-read these notes, so there may be obvious mistakes: tell BLF.

Abstract

We discuss how to perform supersymmetric localization in the presence of boundaries and other extended operators. These are lecture notes for the 2018 IHÉS summer school on *Supersymmetric localization and exact results*.

These lecture notes assume familiarity with supersymmetry at the level of the first few chapters of the book by Wess and Bagger.

1 Lecture 1, July 23 — supersymmetric localization with boundaries in 2d

Extended operators in QFTs include:

- line operators such as Wilson, 't Hooft, . . . ;
- surface operators;
- boundaries, interfaces (also known as domain walls);
- coupled systems of QFTs of different dimensionalities.

Supersymmetric localization can be performed with them. These objects also play an important role in conformal bootstrap and holography. They were initially introduced as order parameters: their expectation values as functions of some parameters can be used to classify phases of QFTs.

Today we'll talk about boundaries in 2d. This is closely related to what Francesco Benini discussed last week. Tomorrow we'll talk about line operators in 4d. This is closely related to what Wolfger Peelaers discussed last week. The general ideas will be emphasized.

References

- general: section 39 of "Mirror symmetry" by Hori etc. (from the Clay Mathematical Institute);

- 3 papers:
  - Sugishita–Terashima https://arxiv.org/abs/1308.1973,
  - Honda–Okuda https://arxiv.org/abs/1308.2217,
  - Hori–Romo http://arxiv.org/abs/1308.2438.

  We shall use notations most similar to Honda–Okuda, themselves similar to Doroud–Gomis–Le Floch–Lee (the S^2 case).

1.1 B-type boundary conditions

Set-up. We consider 2d $\mathcal{N} = (2, 2)$ theories on S^2 or on the hemisphere.

1.1.1 Geometry and supersymmetry

Consider the supergravity background obtained from dimensional reduction of the 4d $\mathcal{N} = 1$ new minimal supergravity, namely the one that has R-symmetry. We call this 2d background the $U(1)_V$ supergravity background because it preserves a $U(1)$ vector-like R-symmetry. The metric is

$$\mathrm{d}s^2 = f(\theta)^2 \mathrm{d}\theta^2 + \ell^2 \sin^2\theta \mathrm{d}\varphi^2 \tag{1}$$

with $\theta \in [0, \pi]$ and periodic $\varphi \in [0, 2\pi)$. We sometimes use the vielbein (zweibein)

$$e^{\hat{1}} = f(\theta)\mathrm{d}\theta, \qquad e^{\hat{2}} = \ell \sin\theta \mathrm{d}\varphi. \tag{2}$$

The function $f \colon [0, \pi] \to \mathbb{R}$ is such that near the pole $\theta = 0$ we have $f(\theta) \sim \ell + O(\theta^2)$ (to avoid any singularity). For example,

- $f = \ell$ constant is the round metric;
- $f = \sqrt{\ell^2 - \cos^2\theta + \widetilde{\ell}^2 \sin^2\theta}$ gives the squashed sphere $S^2_b \colon \{x_0^2 + b^2(x_1^2 + x_2^2) = \ell^2\}$ (perhaps $1/b^2$?).

The $U(1)$ R-symmetry background gauge field is

$$V^R = \frac{1}{2}\left(1 - \frac{\ell}{f(\theta)}\right)\mathrm{d}\varphi \tag{3}$$

(in particular it goes to zero at the pole). Finally,

$$H = \overline{H} = \frac{\#}{f} \tag{4}$$

for some coefficient #.

Generalized Killing spinor equations are

$$(\nabla_\mu - iV^R_\mu)\epsilon = \frac{1}{2f}\gamma_\mu\gamma_3\epsilon \tag{5}$$

$$(\nabla_\mu + iV^R_\mu)\overline{\epsilon} = -\frac{1}{2f}\gamma_\mu\gamma_3\overline{\epsilon} \tag{6}$$

Explicit solutions to the generalized Killing spinor equations are

$$\epsilon = e^{-\frac{i}{2}\theta\gamma_{\hat{2}}} \begin{pmatrix} e^{\frac{i}{2}\varphi} \\ 0 \end{pmatrix}, \qquad \bar{\epsilon} = e^{\frac{i}{2}\theta\gamma_{\hat{2}}} \begin{pmatrix} 0 \\ e^{-\frac{i}{2}\varphi} \end{pmatrix}. \tag{7}$$

Our convention for Gamma matrices is

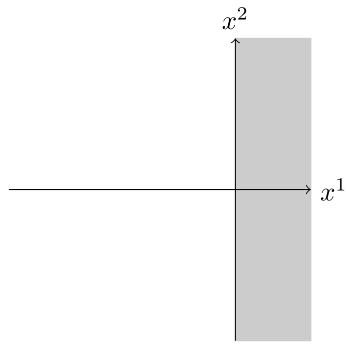
$$\gamma_{\hat{\theta}} = \gamma_{\hat{1}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_{\hat{\varphi}} = \gamma_{\hat{2}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{8}$$

On the round sphere we have $\mathfrak{su}(2|1)$ symmetry, which gets broken by the deformation f down to $\mathfrak{su}(1|1)$. The square of the supercharge involves a rotation in the φ direction. This allows for boundaries at constant θ. For simplicity we put the boundary at $\theta = \pi/2$. Then

$$\epsilon \sim \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \bar{\epsilon} \sim \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{9}$$

up to an R-symmetry gauge transformation.

In coordinates $x^1 = \tilde{\ell}(\theta - \pi/2)$ and $x^2 = \ell\varphi$, then



Near the boundary the supercharges are $Q_1 + Q_2$ ($= Q_+ + Q_-$ in textbook notations) and $\overline{Q}_1 + \overline{Q}_2$ ($= \overline{Q}_- + \overline{Q}_+$). This is known as B-type supersymmetry.

Just by analysing the supersymmetry we see that at the poles A-type supersymmetry is preserved while at the boundary B-type supersymmetry is preserved.[1] We learn that

$$Z_{\text{hemisphere}} \sim \left\langle \text{B-brane} \,\middle|\, \mathbf{1} \text{ in (a,c)-ring} \right\rangle \tag{10}$$

where the B-brane describes the choice of boundary condition, while 1 denotes that we did not insert any operator at the pole, but we could have inserted any operators in the twisted chiral ring (also known as the (a,c) ring).

Question from the audience (asked later): What does this equation mean; so far we only have a gauge theory, not an SCFT. Answer: the gauge theory can flow to an SCFT; then some recent work shows that the equality holds.

[1] By this we mean that a specific 2-supercharge subalgebra of 2d $\mathcal{N} = (2,2)$ supersymmetry is preserved.

Question from Yale Fan maybe (I couldn't see): why do we get an overlap and not a state? Answer (rephrased from Takuya's answer): the path integral over a space with a boundary defines a state on the boundary, here it is what we denoted as $|\mathbf{1}\rangle$; then the partition function with a given boundary condition computes the overlap of that wavefunction with the wavefunction associated to the choice of boundary condition.

1.1.2 Supermultiplets

Vector multiplets for gauge group G: $(A_\mu, \sigma_1, \sigma_2, \lambda, \overline{\lambda}, D)$.

Chiral multiplet in representation R of G: (ϕ, ψ, F).

Some susy transformations are

$$\delta A_\mu = -\frac{i}{2}\left(\bar{\epsilon}\gamma_\mu\lambda + \bar{\lambda}\gamma_\mu\epsilon\right) \tag{11}$$

$$\delta\phi = \bar{\epsilon}\psi \tag{12}$$

$$\delta\psi = i\gamma^\mu\epsilon D_\mu\phi + i\epsilon\sigma_1\phi + \gamma^3\epsilon\sigma_2\phi + \frac{iq}{2r}\gamma^3\epsilon\phi + \bar{\epsilon}F. \tag{13}$$

Here $\bar{\epsilon}\gamma_\mu\lambda = \bar{\epsilon}^\alpha(\gamma_\mu)_\alpha{}^\beta\lambda_\beta$ and $\bar{\epsilon}^\alpha = C^{\alpha\beta}\bar{\epsilon}_\beta$ and components of $C^{\alpha\beta}$ are $C^{12} = -C^{21} = 1$ and $C^{11} = C^{22} = 0$.

Part of the 2d $\mathcal{N} = (2,2)$ supersymmetry is broken by the boundary.

1.1.3 Boundary conditions at $\theta = \pi/2$

For the vector multiplet,

$$\sigma_1 = 0, \qquad D_1\sigma_2 = 0, \qquad A_1 = 0, \qquad F_{12} = 0 \tag{14}$$

$$\bar{\epsilon}\lambda = \epsilon\overline{\lambda} = 0, \qquad D_1(\bar{\epsilon}\gamma_3\lambda) = D_1(\epsilon\gamma_3\overline{\lambda}) = 0, \qquad D_{\hat{1}}(D - iD_{\hat{1}}\sigma_1) = 0. \tag{15}$$

As announced, this preserves half of the supersymmetry.

For the chiral multiplet the main boundary condition is Neumann boundary condition

$$D_1\phi = 0, \qquad \bar{\epsilon}\gamma_3\psi = D_1(\bar{\epsilon}\psi) = 0, \qquad F = 0. \tag{16}$$

An other choice is Dirichlet boundary condition

$$\phi = 0, \qquad \bar{\epsilon}\psi = D_1(\bar{\epsilon}\gamma_3\psi) = 0, \qquad D_{\hat{1}}(e^{-i\varphi}F + iD_{\hat{1}}\phi) = 0. \tag{17}$$

Question from Masahito Yamazaki: are those boundary conditions elliptic? Answer: they are probably not, which is a problem for perturbation theory, but for localization that is not a problem.

1.2 Matrix factorization

1.2.1 Warner term

On the sphere, the action is

$$S_{\text{phys}} = S_{\text{vec}} + S_{\text{chi}} + S_W + S_\vartheta + S_{\text{FI}} \tag{18}$$

where S_{vec} is basically the Yang–Mills action, S_{chi} has a kinetic term for the chiral multiplet, S_ϑ is a topological term $\int F$, S_{FI} is basically $\int \mathrm{d}^2x\, D$, and finally the superpotential term is

$$S_W \simeq \int \Big(F^i \partial_i W(\phi) - \frac{1}{2}\psi^i\psi^j\partial_i\partial_j W + \text{conjugate}\Big). \tag{19}$$

We need to make these terms supersymmetric. From the calculations on the sphere in Francesco Benini's lecture we know that supersymmetry variations are total derivatives, but then these must be compensated by the supersymmetry variations of boundary terms that we add to the action. For most terms that's easy. The exception is

$$\delta S_W \sim \oint \mathrm{d}\varphi \left(\epsilon\gamma^\mu\psi^i\partial_i W + \text{conjugate}\right). \tag{20}$$

This term is the Warner term, it is hard to get as a supersymmetric variation of a boundary term.

1.2.2 Matrix factorization

The way we do that is to insert a Wilson loop in the path integral:

$$Z \sim \int_{\substack{\text{boundary}\\ \text{condition}}} \mathcal{D}(\text{fields}) e^{-S_{\text{physical}}} \operatorname{Tr}_V \operatorname{Pexp} i \oint \mathcal{A}_\varphi \mathrm{d}\varphi. \tag{21}$$

The boundary interaction is

$$\begin{aligned} \mathcal{A}_\varphi \sim{} & A_\varphi + i\sigma_2 + (\text{R-charge}) + (\text{twisted mass}) \\ & + \big\{Q(\phi), \overline{Q}(\phi)\big\} + \#(\psi_1-\psi_2)^i\partial_i Q(\phi) + \text{conjugate}. \end{aligned} \tag{22}$$

Here $Q(\phi)\colon V \to V$ is a linear map on some auxiliary vector space V. The point is to cancel the Warner term:

$$\delta_{\text{SUSY}}\left(e^{-S_W} \operatorname{Tr}_V \operatorname{Pexp} i\oint \mathcal{A}_\varphi \mathrm{d}\varphi\right) = 0. \tag{23}$$

This holds if

$$Q(\phi)^2 = W(\phi) \times \mathrm{id}_V\,. \tag{24}$$

This operator $Q(\phi)$ is called a **matrix factorization** of the superpotential W, or **tachyon profile**. For details see Herbst–Hori–Page ∼2008.

Gauge-invariance requires $g^{-1}Q(g\phi)g = Q(\phi)$ where on the left-hand side, g implicitly acts on V.

R-symmetry requires $e^{ir_*\chi}Q(e^{iR\chi}\phi)e^{-ir_*\chi} = e^{i\chi}Q(\phi)$.

Why is it called matrix factorization? Assume that $V = V^{\text{even}} \oplus V^{\text{odd}}$ and Q is odd, namely $Q = \left(\begin{smallmatrix} 0 & a \\ c & 0\end{smallmatrix}\right)$. Then $Q^2 = W$ requires $ab = ba = W\,\mathrm{id}_V$.

1.3 Hemisphere partition function

We use the same localization action as Francesco Benini did on S^2. For simplicity we work with $G = U(N)$, with complexified FI parameter $t = 2\pi\xi - i\vartheta$ where ϑ is the topological angle.

The localization equations set the field strength to be constant, but $F_{12} = 0$ at the boundary because of boundary conditions, so we find

$$\sigma_1 = F_{\mu\nu} = 0, \qquad \sigma_2 = \text{constant}, \qquad \phi = 0, \qquad F = 0. \tag{25}$$

The classical action is $e^{-S_{\text{physical}}} = e^{t \operatorname{Tr} \sigma}$ where $\sigma = -i\ell\sigma_2$ (this includes twisted masses where appropriate).

The pedestrian way to get the one-loop determinants is by listing the eigenmodes and keeping those that are consistent with the boundary condition.

1.3.1 One-loop determinants

Chiral multiplets with Neumann boundary conditions are

$$Z_{\text{1-loop}}^{\text{Neumann}} = \prod_{w \in \text{weights}(R)} \frac{1}{\prod_{j\geq 0}^{\text{reg.}} (j - i(w\sigma_2 + iq/2))} = \prod_{w \in \text{weights}(R)} \Gamma(w\sigma + q/2) \tag{26}$$

where we used zeta-function regularization. With Dirichlet boundary conditions,

$$Z_{\text{1-loop}}^{\text{Dirichlet}} = \prod_{w \in \text{weights}(R)} \prod_{j\geq 0}^{\text{reg.}} \big(j + 1 + i(w\sigma_2 + iq/2)\big), \tag{27}$$

which zeta-function regularization suggests should be $\prod_w 1/\Gamma(1 - w\sigma - q/2)$.

Claim: zeta function regularization is not always correct (in fact, why should it be? its validity needs to be checked on a case-by-case basis). Specifically, the correct answer is

$$Z_{\text{1-loop}}^{\text{Dirichlet}} = \prod_{w \in \text{weights}(R)} \frac{-2\pi i e^{\pi i (w\cdot\sigma - q/2)}}{\Gamma(1 - w\sigma - q/2)}. \tag{28}$$

(In fact a more precise analysis shows that each factor in the infinite product comes with a phase $e^{i\pi}$; zeta regularization then produces a power of $e^{i\pi}$ that gives the numerator above. The factor 2π likewise comes from a careful zeta-function regularization.) Why?

- There is a duality between Dirichlet boundary condition and Neumann boundary condition plus boundary interaction (related to Atiyah–Bott–Shapiro construction of D0 branes). This imposes the extra factors.
- First principles derivation (unpublished): Pauli–Villars regularization shows that
$$Z_{\text{1-loop}}^{\text{Dirichlet}} \big/ Z_{\text{1-loop}}^{\text{Neumann}} = 1 - e^{2\pi i (w\cdot\sigma - iq/2)}. \tag{29}$$

This ratio is unambiguous if we impose supersymmetry and gauge-invariance.[2]

The vector multiplet one-loop determinant is

$$Z_{\text{1-loop}}^{\text{vector}} = \prod_{\alpha>0} \prod_{j\geq 0}^{\text{reg.}} \left[j^2 + (\alpha\sigma_2)^2\right] = \prod_{\alpha>0} \prod_{\alpha>0} (\alpha\sigma) \sin(\pi\alpha\cdot\sigma). \tag{30}$$

using zeta-function regularization.

The July 24 lecture starts here

Altogether

$$\begin{aligned} Z_{\text{1-loop}} = &\left(\prod_{\alpha>0} \alpha\cdot\sigma \, \sin\pi\alpha\cdot\sigma\right) \\ &\prod_{a\in\text{Neu}} \prod_{w\in\text{weights}(R_a)} \Gamma(w\cdot\sigma + q/2) \\ &\prod_{a\in\text{Dir}} \prod_{w\in\text{weights}(R_a)} \frac{-2\pi i e^{\pi i(w\cdot\sigma+q/2)}}{\Gamma(1-w\cdot\sigma-q/2)}, \end{aligned} \tag{31}$$

where the products range over chiral multiplets with Neumann and with Dirichlet boundary condition respectively, and where $\sigma = -i\ell\sigma_2$ and where twisted masses are introduced by shifting $w\cdot\sigma \to w\cdot\sigma + m_a$.

We call $B = (\text{Neu}, \text{Dir}, (V, \rho, r_*), Q)$ the boundary data. Boundary data is in one to one correspondence with B-branes.

For a gauge group G we find the hemisphere partition function as a function of $t = 2\pi\xi - i\vartheta$:

$$Z_{\text{hem}}(B;t) = \frac{1}{|\text{Weyl}(G)|} \int_{(i\mathbb{R})^{\text{rank}\,G}} \frac{\mathrm{d}^{\text{rank}\,G}\sigma}{(2\pi i)^{\text{rank}\,G}} \, e^{t_{\text{ren}}\cdot\sigma} \, \text{Str}_V \, e^{-2\pi i\sigma} Z_{\text{1-loop}}(\sigma). \tag{32}$$

To be more precise, the contour $(i\mathbb{R})^{\text{rank}\,G}$ comes from the boundary condition of the vector multiplet, which for convergence may need to be tilted compared to this real axis, but for supersymmetry needs to remain a Lagrangian submanifold. The supertrace factor $\text{Str}_V \, e^{-2\pi i\sigma}$ is also called the brane factor, as this is what depends on the choice of matrix factorization (V, ρ, r_*, Q).

1.4 D-brane central charge

What is the meaning for CFT? Namely, for the case where the gauge theory flows in the IR to a CFT. Conjecture by Honda–Okuda and Hori–Romo: the hemisphere partition function is an unnormalized version of the central charge of the D-brane. The conjecture was proven by Bachas and Plencner.

In compactifications of II string theory on a Calabi–Yau$_3$ down to 4d for instance, particles in 4d come from D-branes whose spatial directions are along the Calabi–Yau manifold. In the 4d $\mathcal{N}=2$ superalgebra there are central

[2] Recall that different regularizations are related by counterterms. Different choices of counterterms preserve different symmetries.

charges, and this is the central charge we are talking about. A more intrinsic definition is that the central charge is obtained from the partition function in an infinitely long cigar geometry with an A-twist. This constructs the RR ground state. Then the partition function computes $\langle B|0\rangle_{RR}$.

Note that the fact that it is unnormalized is not a big problem: we are interested in comparing central charges of different D-branes to determine which combination of particles is more stable.

For the non-conformal case, conjecture by Cecotti and Vafa.

Example Calabi–Yau hypersurface in $\mathbb{CP}^{N-1}$. For $N = 3$ this is the torus, for $N = 4$ this is the K3 surface, for $N = 5$ this is the quintic threefold.

Gauge group $U(1)$, N chirals ϕ_i of gauge charge $+1$, one chiral P of gauge charge $-N$. The gauge charges sum up to zero so this model is Calabi–Yau. The superpotential is $W = PG_N(\phi)$ where $G_N(\phi)$ is a generic degree N polynomial.

Let us work out a matrix factorization for the $\mathrm{D}(2N-4)$ brane, namely choose all coordinates to have Neumann boundary condition, none with Dirichlet boundary condition. Introduce fermionic oscillators $\{\eta, \overline{\eta}\} = 1$ and a state $|0\rangle$ with $\eta|0\rangle = 0$. This gives a two-dimensional space V spanned by $|0\rangle$ and $\overline{\eta}|0\rangle$. Then

$$Q(\phi, P) = G_N(\phi)\eta + P\overline{\eta} \tag{33}$$

is such that $Q^2 = W$. One can assign gauge and R-charges on V such that the remaining conditions are obeyed: gauge charge of $|0\rangle$ is $n + N/2$ with $n \in \mathbb{Z}$, R-charge of $|0\rangle$ is 0.

Claim (discussed in Herbst–Hori–Page): the matrix factorization Q describes $\mathcal{O}_M(n)$ over the Calabi–Yau manifold M, which is obtained as a restriction of the standard line bundle $\mathcal{O}(n)$ on $\mathbb{CP}^{N-1}$.

Then

$$Z_{\text{hem}}\big[\mathcal{O}_M(n)\big] = \int_{i\mathbb{R}} \frac{\mathrm{d}\sigma}{2\pi i} e^{-2\pi i n\sigma}(e^{-N\pi i\sigma} - e^{N\pi i\sigma})e^{t\sigma}\Gamma(\sigma)^N\Gamma(1-N\sigma). \tag{34}$$

After some calculations, in the large volume limit $\Re(t) = 2\pi\xi \gg 0$, this goes like

$$Z_{\text{hem}}\big[\mathcal{O}_M(n)\big] \sim \int_M \mathrm{ch}(\mathcal{O}_M(n))e^{B+i\omega}\widehat{\Gamma}(TM) \tag{35}$$

where M is our target space, where $B + i\omega = \frac{-t}{2\pi i}\underline{e}$ in terms of a generator $\underline{e} = i^*h$ of $H^2(M,\mathbb{Z})$ where h is the hyperplane class in $\mathbb{CP}^{N-1}$. Here the Gamma hat class is

$$\widehat{\Gamma}(E) \coloneqq \prod_j \Gamma\left(1 + \frac{ix_j}{2\pi}\right) \tag{36}$$

in terms of Chern roots, themselves such that $\mathrm{ch}(E) = \sum_j e^{x_j}$.

Thus the localization calculation explains the appearance of the Gamma class, which was previously found through more ad-hoc calculations.

Question from the audience: what happens when we change the boundary condition from Neumann to Dirichlet? Well, that changes the one-loop determinant to

$$Z^{\text{Dirichlet}}_{\text{1-loop}} = Z^{\text{Neumann}}_{\text{1-loop}}(1 - e^{2\pi i(w\cdot\sigma - iq/2)}) \tag{37}$$

in other words it is equivalent to changing the brane factors by a factor $(1 - e^{2\pi i(w\cdot\sigma - iq/2)})$.

1.5 Interfaces

Consider a sphere with theory T_1 on one hemisphere and T_2 on the other hemisphere. We can fold this into $T_1 \times \overline{T}_2$ on the hemisphere. Then an interface preserving B-type supersymmetry is equivalent to a B-brane for the $T_1 \times \overline{T}_2$ theory. In the exercises from yesterday we gave a way to write the trivial interface as a matrix factorization for the product theory. This means that the hemisphere partition function with that boundary condition reproduces the sphere partition function.

2 Lecture 2, July 24 — supersymmetric localization with line operators in 4d

`https://arxiv.org/abs/1412.7126` (review)
`https://arxiv.org/abs/1111.4221` Ito–Okuda–Taki
`https://arxiv.org/abs/1801.01986` Brennan–Dey–Moore

Consider a theory with gauge group G. Denote the Cartan algebra by $\mathfrak{t}$ and $\mathfrak{t}^*$ its dual. We have the root and weight lattices $\Lambda_{\text{r}} \subset \Lambda_{\text{w}} \subset \mathfrak{t}^*$, and their duals $\Lambda_{\text{cw}} \supset \Lambda_{\text{cr}}$ inside $\mathfrak{t}$. Namely the weight lattice is dual to the coroot lattice and conversely.

In 4d $\mathcal{N} = 2$, one can have $\frac{1}{2}$-BPS Wilson line operators

$$W_R = \text{Tr}_R \, \text{Pexp} \oint_{\text{Straight line}} (iA + \Re\phi \mathrm{d}s). \tag{38}$$

The possible curves on which the Wilson loop can be put and preserve half of the supersymmetry have been classified for the case of 4d $\mathcal{N} = 4$. For $\mathcal{N} = 2$ this may be an open problem. Here R is an irrep of G, characterized by its highest weight $w \in \Lambda_{\text{w}}$, which is a dominant weight.

A 't Hooft line operator $T(B)$ in 4d $\mathcal{N} = 2$ is defined by a singular boundary condition

$$F \sim \frac{B}{2}\epsilon_{ijk}\frac{x^i}{r}\mathrm{d}x^k \wedge \mathrm{d}x^l = -\frac{B}{2}\sin\theta \mathrm{d}\theta \wedge \mathrm{d}\varphi, \tag{39}$$

$$\phi \sim \frac{iB}{2r}, \tag{40}$$

where x^i and (r, θ, φ) are local coordinates describing transverse directions to the loop (namely the normal bundle). More precisely this is for $\vartheta_{\text{topological}} = 0$; otherwise details change to be consistent with the Witten effect.

The Dirac quantization condition for B is found by considering a gauge where $A \sim \frac{-B}{2}(1 - \cos\theta)\mathrm{d}\varphi$, namely put the Dirac string along $\theta = \pi$. The condition for the Dirac string to be unobservable by an electric charge is that $\frac{1}{2\pi}\int_{S^2} F$ should have integer eigenvalues on arbitrary representations of G. We conclude that $B \in \Lambda_{\mathrm{w}}^* = \Lambda_{\mathrm{cr}}$ (also called magnetic charge lattice). *Note that this does not depend on the matter content, but only on the global structure of the gauge group.*

One can combine the two. This gives a dyonic loop operator, characterized by (B, w) with $B \in \Lambda_{\mathrm{cr}}$ and w in Λ_{w} of the part of the gauge group unbroken by B, modulo the Weyl group.

A more refined classification comes from considering two line operator with labels (B_1, w_1) and (B_2, w_2). Just like an electrically charged particle picks up a phase when going around a Dirac string, and we need to set that phase to 1, loops pick up a phase

$$\exp(2\pi i \big(\langle B_2, w_1\rangle - \langle B_1, w_2\rangle\big). \qquad (41)$$

When this is equal to 1, we say the operators are mutually local.

Aharony, Seiberg, Tachikawa understood that to specify a theory completely we need to choose a *maximal set of mutually local line operators*. This is equivalent to a choice of theta angles, or to a choice of maximal isotropic subgroup of $Z(G) \times Z(G)^*$ where $Z(G)$ is the center of G. This paper is essential reading.

2.1 A_1 theories of class S

Class S theories are built from two M5 branes on $C_{g,n}$ (g is the genus and n the number of punctures).

There is a correspondence by Drukker–Morrison–Okuda between

- homotopy classies of closed curves on $C_{g,n}$;
- line operators charges

Tachikawa worked out that the more refined classification of line operator charges is in one to one correspondence with isotropic subgroups of $H^1(C_{g,n=0}, \mathbb{Z}_2^{3g-3})$.

2.2 Localization with line operators

Line operators can be placed on two possible circles on S_b^4, or along the time circle on $S^1 \times \mathbb{R}^3$.

For Wilson lines, the localization calculation has been done by Pestun for $b = 1$, by Hama–Hosomichi for other b. For ’t Hooft loops and dyonic loops, the localization calculation was done by Gomis–Okuda–Pestun for $b = 1$, not done yet for general b. On the other hand, the AGT correspondence provides a prediction for the answer in general so it may not be so interesting to actually do the localization calculation.

Consider a 't Hooft loop on S^1_b inside S^4_b. Only the loca neighborhood will matter, so $S^1 \times_\lambda \mathbb{R}^3$, where λ is a twist parameter describing how $\mathbb{R}^3$ is rotated when we go around S^1 once. Then the 't Hooft loop expectation value is

$$\langle T_B \rangle_{S^4_b} \supset Z_{\text{mono}}. \tag{42}$$

2.3 Kronheimer's correspondence

We should all thank Okuda and Pestun for having bugged Kronheimer until he finally posted his master thesis on his webpage.

The Bogolmony equations are $\star_3 F = D\phi$.

The antiselfduality equation is $F + \star F = 0$.

Kronheimer showed that

- instantons on Taub–NUT space invariant under some $U(1)$ rotation;
- singular monopoles

are in one-to-one correspondence. See exercises for details.

The multi-Taub–NUT space is

$$\mathrm{d}s^2 = V \mathrm{d}\vec{x}^2 + V^{-1}(\mathrm{d}\psi + \omega)^2, \qquad \psi \sim \psi + 2\pi, \tag{43}$$

$$V = \ell + \sum_j \frac{1}{2|\vec{x} - \vec{x_j}|}, \qquad \mathrm{d}\omega + \star_3 \mathrm{d}V = 0. \tag{44}$$

Just like in Nekrasov's instanton partition function, small instantons contribute. On top of the 't Hooft singularity we can have 't Hooft–Polyakov monopoles that come close to the singularity. This can change B to a smaller weight v. This is called monopole screening or bubbling. Through Kronheimer's correspondence we can map these to small instantons.

The relevant instanton moduli space turns out to be the same as in flat space, so we can use the ADHM construction, and the same localization techniques apply.

Follow-up on July 25

2.4 Interlude on 't Hooft

In 1978 in Nuclear Physics B, 't Hooft introduced an algebra of loop operators in an $\mathfrak{su}(N)$ gauge theory. Let W be a Wilson loop on a curve C_W and T a 't Hooft loop on a curve C_T. Consider the configuration where C_W and C_T are Hopf-linked (simplest linking) on a constant time-slice $\mathbb{R}^3$. Now we can move C_W and C_T slightly in time; it makes sense to compare the two orderings. One finds

$$WT = e^{\frac{2\pi i}{N}} TW \tag{45}$$

namely the fundamental Wilson loop and fundamental 't Hooft loop are *not mutually local*.

What is the modern point of view on this?

The two loop operators cannot both be genuine loop operators at the same time. At least one of the two loops must be the boundary of some topological surface operator.

Let us be more precise about the global structure of the group.

- Consider a gauge theory with gauge group $G = SU(N)$. Then the fundamental Wilson loop is well-defined. However, the 't Hooft loop must be the boundary of a topological surface operator, a Dirac string on $\mathbb{R}_{t>0} \times C_T$ (the usual Dirac string people are used to when there is a monopole configuration).

- Consider a gauge theory with gauge group $G = SU(N)/\mathbb{Z}_N$ and with zero discrete theta angle. Then naively $\operatorname{Tr}_\square \operatorname{Pexp} i \oint A$ is not defined because the fundamental representation is not a representation of the group $SU(N)/\mathbb{Z}_N$. But the expression is written in terms of the connection A, so it actually seems to make sense to take the trace in any representation. Gauge-invariance fails only when we consider some gauge transformations that are only defined on patches. If we realize C_W as the boundary of some surface, then the Wilson loop is well-defined and gauge-invariant (but it depends on the surface): the idea is to define it using a gauge field that is defined in one patch containing the whole surface.

The take-home message is that some line operators are not genuine and arise as boundaries of surface operators.

2.5 Kronheimer correspondence

$$\mathrm{d}s^2 = V \mathrm{d}\vec{x}^2 + V^{-1}(\mathrm{d}\psi + \omega)^2 \tag{46}$$

with coordinate $\psi \sim \psi + 2\pi$, and where the potential is $V = \ell + \sum_j \frac{1}{2|\vec{x} - \vec{x}_j|}$ and where ω is defined by $\mathrm{d}\omega + \star_3 \mathrm{d}V = 0$.

The partition function in the presence of a 't Hooft loop operator is (for $b \neq 1$ it is a guess)

$$\langle T_B \rangle_{S^4_b} = \int_{\mathfrak{t}} \mathrm{d}a \sum_v Z^{S^1\times\mathbb{R}^3}_{\text{1-loop}} Z_{\text{monopole}} Z^{S^4_b}_{\text{1-loop}} \Big| Z_{\text{instanton}} \Big|^2 \tag{47}$$

The sum over v comes from screening/bubbling monopoles that weaken the singularity. The Z_{monopole} is an analogue of the instanton partition function.

The similar calculation on $S^1 \times_\lambda \mathbb{R}^3$, namely the following trace, is expressed in terms of the same one-loop determinant and monopole contribution. Note that the 't Hooft loop operator changes the Hilbert space:

$$\langle T_B \rangle_{S^1\times_\lambda\mathbb{R}^3} = \operatorname{Tr}_{\mathcal{H}(T_B)} \Big((-1)^F e^{-2\pi R H} e^{2\pi i \lambda (J_3 + I_3)} e^{-2\pi i m_f F_f} \Big) \tag{48}$$

$$= \sum_v e^{2\pi i v \mathbf{b}} Z^{S^1\times\mathbb{R}^3}_{\text{1-loop}}(a, m_f, b, v) Z_{\text{monopole}}(\mathbf{a}, m_f, b; B, v) \tag{49}$$

where **a** depends on one vector multiplet real scalar and the holonomy around the circle, while **b** depends on other vector multiplet real scalar and on the graviphoton.