5 Exercise session 5, July 23

5.1 Exercises for Maxim Zabzine's lecture

Exercise 5.1. Redo all calculations in Maxim Zabzine's lecture in the manifold setting, in particular the determinant

$$\alpha_0(0)\frac{\mathrm{Pf}(S)}{\sqrt{\det H}} = \alpha_0(0)\frac{1}{\sqrt{\det \partial_\mu V^\rho(0)}}. \tag{1}$$

Exercise 5.2. Redo all calculations in Maxim Zabzine's lecture in the Grassmann odd vector bundle setting, in particular check

$$\frac{\det^{1/4}(-R_1^2+\frac{1}{4}DD^\dagger)}{\det^{1/4}(-R_0^2+\frac{1}{4}D^\dagger D)} = \frac{\det^{1/4}_{\ker DD^\dagger}(-R_1^2)}{\det^{1/4}_{\ker D^\dagger D}(-R_0^2)} = \frac{\det^{1/2} R_1}{\det^{1/2} R_0} \qquad \text{up to phases.} \tag{2}$$

One strategy is as follows. First note that R_1^2 and $DD^\dagger$ commute and can be co-diagonalized (similarly for R_0^2 and $D^\dagger D$). Note that $(-R_1^2+\frac{1}{4}DD^\dagger)$ and $(-R_0^2+\frac{1}{4}D^\dagger D)$ act on different spaces. Using D, map eigenvectors of $DD^\dagger$ in one space to eigenvectors of $D^\dagger D$ in the other space, with the same eigenvalue. Do this in reverse to get back from the kernels to the whole spaces.

5.2 Exercises for Nikita Nekrasov's lecture

Exercise 5.3. Exercise: study the Airy function

$$A_\hbar(x) = \int \mathrm{d}t e^{(i/\hbar)(tx-t^3/3)}, \tag{3}$$

find its two critical points and its Lefschetz thimbles. The Airy function obeys a second-order differential equation.

5.3 Exercises for Takuya Okuda's lectures

Exercise 5.4. Show that the space of field configurations obeying the following boundary conditions (*Neumann boundary condition*) at $x^1=0$ is invariant under B-type supersymmetry transformations generated by $\epsilon \propto \bar{\epsilon} \propto \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ in flat space:

$$\partial_1\phi = 0\,, \quad \psi_1+\psi_2 = 0\,,$$

$$\partial_1(\psi_1-\psi_2) = 0\,, \quad \mathrm{F} = 0\,.$$

For SUSY transformations see Appendix A of 1308.2217.

Exercise 5.5. Show that the space of field configurations obeying the following boundary conditions (*Dirichlet boundary condition*) at $x^1=0$ is invariant under the same transformations:

$$\phi = \text{const.}\,, \quad \psi_1-\psi_2 = 0\,,$$

$$\partial_1(\psi_1+\psi_2) = 0\,, \quad \partial_1(\mathrm{F}+i\partial_1\phi) = 0\,.$$

Exercise 5.6. Let G denote the homogeneous polonomial

$$G(\boldsymbol{x}) = x_1^d + \ldots x_N^d\,,$$

where $\boldsymbol{x}=(x_1,\ldots,x_N)$. We can write

$$G(\boldsymbol{x}) - G(\boldsymbol{y}) = \sum_{j=1}^N (x_j-y_j)A_j(\boldsymbol{x},\boldsymbol{y})\,,$$

for some A_j. (Actually this is true for any homogeneous G.) Let p and q be complex variables, and assume that "operators" α and β satisfy the relations

$$\alpha^d = p\,, \quad \beta^d = q\,.$$

Let us introduce fermionic oscillators

$$\{\eta_j, \bar{\eta}_k\} = \delta_{jk}\,, \quad \{\eta_j, \eta_k\} = \{\bar{\eta}_j, \bar{\eta}_k\} = 0\,.$$

with the Clifford vacuum defined by $\eta_j|0\rangle = 0$. Show that

$$Q_0(p, \boldsymbol{x}, q, \boldsymbol{y}) := \sum_{j=1}^{N} \Big((\alpha x_j - \beta y_j)\overline{\pi}_j + A_j(\alpha \boldsymbol{x}, \beta \boldsymbol{y})\pi_j \Big)$$

preserves the space V spanned by

$$\overline{\eta}_{i_1} \dots \overline{\eta}_{i_s} \alpha^a \beta^b |0\rangle$$

with $1 \leq i_1 < \ldots < i_s \leq N$, $0 \leq s \leq N$, $0 \leq a < d$, $0 \leq b < d$, $a + b \equiv s \mod d$. Also show that the restriction to V

$$Q(p, \boldsymbol{x}, q, \boldsymbol{y}) := Q_0|_V$$

is a matrix factorization of $W = p\,G(\boldsymbol{x}) - q\,G(\boldsymbol{y})$:

$$Q(p, \boldsymbol{x}, q, \boldsymbol{y})^2 = (p\,G(\boldsymbol{x}) - q\,G(\boldsymbol{y}))\mathrm{id}_V\,.$$

If you are bored, compute the hemisphere partition function for Q (with appropriate gauge and R-charge assignments) and check that it coincides with the sphere partition function for a Calabi-Yau hypersurface (for $d = N$). (See 0806.4734 by Brunner, Jockers, and Roggenkamp.)

Exercise 5.7. Show that the SQED hemisphere partition function can be expanded in vortex partition functions. (See Problem set 3 for the sphere case.)