6 Exercise session 6, July 24

6.1 Exercises for Maxim Zabzine's lecture

Exercise 6.1. Check that in 2d, $F = 0$ with $\mathrm{d}^\dagger A = 0$ is an elliptic problem. Hint: $\det\left(\begin{smallmatrix} \xi_2 & -\xi_1 \\ \xi_1 & \xi_2 \end{smallmatrix}\right) = \xi_1^2 + \xi_2^2$.

Exercise 6.2. Check that the following 4d problem (instantons) is elliptic: $F^+ \coloneqq \frac{1}{2}(1+\star)F = 0$ and $\mathrm{d}^\dagger A = 0$.

Exercise 6.3. Check that the following 3d problem is elliptic: $F + \star \mathrm{d}_A \sigma = 0$ and $\mathrm{d}^\dagger A = 0$, where σ is an adjoint scalar.

Exercise 6.4. Check that the following problem is elliptic: $\overline{\partial} X^i = 0$ with $X \colon \Sigma \to M$ with Σ a Riemann surface and M compact.

Exercise 6.5. Check in your favorite dimension that the Dirac operator is elliptic.

If you have time: learn about the Todd class and other characteristic classes.

6.2 Exercises for Nikita Nekrasov's lecture

Exercise 6.6. Check that the instanton equation is conformally invariant.

Exercise 6.7 (Exercise 1.6.7 in Nekrasov's notes)**.** Show that, $F_A^+ = 0 \Leftrightarrow F_A^{0,2} = 0, F_A^{1,1} \wedge \varpi = 0$, where

$$2i\varpi = dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 \tag{1}$$

Exercise 6.8 (Exercise 1.6.8 in Nekrasov's notes)**.** Show that, for $G = SU(N)$,

$$k = -\frac{1}{8\pi^2} \int_M \mathrm{Tr} F_A \wedge F_A > 0 \tag{2}$$

one has the index theorem:

$$\mathrm{dim} ker_{L^2} \not{D}_A^* - \mathrm{dim} ker_{L^2} \not{D}_A = k \tag{3}$$

Exercise 6.9 (Exercise 1.6.9 in Nekrasov's notes)**.** Show that

$$\not{D}_A \left(\eta \oplus \chi\right) = 0 \Longrightarrow \eta = \chi = 0 \tag{4}$$

Hint: The equations read, in components: $D_{\bar{\alpha}}\eta + \varepsilon_{\bar{\alpha}\bar{\beta}} g^{\beta\bar{\beta}} D_\beta \chi = 0$. Derive from this $\Delta_A \eta = 0$, $\Delta_A \chi = 0$.

Exercise 6.10 (Exercise 1.6.14 in Nekrasov's notes)**.** Show that

$$\psi = \left(\nu_+^\dagger d\bar{z}_1 + \nu_-^\dagger d\bar{z}_2\right) \Delta^{-1} \tag{5}$$

solves Dirac equation, i.e. compute:

$$\begin{aligned} D_{\bar{1}}\psi_{\bar{2}} - D_{\bar{2}}\psi_{\bar{1}} &= \Xi^\dagger \left(\bar{\partial}_{\bar{1}} \left(\Xi \nu_-^\dagger \Delta^{-1}\right) - \bar{\partial}_{\bar{2}} \left(\Xi \nu_+^\dagger \Delta^{-1}\right)\right) \\ D_1 \psi_{\bar{1}} + D_2 \psi_{\bar{2}} &= \Xi^\dagger \left(\partial_1 \left(\Xi \nu_+^\dagger \Delta^{-1}\right) + \partial_2 \left(\Xi \nu_-^\dagger \Delta^{-1}\right)\right) \end{aligned} \tag{6}$$

6.3 Exercises for Takuya Okuda's lectures

Exercise 6.11. The multi-centered Taub-NUT space, is defined by the metric

$$ds^2 = V d\vec{x}^2 + V^{-1}(d\psi + \omega)^2 \,, \tag{7}$$

$$V = l + \sum_j \frac{1}{2|\vec{x} - \vec{x}_j|} \,, \tag{8}$$

$$d\omega = - *_3 dV \text{ on } \mathbf{R}^3, \tag{9}$$

where $l > 0$ is a constant and $\psi \sim \psi + 2\pi$. From the three-dimensional fields (A, Φ) with singularities

$$A \sim \frac{B_j}{2} \cos\theta d\varphi\,, \tag{10}$$

$$\Phi \sim \frac{B_j}{2r} \qquad \text{near } \vec{x} = \vec{x}_j\,, \tag{11}$$

where (r, θ, φ) are the spherical coordinates on a 3-ball centered at $\vec{x} = \vec{x}_j$, we construct a four-dimensional gauge connection

$$\mathcal{A} \equiv g\left(A + \Phi\frac{d\psi + \omega}{V}\right) g^{-1} - igdg^{-1} \tag{12}$$

and its curvature $\mathcal{F} = d\mathcal{A} + i\mathcal{A}\wedge\mathcal{A}$. The singularities in A and Φ cancel in (12) to define a smooth four-dimensional gauge field $\mathcal{A}$. Here g is a suitable singular gauge transformation that locally behaves as $g \sim e^{iB_j\psi}$ near $\vec{x} = \vec{x}_j$ so that $\mathcal{A}$ is smooth there. For a single singular monopole we can take $g = e^{iB\psi}$. For multiple singular monopoles, see the appendix of 1806.00024.

Show that the Bogomolny equations

$$*_3 F = D\Phi \tag{13}$$

are equivalent to the anti-self-dual equations

$$*_4 \mathcal{F} + \mathcal{F} = 0 \tag{14}$$

for the orientation specified by (volume form)$= f(d\psi + \omega) \wedge dx^1 \wedge dx^2 \wedge dx^3$ for some $f > 0$.

Exercise 6.12. For functions of a and b, we define the *Moyal product* by

$$(f * g)(a, b) \equiv e^{i\frac{\lambda}{8\pi}(\partial_b\cdot\partial_{a'} - \partial_a\cdot\partial_{b'})} f(a, b) g(a', b')\Big|_{a'=a, b'=b}\,, \tag{15}$$

where λ is some parameter.

Define

$$\ell_{1,0}(a, b) := (e^{2\pi i b} + e^{-2\pi i b}) \left(\frac{\sin(2\pi a + \pi m)\sin(2\pi a - \pi m)}{\sin\left(2\pi a + \frac{\pi}{2}\lambda\right)\sin\left(2\pi a - \frac{\pi}{2}\lambda\right)}\right)^{1/2}\,. \tag{16}$$

Show that $\ell_{1,0} * \ell_{1,0}$ is given by

$$\begin{aligned}&(e^{4\pi i b} + e^{-4\pi i b})\frac{\prod_{s_1,s_2=\pm1}\sin^{1/2}\left(2\pi a + s_1\pi m + s_2\frac{\pi}{2}\lambda\right)}{\sin^{1/2}(2\pi a + \pi\lambda)\sin^{1/2}(2\pi a - \pi\lambda)\sin(2\pi a)} \\ &\quad + \sum_{s=\pm}\frac{\prod_{\pm}\sin\pi(2a \pm m + s\lambda/2)}{\sin(2\pi a)\sin\pi(2a + s\lambda)}\,.\end{aligned} \tag{17}$$