Instantons

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Introduction to instantons. Unnamed sections are the *exercises*.

Contents

1.1. Path integral as a period. Lefschetz thimbles. Complex critical points

Let $\mathcal{X}$ be dimension n complex manifold. Let $S : \mathcal{X} \longrightarrow \mathbb{C}$ be a holomorphic function , h a hermitian metric on $\mathcal{X}$, $\hbar \in \mathbb{C}$, and $\Omega_{\mathcal{X}}$ a holomorphic top degree form. Let $\mathcal{X}_{\infty} \subset \mathcal{X}$ be the region where $\mathrm{Re}(S/\hbar) \gg 0$, and $[\Gamma] \in H_n(\mathcal{X}, \mathcal{X}_{\infty})$. We study

$$Z_{\Gamma}(\hbar) = \int_{\Gamma} \Omega_{\mathcal{X}} \, e^{-\frac{S}{\hbar}} \tag{1.1}$$

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1.1.1.

Consider

$$Z_{\Gamma}(x) = \int_{\Gamma} dt\, e^{\frac{\mathrm{i}}{\hbar}\left(tx - \frac{1}{3}t^3\right)} \tag{1.2}$$

Find the convergence cycles, and Lefschetz thimbles for real $\hbar$.

1.2. Example of the anharmonic oscillator. Critical points as rational foliations. Generalizations to many degrees of freedom. Instanton-antiinstanton gas

1.2.1. *Definitions*

Define

$$U(x) = \frac{\lambda}{4}(x^2 - v^2)^2\,, \qquad H(p,x) = \frac{1}{2}p^2 + U(x) = E \tag{1.3}$$

Pass to the dimensionless variables: $x = v\xi$, $E = U_0\epsilon$, $p = \sqrt{2U_0}\rho$ with

$$U_0 = \frac{\lambda}{4}v^4 \tag{1.4}$$

The complex energy level $\mathcal{C}_{\epsilon}$ is the curve:

$$\rho^2 + (\xi^2 - 1)^2 = \epsilon \tag{1.5}$$

1.2.2.

Show it is isogenous to the elliptic curve

$$\mathrm{y}^2 = 4\,\mathrm{x}^3 - g_2\,\mathrm{x} - g_3 \tag{1.6}$$

with g_2, g_3 functions of ϵ:

$$g_2 = \frac{4}{3}\,(3\epsilon + 1)\ , \qquad g_3 = \frac{8}{27}\,(9\epsilon - 1) \tag{1.7}$$

and discriminant $\Delta = g_2^3 - 27g_3^2 = 64\epsilon(\epsilon - 1)^2$.

$\mathrm{y} = 2\mathrm{i}\rho\xi$, $\mathrm{x} = \xi^2 - \frac{2}{3}$

1.2.3. *Useful differentials*

Action differential

$$pdx = \frac{3S_0}{4}\rho d\xi \tag{1.8}$$

where

$$S_0 = \frac{4v}{3}\sqrt{2U_0} \tag{1.9}$$

Timing differential

$$\frac{dx}{p} = \frac{v}{\sqrt{2U_0}}\frac{d\xi}{\rho} = \frac{\mathrm{i}v}{\sqrt{2U_0}}\frac{d\mathrm{x}}{\mathrm{y}} \tag{1.10}$$

1.2.4.

Classification of solutions, e.g. periodic trajectories with time period T: each trajectory corresponds to some ϵ, and represents a homology cycle $\gamma \leftrightarrow (n,m) \in H_1(\mathcal{C}_\epsilon, \mathbb{Z})$, such that

$$\oint_\gamma \frac{dx}{p} = T \tag{1.11}$$

For large T expect $\epsilon \to 0$. Show, without doing any calculations[†], that (with some choice of the basis), when $\epsilon \to 0$,

$$\oint_\gamma \frac{dx}{p} = \left(n + m \frac{2}{2\pi \mathrm{i}} \log\left(\epsilon/\epsilon_0\right) \right) \tau_0 + O(\epsilon) \tag{1.12}$$

where ϵ_0 is of order 1, and

$$\tau_0 = \frac{\pi v}{\sqrt{2U_0}} \tag{1.13}$$

is the period of small oscillations near the classical minimum.

1.2.5.

Approximate the $\epsilon \to 0$ trajectory by the sequence of instantons and antiinstantons, i.e. solutions to the first-order equations:

$$\rho = \pm \mathrm{i}(1 - \xi^2) \tag{1.14}$$

1.2.6. *Many-body systems*

Canonical variables (p_a, x^a), algebraic integrability $H_a(p,x)$, $\{H_a, H_b\} = 0$, fibers (after some partial compactification) $\vec{H}^{-1}(\vec{E})$ being the abelian varieties. Action-angle variables, $(\mathrm{a}_a, \varphi^a)$, period matrix $\tau_{ab}(\mathrm{a})$.

$$d\vec{p} \wedge d\vec{x} = \sum_a d\mathrm{a}_a \wedge d\varphi^a \tag{1.15}$$

1.2.7.

Classification of solutions, e.g. periodic trajectories with time periods T^a: each trajectory corresponds to $\vec{E}$, with $\vec{H} = \vec{E}$, $\mathcal{C}_{\vec{E}} = \vec{H}^{-1}(\vec{E})$, it represents a homology cycle $\gamma \leftrightarrow (\vec{n}, \vec{m}) \in H_1(\mathcal{C}_{\vec{E}}, \mathbb{Z})$, such that

$$\frac{\partial}{\partial E_a} \oint_\gamma \vec{p} d\vec{x} = T^a \tag{1.16}$$

[†]Computing an arc length does not count

1.3. Supersymmetric quantum mechanics. Instantons as the Q-fixed points. Morse theory

One dimensional quantum mechanics. Fields $(x, p; \psi, \bar{\psi})$, supersymmetry:

$$\delta x = \psi \,, \ \delta\bar{\psi} = p \tag{1.17}$$

The Lagrangian

$$L = \delta\left(\bar{\psi}\left(\dot{x} + \frac{\mathrm{i}}{2}p - V'(x)\right)\right) \tag{1.18}$$

Critical points:

$$\begin{aligned}
&\dot{x} + \mathrm{i}p - V'(x) = 0 \\
&-\dot{p} + V'''(x)\bar{\psi}\psi - V''(x)p = 0 \\
&\dot{\psi} - V''(x)\psi = 0 \\
&-\dot{\bar{\psi}} - V''(x)\bar{\psi} = 0
\end{aligned} \tag{1.19}$$

Show that

$$\begin{aligned}
&b = \bar{\psi}\psi \\
&E = \frac{1}{2}\left(p + \mathrm{i}V'\right)^2 + U(x) \\
&U(x) = \frac{1}{2}\left(V'(x)\right)^2 - \mathrm{i}bV''(x)
\end{aligned} \tag{1.20}$$

are conserved quantities $\dot{b} = \dot{E} = 0$. Solve the equations of motion for $V(x)$ given by a cubic polynomial in x.

Several degrees of freedom:

$$\begin{aligned}
&\delta x^a = \psi^a \,, \qquad \delta\bar{\psi}_a = p_a \,, \\
&\delta\psi^a = 0 \,, \qquad \delta p_a = 0 \\
&\mathcal{L} = \delta\left(\bar{\psi}_a\left(\dot{x}^a + \frac{\mathrm{i}}{2}g^{ab}\left(p_b - \gamma^c_{bd}\psi^d\bar{\psi}_c\right) - g^{ab}\partial_b V\right)\right)
\end{aligned} \tag{1.21}$$

Localisation: take the limit $g^{ab} \to 0$, $V \to \infty$, so that $\mathcal{V}^a = g^{ab}\partial_b V$ stays finite. Describe the space of states in this limit. *Note* one gets a priori different spaces of *in-* and *out-* states, why?

The evolution operator, in the limit. Relation to Morse theory.

Lagrangian approach: localisation locus in the space of fields: solutions to

$$\dot{x}^a = \mathcal{V}^a \tag{1.22}$$

i.e. gradient trajectories, connecting critical points p, where $\mathcal{V}^a(p) = 0$. Careful analysis for quadratic V shows, that the limiting states are differential forms on a patch $\mathcal{U}_p \ni p$, which have the following structure:

$$\mathcal{U}_p \backslash \{p\} \approx X_p^{n-m_p} \times Y_p^{m_p} \tag{1.23}$$

where

$$V - V(p) \approx \sum_{i=1}^{n-m_p} x_i^2 - \sum_{j=1}^{m_p} y_j^2 \tag{1.24}$$

So,

$$\mathcal{H}^{\rm in} = \bigoplus_p A^{\bullet}(Y_p^{m_p}) \otimes A^{\bullet}(\left(X_p^{n-m_p}\right)^{\vee})\delta_{Y_p^{m_p}} \tag{1.25}$$

1.4. Generalizations to infinite-dimensional systems: two dimensional sigma models and four dimensional gauge theories

1.4.1. *Sigma model action*

governs the maps $X : \Sigma \to \mathcal{X}$, for the target space $\mathcal{X}$ equipped with the metric G and a two-form (a connection on a gerbe) B:

$$S = \frac{1}{2}\int_{\Sigma} G_{MN} dX^M \wedge \star dX^N + \frac{\rm i}{2}\int_{\Sigma} B_{MN} dX^M \wedge dX^N \tag{1.26}$$

Define

$$E_{MN} = G_{MN} + {\rm i}B_{MN} \tag{1.27}$$

In two dimensions, for Euclidean Σ, $\star^2 = -1$ when acting on 1-forms. Assume an almost complex structure $J : T\mathcal{X} \to T\mathcal{X}$, $J^2 = -1$, such that G and B are related via:

$$B_{MN} = J_M^{M'} J_N^{N'} B_{M'N'}\,, \qquad G_{MN} = B_{MK} J_N^K\,, \tag{1.28}$$

so that

$$S = \frac{\rm i}{2}\int_{\Sigma} B_{MN} dX^M \wedge \left(\star\left(-{\rm i}J_K^N dX^K\right) + dX^N\right)\ . \tag{1.29}$$

$$dX^M = \partial X^M + \bar\partial X^M\,, \qquad \star dX^M = {\rm i}\left(\partial X^M - \bar\partial X^M\right) \tag{1.30}$$

1.4.2.

For $\Sigma = S^1 \times \mathbb{R}^1$ map the sigma model to the bosonic sector of a supersymmetric quantum mechanical system. What is the target space and what is the prepotential V?

1.4.3. *Yang-Mills action*

governs connections A on a principal G-bundle over the four dimensional Euclidean spacetime M^4:

$$S_{YM} = \frac{1}{4g_{YM}^2}\int_M {\rm Tr} F_A \wedge \star F_A + \frac{{\rm i}\vartheta}{8\pi^2}\int_M {\rm Tr} F_A \wedge F_A \tag{1.31}$$

In four dimensions $\star^2 = 1$ when acting on 2-forms.

1.4.4.

Take $M^4 = N^3 \times \mathbb{R}^1$. Map the gauge theory to the bosonic sector of a supersymmetric quantum mechanical system. What is the target space and what is the prepotential V?

1.5. Instantons in sigma models. Non-linear vs. gauged linear sigma models. Vortices, freckles, bubbles. Worldsheet instantons in string theory. Stable maps

1.5.1.

Bogomolny trick: define $\Pi = \frac{1}{2}(1 - \mathrm{i}J)$, $\bar{\Pi} = \frac{1}{2}(1 + \mathrm{i}J)$, and rewrite S as:

$$S = \|\Pi\bar{\partial}X\|^2 + \frac{\mathrm{i}}{2}\int X^*\omega \tag{1.32}$$

where

$$\omega_{MN} = B_{MN} + \mathrm{i}\left(G_{MN'}J_N^{N'} - G_{NM'}J_M^{M'}\right) \tag{1.33}$$

Show that, for closed B, $dB = 0$, the absolute minimum of the real part of the action in a given topological sector is achieved by the pseudoholomorphic maps $\Pi_M^N \bar{\partial} X^M = 0$.

1.5.2.

This is wrong. Correct this statement.

1.5.3. *Gauged linear sigma model*

Assume the target space Y has a group H of isometries, and at the same time symmetries of B. Couple the sigma model to H-gauge fields:

$$S = \frac{1}{2}\int_\Sigma G_{MN}\nabla X^M \wedge \star \nabla X^N + \frac{\mathrm{i}}{2}\int_\Sigma B_{MN}\nabla X^M \wedge \nabla X^N + \frac{1}{4e^2}\int_\Sigma \mathrm{Tr} F_A \wedge \star F_A + e^2 \int_\Sigma \|\mu\|^2 \tag{1.34}$$

where

$$\nabla X^M = dX^M + A^a V_a^M\ ,\ \nabla = \nabla' + \nabla''\ , \tag{1.35}$$

with

$$\nabla'' = \bar{\partial} + \bar{A} \tag{1.36}$$

$V_a \in Vect(Y)$ generate the H-action, $V_a^M \partial_M V_b^N - V_b^M \partial_M V_a^N = f_{ab}^c V_c^N$, and μ is the moment map:

$$d\mu^a = \iota_{V_a} B \tag{1.37}$$

1.5.4.

Assume there exists an H-invariant J, $J^2 = -1$, such that $B = GJ$, assume $dB = 0$ (Kähler manifold) and perform the gauged Bogomolny trick:

$$S = \|\Pi\nabla'' X\|^2 + \frac{1}{4e^2}\|F_A + 2e^2\mu\|^2 + \mathrm{i}\int X^*\omega \tag{1.38}$$

1.5.5. *Gauged instantons = vortex equations*

$$\nabla'' Z = 0\,,\ F_A + 2e^2\mu = 0 \tag{1.39}$$

comparing solutions to those of nonlinear sigma model $e^2 = \infty$.

Example of $\mathbb{CP}^{N-1}$.

Nonlinear/gauged linear sigma model instantons of degree d

$$(Z_1(\xi_0, \xi_1) : \ldots : Z_N(\xi_0, \xi_1)) \tag{1.40}$$

where

$$Z_a(\xi_0, \xi_1) = \sum_{j=0}^{d} Z_{a,d}\xi_0^j \xi_1^{d-a} \tag{1.41}$$

the coefficients $Z_{a,d}$ are defined up to an overall multiplier $\in \mathbb{C}^\times$, and are required to obey:

NLSM) For any point $(\xi_0 : \xi_1) \in \mathbb{CP}^1$ there exists at least one $a = 1, \ldots, N$, such that $Z_a(\xi_0, \xi_1) \neq 0$;

GLSM) There exists at least one point $(\xi_0 : \xi_1) \in \mathbb{CP}^1$ and there exists at least one $a = 1, \ldots, N$, such that $Z_a(\xi_0, \xi_1) \neq 0$;

Given the set of polynomials $Z_a(\xi_0, \xi_1)$ the solution to the vortex equations is found by finding the Hermitian metric $e^{-2\chi}$ on the line bundle L:

$$A = -\partial\chi,\ \bar{A} = \bar{\partial}\chi \tag{1.42}$$

where

$$\Delta\chi + 2e^2\left(e^{-2\chi}\sum_{a=1}^{N}|Z_a(\xi)|^2 - r\right) = 0 \tag{1.43}$$

When $e^2 \to \infty$ the solution chooses χ so that

$$\frac{1}{r}\sum_{a=1}^{N}|Z_a(\xi)|^2 \approx e^{2\chi} \tag{1.44}$$

However, near the common zero of all Z_a's, e.g. near $\xi = 0$, the equation looks as follows

$$\partial_\xi\bar{\partial}_{\bar{\xi}}\chi + 2e^2\left(c_1|\xi|^2 e^{-2\chi} - r\right) = 0 \tag{1.45}$$

which has a solution: $\chi(\xi, \bar{\xi}) \approx 2e^2 r|\xi|^2 - \frac{c_1}{2}e^2|\xi|^4 + \ldots$,

1.6. Instantons in gauge theory. ADHM construction, reciprocity

1.6.1.

Bogomolny trick:

$$S_{YM} = \frac{1}{4g_{YM}^2}\|F_A^+\|^2 + \frac{2\pi \mathrm{i}\tau}{8\pi^2}\int_M \mathrm{Tr} F_A \wedge F_A \tag{1.46}$$

Instantons, i.e. the solutions to

$$F_A^+ \equiv \frac{1}{2}\left(1+\star\right) F_A = 0 \tag{1.47}$$

1.6.2.

Prove conformal invariance of the Yang-Mills action.

1.6.3.

Prove that the round metric on S^4,

$$ds^2 = R_0^2\left(d\theta_1^2 + \sin^2(\theta_1)\left(d\theta_2^2 + \sin^2(\theta_2)\left(d\theta_3^2 + \sin^2(\theta_3)d\theta_4^2\right)\right)\right) \tag{1.48}$$

restricted onto $S^4 \backslash \infty \approx \mathbb{R}^4$ is conformally equivalent to flat metric on $\mathbb{R}^4$.

1.6.4. S^4 *vs* $\mathbb{R}^4$

The finite action instanton solution on $\mathbb{R}^4$ extend to S^4. $A \to g^{-1}dg + g^{-1}\alpha g$, as $|x| \to \infty$, where $g : S^3 \to G$, and $|d\alpha + \alpha\wedge\alpha| \sim O(\frac{1}{|x|^{2+\epsilon}}) \Longrightarrow |\alpha| \sim O(\frac{1}{|x|^{1+\epsilon}})$.

1.6.5. *Explicit solutions on* $M^3 \times \mathbb{R}^1$

Ansatz: $A = \Theta f(t)$, where Θ-flat connection on M^3,

$$F_A = \dot f dt \wedge \Theta + f d\Theta + f^2\Theta^2 = \dot f dt\wedge\Theta + f(f-1)\Theta^2 \tag{1.49}$$

$$S_{YM} = \frac{g_\Theta}{2}\dot f^2 + \frac{h_\Theta}{4} f^2 (f-1)^2 \tag{1.50}$$

1.6.6. *Dirac operator on* $\mathbb{R}^4$

Let E denote a rank N complex vector bundle over S^4. We restrict it on $\mathbb{R}^4$.

$$\begin{aligned} \not{D}_A &= \sigma^m\left(\partial_m + A_m\right) : \Gamma(S_+\otimes E) \to \Gamma(S_-\otimes E) \\ \not{D}_A^* &= \bar\sigma^m\left(\partial_m + A_m\right) : \Gamma(S_-\otimes E) \to \Gamma(S_+\otimes E) \end{aligned} \tag{1.51}$$

Let $M = \mathbb{R}^4 \approx \mathbb{C}^2$. Identify, by twisting with $K_M^{\frac{1}{2}}$:

$$S_+ = \Omega^{0,0}\oplus\Omega^{0,2} \qquad S_- = \Omega^{0,1} \tag{1.52}$$

then

$$\not{D}_A = \bar\partial_A + \bar\partial_A^\dagger \tag{1.53}$$

1.6.7.

Show that, $F_A^+ = 0 \Leftrightarrow F_A^{0,2} = 0, F_A^{1,1} \wedge \varpi = 0$, where

$$2\mathrm{i}\varpi = dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 \tag{1.54}$$

1.6.8.

Show that, for $G = SU(N)$,

$$k = -\frac{1}{8\pi^2}\int_M \mathrm{Tr} F_A \wedge F_A > 0 \tag{1.55}$$

one has the index theorem:

$$\mathrm{dim} ker_{L^2} \not{D}_A^* - \mathrm{dim} ker_{L^2} \not{D}_A = k \tag{1.56}$$

1.6.9.

and that

$$\not{D}_A (\eta \oplus \chi) = 0 \Longrightarrow \eta = \chi = 0 \tag{1.57}$$

Hint: The equations read, in components: $D_{\bar{\alpha}}\eta + \varepsilon_{\bar{\alpha}\bar{\beta}} g^{\beta\bar{\beta}} D_\beta \chi = 0$. Derive from this $\Delta_A \eta = 0$, $\Delta_A \chi = 0$.

1.6.10. *ADHM data*

Let $K \approx \mathbb{C}^k = ker_{L^2} \not{D}_A^*$. Define

$$B_\alpha : K \to K\,,\ B_\alpha^\dagger : K \to K \tag{1.58}$$

by

$$B_\alpha = P z_\alpha \cdot\,,\ B_\alpha^\dagger = P \bar{z}_\alpha \cdot \tag{1.59}$$

where $P : \Gamma_{L^2}(S_- \otimes E) \longrightarrow K$ is the orthogonal projection. Let $\psi \in K$, represented by an L^2-normalizable $(0,1)$-form ψ, valued in E. It obeys:

$$\bar{\partial}_A \psi = 0\,,\ \bar{\partial}_A^\dagger \psi = 0 \tag{1.60}$$

In components:

$$\psi = \psi_{\bar{1}} d\bar{z}_1 + \psi_{\bar{2}} d\bar{z}_2 \tag{1.61}$$

the Dirac equation reads:

$$\begin{aligned} D_{\bar{1}}\psi_{\bar{2}} - D_{\bar{2}}\psi_{\bar{1}} = 0\,, \\ D_1\psi_{\bar{1}} + D_2\psi_{\bar{2}} = 0\,, \end{aligned} \tag{1.62}$$

For large $r^2 = |z|^2 \to \infty$

$$\psi_{\bar{\alpha}} \sim -D_{\bar{\alpha}}\left(\frac{\xi_0^\dagger I^\dagger}{|z|^2}\right) - \varepsilon_{\bar{\alpha}\bar{\beta}} g^{\beta\bar{\beta}} D_\beta \left(\frac{\xi_0^\dagger J}{|z|^2}\right) \tag{1.63}$$

1.6.11. *ADHM equations*

$$[B_1, B_2] + IJ = 0 \tag{1.64}$$

$$[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = 0 \tag{1.65}$$

Hyperkahler moment maps. Hyperkahler structure. $\varpi_I, \varpi_J, \varpi_K$

1.6.12. *ADHM construction*

Define

$$\mathcal{D}^\dagger = \begin{pmatrix} B_1 - z_1 & B_2 - z_2 & I \\ -B_2^\dagger + \bar{z}_2 & B_1^\dagger - \bar{z}_1 & -J^\dagger \end{pmatrix} \tag{1.66}$$

Then

$$\mathcal{D}^\dagger \mathcal{D} = \Delta \otimes 1_{\mathbb{C}^2} \tag{1.67}$$

Let $\boldsymbol{\Xi}^\dagger = \left(\nu_+^\dagger \ \nu_-^\dagger \ \xi^\dagger \right)$, with $\nu_\pm : N \to K$, $\xi : N \to N$, solve:

$$\mathcal{D}^\dagger \boldsymbol{\Xi} = 0 \tag{1.68}$$

and be normalized

$$\boldsymbol{\Xi}^\dagger \boldsymbol{\Xi} = 1_N \tag{1.69}$$

Then A defined by:

$$A = \boldsymbol{\Xi}^\dagger d\boldsymbol{\Xi} \tag{1.70}$$

is antiselfdual. Indeed,

$$F_A = d\boldsymbol{\Xi}^\dagger d\boldsymbol{\Xi} + \boldsymbol{\Xi}^\dagger d\boldsymbol{\Xi}\boldsymbol{\Xi}^\dagger d\boldsymbol{\Xi} = d\boldsymbol{\Xi}^\dagger \left(1 - \boldsymbol{\Xi}\boldsymbol{\Xi}^\dagger\right) d\boldsymbol{\Xi} = \\ d\boldsymbol{\Xi}^\dagger \mathcal{D} \frac{1}{\Delta} \mathcal{D}^\dagger d\boldsymbol{\Xi} = \boldsymbol{\Xi}^\dagger \left(d\mathcal{D} \frac{1}{\Delta} d\mathcal{D}^\dagger \right) \boldsymbol{\Xi} \tag{1.71}$$

1.6.13. *Reciprocity*

$$\psi = \left(\nu_+^\dagger d\bar{z}_1 + \nu_-^\dagger d\bar{z}_2 \right) \Delta^{-1} \tag{1.72}$$

solves Dirac equation.

1.6.14.

Check that, i.e. compute:

$$\begin{aligned} D_{\bar{1}}\psi_{\bar{2}} - D_{\bar{2}}\psi_{\bar{1}} &= \Xi^{\dagger}\left(\bar{\partial}_{\bar{1}}\left(\Xi\nu_{-}^{\dagger}\Delta^{-1}\right) - \bar{\partial}_{\bar{2}}\left(\Xi\nu_{+}^{\dagger}\Delta^{-1}\right)\right) \\ D_{1}\psi_{\bar{1}} + D_{2}\psi_{\bar{2}} &= \Xi^{\dagger}\left(\partial_{1}\left(\Xi\nu_{+}^{\dagger}\Delta^{-1}\right) + \partial_{2}\left(\Xi\nu_{-}^{\dagger}\Delta^{-1}\right)\right) \end{aligned} \tag{1.73}$$

Hints: Use

$$\Xi\Xi^{\dagger} = 1_{K\otimes\mathbb{C}^2\oplus N} - \mathcal{D}\frac{1_{\mathbb{C}^2}}{\Delta}\mathcal{D}^{\dagger} \tag{1.74}$$

to compute,

$$\begin{aligned} \nu_{+}\nu_{+}^{\dagger}\Delta^{-1} &= \Delta^{-1} + \partial_1\Delta\bar{\partial}_{\bar{1}}\Delta^{-1} + \bar{\partial}_{\bar{2}}\Delta\partial_2\Delta^{-1}\,, & \nu_{+}\nu_{-}^{\dagger}\Delta^{-1} &= \partial_1\Delta\bar{\partial}_{\bar{2}}\Delta^{-1} - \bar{\partial}_{\bar{2}}\Delta\partial_1\Delta^{-1}\,, \\ \nu_{-}\nu_{+}^{\dagger}\Delta^{-1} &= \partial_2\Delta\bar{\partial}_{\bar{1}}\Delta^{-1} - \bar{\partial}_{\bar{1}}\Delta\partial_2\Delta^{-1}\,, & \nu_{-}\nu_{-}^{\dagger}\Delta^{-1} &= \Delta^{-1} + \bar{\partial}_{\bar{1}}\Delta\partial_1\Delta^{-1} + \partial_2\Delta\bar{\partial}_{\bar{2}}\Delta^{-1}\,, \\ \xi\nu_{+}^{\dagger}\Delta^{-1} &= -I^{\dagger}\bar{\partial}_{\bar{1}}\Delta^{-1} - J\partial_2\Delta^{-1}\,, & \xi\nu_{-}^{\dagger}\Delta^{-1} &= -I^{\dagger}\bar{\partial}_{\bar{2}}\Delta^{-1} + J\partial_1\Delta^{-1}\,. \end{aligned} \tag{1.75}$$

Thus, cf. (1.73)

$$\begin{aligned} D_{\bar{1}}\psi_{\bar{2}} - D_{\bar{2}}\psi_{\bar{1}} &= \nu_{+}^{\dagger}\gamma_{+} + \nu_{-}^{\dagger}\gamma_{-} + \xi^{\dagger}\gamma_0 = 0 \\ D_1\psi_{\bar{1}} + D_2\psi_{\bar{2}} &= \nu_{+}^{\dagger}\tilde{\gamma}_{+} + \nu_{-}^{\dagger}\tilde{\gamma}_{-} + \xi^{\dagger}\tilde{\gamma}_0 = 0 \end{aligned} \tag{1.76}$$

with

$$\begin{aligned} \gamma_{+} &= \bar{\partial}_{\bar{1}}\left(\nu_{+}\nu_{-}^{\dagger}\Delta^{-1}\right) - \bar{\partial}_{\bar{2}}\left(\nu_{+}\nu_{+}^{\dagger}\Delta^{-1}\right) = (B_2 - z_2)\Box\Delta^{-1}\,, \\ \gamma_{-} &= \bar{\partial}_{\bar{1}}\left(\nu_{-}\nu_{-}^{\dagger}\Delta^{-1}\right) - \bar{\partial}_{\bar{2}}\left(\nu_{-}\nu_{+}^{\dagger}\Delta^{-1}\right) = -(B_1 - z_1)\Box\Delta^{-1}\,, \\ \gamma_0 &= \bar{\partial}_{\bar{1}}\left(\xi\nu_{-}^{\dagger}\Delta^{-1}\right) - \bar{\partial}_{\bar{2}}\left(\xi\nu_{+}^{\dagger}\Delta^{-1}\right) = J\Box\Delta^{-1}\,, \\ \tilde{\gamma}_{+} &= \partial_2\left(\nu_{+}\nu_{-}^{\dagger}\Delta^{-1}\right) + \partial_1\left(\nu_{+}\nu_{+}^{\dagger}\Delta^{-1}\right) = -(B_1^{\dagger} - \bar{z}_1)\Box\Delta^{-1}\,, \\ \tilde{\gamma}_{-} &= \partial_2\left(\nu_{-}\nu_{-}^{\dagger}\Delta^{-1}\right) + \partial_1\left(\nu_{-}\nu_{+}^{\dagger}\Delta^{-1}\right) = -(B_2^{\dagger} - \bar{z}_2)\Box\Delta^{-1}\,, \\ \tilde{\gamma}_0 &= \partial_2\left(\xi\nu_{-}^{\dagger}\Delta^{-1}\right) + \partial_1\left(\xi\nu_{+}^{\dagger}\Delta^{-1}\right) = -I^{\dagger}\Box\Delta^{-1}\,, \end{aligned} \tag{1.77}$$

where

$$\Box = \partial_1\bar{\partial}_{\bar{1}} + \partial_2\bar{\partial}_{\bar{2}} \tag{1.78}$$

and use (1.68)

1.7. Nakajima-Gieseker compactification, noncommutative instantons

Deform the ADHM equations to

$$[B_1, B_1^{\dagger}] + [B_2, B_2^{\dagger}] + II^{\dagger} - J^{\dagger}J = \zeta\cdot 1_K \tag{1.79}$$

1.7.1.

Prove stability theorem: ADHM equations with $\zeta > 0$ modulo $U(K)$ are equivalent to t the complex equation plus stability $\mathbb{C}[B_1, B_2]I(N) = K$ modulo $GL(K)$.

1.8. Supersymmetric localisation, fixed points, colored partitions, compactness theorem

$\mathcal{N}=2$ $d=4$ twisted theory: $(A,\psi,\sigma;\chi,\eta,\bar\sigma)$. $\mathcal{Q}$-supercharge.

ADHM version of the $\mathcal{N}=2$ multiplet

Ω-deformation. $\varepsilon_1,\varepsilon_2$-parameters.

Fixed points in the ADHM description:

$$\begin{aligned} \varepsilon_\alpha B_\alpha &= [\phi, B_\alpha] \\ 0 &= \phi I - I\mathrm{a} \\ (\varepsilon_1+\varepsilon_2)J &= \mathrm{a}J - J\phi \end{aligned} \tag{1.80}$$

Define $N_\alpha = \ker(\mathrm{a}-a_\alpha)$, $K_\alpha = \mathbb{C}[B_1,B_2]I(N_\alpha)$. Since $\phi|_{K_\alpha}\subset a_\alpha+\varepsilon_1\mathbb{Z}_{\geq 0}+\varepsilon_2\mathbb{Z}_{\geq 0}$, for generic a, $K_\alpha\cap K_\beta=\emptyset$, for $\alpha\neq\beta$, and $J=0$. Define $\lambda^{(\alpha)}$ by: $1\leq j\leq\lambda_i^{(\alpha)}\Leftrightarrow B_1^{i-1}B_2^{j-1}I(N_\alpha)\neq 0$. For generic $\varepsilon_1,\varepsilon_2$ these vectors are linearly independent (different ϕ eigenvalues) hence form a basis of K.

For non-generic $\varepsilon_1,\varepsilon_2$, as long as both $\varepsilon_1\neq 0$ and $\varepsilon_2\neq 0$, and $a_\alpha - a_\beta\notin\varepsilon_1\mathbb{Z}_{>0}+\varepsilon_2\mathbb{Z}_{>0}$ we claim the set of fixed points is compact.

It suffices to prove it for $\varepsilon_1=p$, $\varepsilon_2=q$, with $p,q\in\mathbb{Z}_{>0}$. We still prove $J=0$ using that a's are generic. It suffices to consider the case $N=1$ (again, for generic a). Let $K_n=\ker(\phi-n)$. Then $B_1(K_n)\subset K_{n+p}, B_2(K_n)\subset K_{n+q}$.

Since $\mathrm{Tr}II^\dagger=\zeta k$, $\mathrm{Tr}_{K_n}II^\dagger\leq\zeta k$. Let $k_n=\mathrm{dim}K_n$.

Define

$$\delta_n = \frac{1}{\zeta}\mathrm{Tr}_{K_n}\left(B_1B_1^\dagger + B_2B_2^\dagger + II^\dagger\right)\leq k_n+\delta_{n+p}+\delta_{n+q} \tag{1.81}$$

1.8.1.

Define generalized Fibonacci numbers: for $p>q>0$,

$$\begin{aligned} &F_n = 0\,, \qquad 1-p\leq n\leq 0 \\ &F_1 = 1, \\ &F_n = F_{n-p}+F_{n-q}\,, \qquad n>0 \end{aligned} \tag{1.82}$$

Find a formula for F_n. *Hint:* use the solutions to the equation

$$x^p = x^{p-q}+1 \tag{1.83}$$

1.8.2. *Proof of compactness theorem*

By induction:

$$\delta_n \leq \sum_{n'\geq 0} k_{n+n'}F_{n'+1} \tag{1.84}$$

Indeed, $\delta_{n'}=0$ for $n'\gg 0$, and, assuming the above for all $n'\geq n$,

$$\begin{aligned} \delta_{n-1}\leq k_{n-1}+\delta_{n-1+p}+\delta_{n-1+q}\leq k_{n-1}+\sum_{n'\geq 0}(k_{n-1+p+n'}+k_{n-1+q+n'})F_{n'+1}\leq \\ k_{n-1}+\sum_{n''\geq q}k_{n-1+n''}F_{n''+1}=\sum_{n''\geq 0}k_{n-1+n''}F_{n''+1} \end{aligned} \tag{1.85}$$

Therefore

$$\sum_{n\geq 0}\delta_n \leq k\sum_{n'=0}^{k} F_{n'+1} = O(ke^{c_1 k}) \tag{1.86}$$

For $q > 0 > p$ define an operator $H = B_1^q B_2^{-p}$ and use Jacobsen-Morozov.

1.9. Crossed instantons, qq-characters, non-perturbative Dyson-Schwinger equations. Seiberg-Witten geometry

1.9.1. $\mathcal{N} = 4$ *theory*

Additional fields. The bosons: $(A, B^+, C; \sigma, \bar{\sigma})$

Modified instanton equations:

$$\begin{aligned} &F_A^+ + [C, B^+] + [B, B]^+ = 0 \\ &D_A^* B + D_A C = 0 \end{aligned} \tag{1.87}$$

Modified ADHM data: (B_a, I, J), $a = 1, 2, 3, 4$.

Modified ADHM equations:

$$\begin{aligned} &[B_1, B_3] + [B_4, B_2]^\dagger = 0 \\ &[B_1, B_4] + [B_2, B_3]^\dagger = 0 \\ &[B_1, B_2] + IJ + [B_3, B_4]^\dagger = 0 \\ &\sum_{a=1}^{4}[B_a, B_a^\dagger] + II^\dagger - J^\dagger J = \zeta \cdot 1_K \end{aligned} \tag{1.88}$$

$$\begin{aligned} &B_3 I + B_4^\dagger J^\dagger = 0 \\ &B_4 I - B_3^\dagger J^\dagger = 0 \end{aligned}$$

1.9.2.

Prove the algebraic stability condition: for $\zeta > 0$ these equations are equivalent to: $[B_a, B_c] = 0$, $a = 1, 2$, $c = 3, 4$, $[B_1, B_2] + IJ = 0$, $[B_3, B_4] = 0$, $B_3 I = B_4 I = 0$, $JB_3 = JB_4 = 0$, $\mathbb{C}[B_1, B_2, B_3, B_4]I(N) = K$.

1.9.3.

Prove the vanishing theorem: $B_3, B_4 = 0$ on the solutions to the equations above.

1.9.4. *Crossed instantons*

$(B_a, I, J, \tilde{I}, \tilde{J})$, $a = 1, 2, 3, 4$.

$$I : N \to K\,,\ \ J : K \to N\,,\ \ \tilde{I} : \tilde{N} \to K\,,\ \ \tilde{J} : K \to \tilde{N} \tag{1.89}$$

Crossed ADHM equations:

$$\begin{aligned}
&[B_1, B_3] + [B_4, B_2]^\dagger = 0 \\
&[B_1, B_4] + [B_2, B_3]^\dagger = 0 \\
&[B_1, B_2] + IJ + \left([B_3, B_4] + \tilde{I}\tilde{J}\right)^\dagger = 0 \\
&\sum_{a=1}^{4}[B_a, B_a^\dagger] + II^\dagger + \tilde{I}\tilde{I}^\dagger - J^\dagger J - \tilde{J}^\dagger\tilde{J} = \zeta \cdot 1_K
\end{aligned} \tag{1.90}$$

$$\begin{aligned}
&B_3 I + B_4^\dagger J^\dagger = 0 \\
&B_4 I - B_3^\dagger J^\dagger = 0 \\
&B_1 \tilde{I} + B_2^\dagger \tilde{J}^\dagger = 0 \\
&B_2 \tilde{I} - B_1^\dagger \tilde{J}^\dagger = 0
\end{aligned}$$

1.9.5.

Prove the stability theorem: the crossed ADHM equations modulo $U(K)$ are equivalent to the complex equations,

$$[B_1, B_3] = [B_2, B_3] = [B_1, B_4] = [B_2, B_4] = 0\ , \tag{1.91}$$

$$[B_1, B_2] + IJ = 0\,, \qquad [B_3, B_4] + \tilde{I}\tilde{J} = 0\ , \tag{1.92}$$

with $K_{12} = \mathbb{C}[B_1, B_2]I(N)$, $K_{34} = \mathbb{C}[B_3, B_4]\tilde{I}(\tilde{N})$ with $K_{12} + K_{34} = K$, and $B_1(K_{34}) = B_2(K_{34}) = 0$, $B_3(K_{12}) = B_4(K_{12}) = 0$.

Compactness theorem: for any subtorus $T \subset U(1)^3 \subset SU(4)$, such that $\varepsilon_a \neq 0$ for any $a = 1, 2, 3, 4$, and generic a, ã, the set of fixed points is compact.

1.9.6. $\widehat{A}_0$*-type qq-characters*

Integration over the Hilbert scheme of points on $\mathbb{C}^2$ using localisation. Fixed points as Young diagrams. Tangent space at the fixed point λ. Weight decomposition: arms-legs formula.

Crossed instantons of rank $(1, n)$. Integrating out the phantom part $\Longrightarrow$ the fundamental qq-character. Explicit formula.

Here we give the expression for the fundamental character $\mathcal{X}_1(x) \equiv \mathcal{X}_{1,0}(x)$:

$$\begin{aligned}
\mathcal{X}_1(x) &= \sum_{\lambda} \mathfrak{q}^{|\lambda|} \prod_{\square\in\lambda} \mathbb{S}(\mathtt{m}h_\square + \varepsilon a_\square) \cdot \frac{\prod_{\square\in\partial_+\lambda} \mathcal{Y}(x + \sigma_\square + \varepsilon)}{\prod_{\square\in\partial_-\lambda} \mathcal{Y}(x + \sigma_\square)} = \\
&= \mathcal{Y}(x+\varepsilon) \sum_{\lambda} \mathfrak{q}^{|\lambda|} \prod_{\square\in\lambda} \mathbb{S}(\mathtt{m}h_\square + \varepsilon a_\square) \cdot \prod_{\square\in\lambda} \frac{\mathcal{Y}(x + \sigma_\square - \mathtt{m})\mathcal{Y}(x + \sigma_\square + \mathtt{m} + \varepsilon)}{\mathcal{Y}(x + \sigma_\square)\mathcal{Y}(x + \sigma_\square + \varepsilon)} = \\
&= \mathcal{Y}(x+\varepsilon) + \ \mathfrak{q}\ \mathbb{S}(\mathtt{m})\, \frac{\mathcal{Y}(x - \mathtt{m})\mathcal{Y}(x + \varepsilon + \mathtt{m})}{\mathcal{Y}(x)} + \ldots
\end{aligned} \tag{1.93}$$

Here

$$\sigma_{\square} = \mathtt{m}(i-j) + \varepsilon(1-j) \tag{1.94}$$

is the content of $\square$ defined relative to the pair of weights $(\mathtt{m}, -\mathtt{m}-\varepsilon)$, and

$$\mathbb{S}(x) = 1 + \frac{\varepsilon_1 \varepsilon_2}{x(x+\varepsilon)} \tag{1.95}$$

1.9.7. *Quiver theories from Orbifolds*

$\Gamma \subset SU(2)$ a finite subgroup. Imposing orbifold projection to produce quiver theory or an ALE theory. Crossed instantons on $\mathbb{C}^2/\Gamma \times \mathbb{C}^2$ and qq-characters of quiver gauge theories. Limit to finite quivers.

1.9.8. *Regularity from compactness*

Non-perturbative Dyson-Schwinger equations.

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