Curved-space supersymmetry

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Nobody (even the typist) proof-read these notes, so there may be obvious mistakes: tell BLF.

Abstract

We discuss aspects of supersymmetry on curved-space, with an emphasis on 4d $\mathcal{N}=1$ theories. Topics include supercurrents, rigid supergravity, curved superspace (for S^4 and $S^3\times\mathbb{R}$), a classification of 4d $\mathcal{N}=1$ supersymmetric backgrounds and the dependence of supersymmetric observables on the background geometry. Some of these topics are also described for 3d $\mathcal{N}=2$ and 4d $\mathcal{N}=2$ supersymmetries. These are lecture notes for the 2018 IHÉS summer school on *Supersymmetric localization and exact results.*

These lecture notes assume familiarity with supersymmetry at the level of the first few chapters of the book by Wess and Bagger.

1 Lecture 1, July 16 — supercurrents

Supersymmetric localization is a very powerful technique to compute exactly observables in even very strongly coupled gauge theories. It is often important to deform the theory from its flat-space definition, to define the theory on curved spaces. In particular placing the theory on a compact manifold puts a IR cutoff. Many applications. A few older ones.

- Put supersymmetric field theory on T^4, compute Witten index $\mathrm{Tr}(-1)^F$.
- Topologically twist 4d $\mathcal{N}=2$ theory so that it can live on any compact manifold of our choosing.
- Omega background, very successfully used by Nekrasov to compute instanton partition functions.

More recent ones

- 4d $\mathcal{N}=2$ theories on S^4 (Pestun [**?**]) computed the partition function and expectation value of Wilson lines. Note that the theory on S^4 preserves 8 supercharges, much more than what was preserved by the topological twist.

- 3d $\mathcal{N}=2$ theories with $U(1)_R$ symmetry on S^3 (first Kapustin–Willett–Yaakov, Hama–Hosomichi–Lee, Jafferis).
- 2d $\mathcal{N}=(2,2)$ theories on S^2 (Benini–Cremonesi, Doroud–Gomis–Le Floch–Lee).
- 5d $\mathcal{N}=1$ on S^5.
- 4d $\mathcal{N}=1$ with $U(1)_R$ on $S^3 \times S^1$ (Romelsberger), namely $\mathrm{Tr}(-1)^F e^{-\beta H}$ where the trace ranges over the Hilbert space of the theory on S^3.

All of these geometries can be deformed. Some deformations do not change supersymmetric observables; others do. What do observables depend on?

We are led to three questions.

- Given a supersymmetric field theory, on which geometries can it be placed, preserving some supersymmetry?
- What is the structure of the resulting theory? (E.g., what couplings can we write down on curved manifolds.)
- How do supersymmetric observables depend on the geometry?

1.1 Preliminaries

When placing a theory on curved space, namely changing the metric from the flat metric $\eta_{\mu\nu}$ to $\eta_{\mu\nu}+h_{\mu\nu}$, the Lagrangian of the theory changes by $h_{\mu\nu}T^{\mu\nu}$, by definition of the stress-energy tensor.

In a supersymmetric field theory $T_{\mu\nu}$ is part of a multiplet, called a *supercurrent* (see Komargodski–Seiberg `https://arxiv.org/abs/1002.2228`, Dumitrescu–Seiberg `https://arxiv.org/abs/1106.0031` and references therein).

In particular we will consider 4d $\mathcal{N}=1$

$$\{Q_\alpha, \overline{Q}_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu + \cdots \tag{1}$$

$$\{Q_\alpha, Q_\beta\} = 0 + \cdots, \{\overline{Q}_\alpha, \overline{Q}_\beta\} = 0 + \cdots \tag{2}$$

where the dots are string and domain wall charges that we will see later (similar to central charges of 4d $\mathcal{N}=2$).

In a local field theory, translation-invariance implies the existence of a conserved real tensor $T_{\mu\nu}$; Poincaré invariance implies that $T_{\mu\nu}$ can be improved to be symmetric. Its conserved charge is momentum $P_\mu = \int \mathrm{d}^3x T^0_\mu$. It is not unique, for instance $T_{\mu\nu} \to T_{\mu\nu} + \frac{1}{2}(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2)u$ for any scalar u preserves symmetry and conservation.

In the same multiplet as $T_{\mu\nu}$ there is also $S_{\alpha\mu}$ and $\overline{S}_{\dot\alpha\mu}$, conserved ($\partial^\mu S_{\mu\alpha} = \partial^\mu \overline{S}_{\mu\dot\alpha} = 0$) and such that $Q_\alpha = \int \mathrm{d}^3x S^0_\alpha$. Again, it is not unique: improvements take the form $S_{\alpha\mu} \to S_{\alpha\mu} + 2(\sigma_{\mu\nu})_\alpha{}^\beta \partial^\nu \eta_\beta$.

We will search for a supermultiplet containing $(T_{\mu\nu}, S_{\mu\alpha}, \overline{S}_{\mu\dot{\alpha}})$ and such that no other operator in the multiplet has spin > 1. The largest-spin component of a supermultiplet resides in its $\theta\sigma^\nu\overline{\theta}$ component. Here we want it to be $T_{\mu\nu}$, so

$$\mathbb{S}_\mu = \cdots + T_{\mu\nu}\theta\sigma^\nu\overline{\theta} + \cdots \tag{3}$$

We want operators to be well-defined. We would like to focus on indecomposable multiplets, namely it cannot be written as a direct sum of multiplets. This is different from irreducibility, namely we allow multiplets that contain smaller multiplets. (In fact the supersymmetry algebra is not semi-simple, which means there are representations that are not irreducible nor direct sums of irreducible ones.)

1.2 S-multiplet

What does the S-multiplet consist of? This is a long calculation ($\Longrightarrow$ Exercise). There are 16 bosonic and 16 fermionic physical degrees of freedom. Bosons.

- $T_{\mu\nu}$ (has 10 (by symmetry) minus 4 (by conservation) equals 6 physical components)
- $F_{[\mu\nu]}$ closed 2-form namely $\partial_{[\mu}F_{\nu\rho]} = 0$ (has 3 physical components[1])
- Complex Y_μ with $\partial_{[\mu}Y_{\nu]} = 0$ (has 2 physical components)
- A real scalar A (has 1 physical component)
- A current that is generically not conserved j_μ (has 4 physical components)

Fermions.

- $S_{\mu\alpha}$ and $\overline{S}_{\mu\dot{\alpha}}$ conserved $6 + 6$ components
- ψ_α, $\overline{\psi}_{\dot{\alpha}}$ has $2 + 2$ components

Constraints

$$\overline{D}^{\dot{\alpha}}\mathbb{S}_{\alpha\dot{\alpha}} = \chi_\alpha + Y_\alpha \tag{4}$$

$$\overline{D}_{\dot{\alpha}}\chi_\alpha = 0 \tag{5}$$

$$D^\alpha\chi_\alpha = \overline{D}_{\dot{\alpha}}\overline{\chi}^{\dot{\alpha}} \tag{6}$$

$$D_\alpha Y_\beta + D_\beta Y_\alpha = 0, \quad \overline{D}^2 Y_\alpha = 0 \tag{7}$$

[1] The two-form F has 6 components; the condition $\mathrm{d}F = 0$ has 4 equations; but they are related by the single equation $\mathrm{d}^2F = 0$, leading to $6 - 4 + 1 = 3$ components. Another way to count is that the closed two-form F is locally $\mathrm{d}A$ where A has 4 components but one gauge redundancy $A \to A + \mathrm{d}\Lambda$, giving $4 - 1 = 3$ components. See exercise 3.2.

(sometimes in the literature the last line is written $Y_\alpha = D_\alpha X$ and $\overline{D}_{\dot\alpha} X = 0$, but that is less general). In components

$$\begin{aligned}\mathbb{S}_\mu = j_\mu - i\theta\left(S_\mu - \frac{i}{\sqrt{2}}\sigma_\mu\psi\right) + i\overline{\theta}(\overline{S}_\mu + \cdots) + \frac{i}{2}\theta^2 Y_\mu - \frac{i}{2}\overline{\theta}^2\overline{Y}_\mu \\ + \theta\sigma^\nu\overline{\theta}\left(2T_{\mu\nu} - \eta_{\mu\nu}A - \frac{1}{8}\epsilon_{\nu\mu\rho\sigma}F^{\rho\sigma} - \frac{1}{2}\epsilon_{\nu\mu\rho\sigma}\partial^\rho j^\sigma\right) + \ldots\end{aligned} \tag{8}$$

Exercise: check that the constraints on the superfield $\mathbb{S}_{\alpha\dot\alpha}$ imply that $T_{\mu\nu}$ is conserved, $F_{\mu\nu}$ is a closed 2-form etc.

1.3 Back to the superalgebra

Using the above superfield, acting with a supercharge, we find

$$\{\overline{Q}_{\dot\alpha}, S_{\mu\alpha}\} = 2\sigma^\nu_{\alpha\dot\alpha}(T_{\mu\nu} + C_{[\mu\nu]}) + \text{Schwinger terms.} \tag{9}$$

Schwinger terms are total derivative (topological) terms. Here $C_{\mu\nu} = \frac{-1}{16}\epsilon_{\mu\nu\rho\lambda}F^{\rho\lambda}$ is conserved (because F is closed). Such an antisymmetric conserved current is a string current. Integrating the zeroth component we find

$$\{\overline{Q}_{\dot\alpha}, Q_\alpha\} = 2\sigma^\nu_{\alpha\dot\alpha}(P_\nu + Z_\nu) \tag{10}$$

where $Z_\nu = \int \mathrm{d}^3x C^0_\nu$ is a string charge. The reason the algebra is never written this way is that the charge is infinite for a straight (infinitely long) string. However, the charge density that shows up in $\{\overline{Q}_{\dot\alpha}, S_{\mu\alpha}\}$ is finite.

Similarly,

$$\{Q_\alpha, S_{\mu\beta}\} = \sigma^{\mu\nu}_{\alpha\beta}C_{[\mu\nu\rho]} \tag{11}$$

where $C_{[\mu\nu\rho]} = \epsilon_{\mu\nu\rho\lambda}\overline{Y}^\lambda$ is conserved; it is a *domain wall current*. Again, the charge would be infinite for any object charged under it. We find

$$\{Q_\alpha, Q_\beta\} = \sigma^{\mu\nu}_{\alpha\beta}Z_{\mu\nu} \tag{12}$$

Note that this is superficially similar to the central charge of 4d $\mathcal{N} = 2$, but the difference is that $Z_{\mu\nu}$ is a charge for extended objects.

Schwinger terms are total derivative terms that do not contribute to the conserved charge, determined by the structure of the S-multiplet. They are given in the paper by Dumitrescu–Seiberg cited above.

1.4 Improvements

In terms of a real superfield U with $U = u + \theta\eta + \overline{\theta}\overline{\eta} + \theta^2 N + \overline{\theta}^2\overline{N} - (\theta\sigma^\mu\overline{\theta})V_\mu + \cdots$ (actually U is well-defined up to a constant)

$$S_{\alpha\dot\alpha} \to S_{\alpha\dot\alpha} + [D_\alpha, \overline{D}_{\dot\alpha}]U \tag{13}$$

$$\chi_\alpha \to \chi_\alpha + \frac{3}{2}\overline{D}^2 D_\alpha U \tag{14}$$

$$Y_\alpha \to Y_\alpha + \frac{1}{2}D_\alpha\overline{D}^2 U \tag{15}$$

Note that U is only defined up to a constant shift. How does that improvement transformation act on components? For $T_{\mu\nu}$ and $S_{\mu\alpha}$ we reproduce the improvements above (with the same notations). Other components transform as

$$F_{\mu\nu} \to F_{\mu\nu} - 6(\partial_\mu V_\nu - \partial_\nu V_\mu) \tag{16}$$
$$Y_\mu \to Y_\mu - 2\partial_\mu N. \tag{17}$$

In some theories, improvements allow setting some of the components to zero, in other words shortening the S-multiplet even further.

1.5 Example: Wess–Zumino models

Wess-Zumino models are theories built from a chiral superfield Φ^i (and anti-chiral field $\overline{\Phi}^i$). The Lagrangian is described

- by a Kähler potential $K(\Phi^i, \overline{\Phi}^i)$ defined up to Kähler transformation $K \to K + \Lambda(\Phi^i) + \overline{\Lambda}(\overline{\Phi}^i)$ (which leaves $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$ invariant) and
- by a superpotential $W(\Phi^i)$ defined up to a constant.

The S-multiplet and its friends are

$$S_{\alpha\dot{\alpha}} = 2g_{i\bar{j}} D_\alpha \Phi^i \overline{D}_{\dot{\alpha}} \overline{\Phi}^{\bar{j}} \tag{18}$$
$$\chi_\alpha = \overline{D}^2 D_\alpha K \tag{19}$$
$$Y_\alpha = 4 D_\alpha W \tag{20}$$

In components we find the expected $T_{\mu\nu}$ and for instance

$$F_{\mu\nu} \sim i g_{i\bar{j}} \partial_{[\mu} \Phi^i \partial_{\nu]} \overline{\Phi}^{\bar{j}} \tag{21}$$

Note that this is the pull-back of the Kähler form $\Omega = i g_{i\bar{j}} \mathrm{d}\Phi^i \wedge \mathrm{d}\overline{\Phi}^{\bar{j}}$. Locally, $\Omega = \mathrm{d}A$ with $A = -\frac{i}{2}\partial_i K \mathrm{d}\Phi^i + \frac{i}{2}\partial_{\bar{j}} K \mathrm{d}\overline{\Phi}^{\bar{j}}$. Importantly, A is not invariant under Kähler transformations. When the Kähler potential is not globally well-defined, $F_{\mu\nu}$ is closed but not exact.

Example: $\mathbb{CP}^1$ model has $K = f^2 \log(1 + \overline{\Phi}\Phi)$, not globally well-defined, so we find that $F_{\mu\nu}$ is closed but not exact, so it cannot be improved away.

In components we also find

$$Y_\mu = \partial_i W \partial_\mu \Phi^i \tag{22}$$

so this is closed but not exact when W is not globally well-defined. Again, it cannot be improved away. Example: $\Phi \sim \Phi + 1$ and $W = \Phi$.

2 Lecture 2, July 17 — coupling to supergravity

Recall that the S-multiplet contains

- $T_{\mu\nu}$ (conserved symmetric stress-energy tensor),
- $C_{\mu\nu} = \frac{-1}{16}\epsilon_{\mu\nu\rho\lambda}F^{\rho\lambda}$ (conserved string current),
- $C_{\mu\nu\rho} = -\epsilon_{\mu\nu\rho\lambda}Y^{\lambda}$ (conserved domain wall current),
- j_μ (generically non-conserved R-current),
- A (real scalar),
- $S_{\mu\alpha}$, $\overline{S}_{\mu\dot{\alpha}}$ (conserved supersymmetry currents),
- ψ_α, $\overline{\psi}_{\dot{\alpha}}$ fermions.

The supersymmetry algebra contains string and brane charges Z_μ and $Z_{\mu\nu}$ as

$$\{Q_\alpha, \overline{Q}_{\dot{\alpha}}\} = 2\sigma^{\mu}_{\alpha\dot{\alpha}}(P_\mu + Z_\mu), \qquad \{Q_\alpha, Q_\beta\} = \sigma^{\mu\nu}_{\alpha\beta}Z_{\mu\nu}. \tag{23}$$

2.1 Shortened S-multiplets

In some theories some components of the S-multiplet can be set to zero by an improvement transformation.

- Ferrara–Zumino multiplet: if $F_{\mu\nu}$ is exact (not just closed) then it can be set to zero we can set $\chi_\alpha = 0$ by an improvement.

- Assume that $\chi_\alpha = \frac{-3}{2}\overline{D}^2 D_\alpha U$. Then we can set $\chi_\alpha = 0$ by an improvement.

  Thus $F_{\mu\nu} = 0$ (the string current is trivial), so before the improvement, F had to be exact, and we also find $A = 4T^{\mu}_{\mu}$ and $\psi_\alpha \sim (\sigma^\mu \overline{S})_\alpha$. This multiplet is called the Ferrara–Zumino multiplet. It has 12 bosonic and 12 fermionic degrees of freedom.

- Assume that $Y_\alpha = \frac{-1}{2}D_\alpha\overline{D}^2 U$ so that we can improve to $Y_\alpha = 0$. Then $A = 0$, $Y_\mu = 0$ (the domain wall current is zero), $\psi_\alpha = 0$, $\partial^\mu j_\mu = 0$. This last current, now conserved, is actually a conserved R-symmetry current.

  Reversing the logic, if a theory has a conserved $U(1)_R$ current then it has that R-symmetry multiplet.

- Both reduction can happen at the same time. Then we find $T^{\mu}_{\mu} = A = 0$, hence the theory must be a SCFT, and we also find $\partial^\mu j_\mu = 0$, $F_{\mu\nu} = 0$, $Y_\nu = 0$, $\psi_\alpha = 0$, $\sigma^\mu \overline{S}_\mu = 0$. This multiplet has $8 + 8$ components.

  Any mass-deformation, such as a superpotential mass term, will break superconformal invariance hence forbids the existence of such a short multiplet.

Example: in a WZ model where the Kähler potential cannot be globally well-defined (e.g., when the target is compact), F cannot be exact so the theory cannot have a Ferrara–Zumino multiplet.

Example: in a $U(1)$ gauge theory with FI term $\big($namely $\mathcal{L} = \frac{1}{4e^2}\int d^2\theta W^\alpha W_\alpha +$ c.c. $+ \xi \int d^4\theta V\big)$, we have $F_{\mu\nu} = \xi(da)_{\mu\nu}$. Since in some configurations the gauge potential a_μ is not globally well-defined, we learn that there cannot be a Ferrara–Zumino multiplet. In SQED we had an exercise where it was shown that there can be string configurations with non-zero string charge.

Example: a WZ model with $\Phi \sim \Phi + 1$ with non-globally well-defined $W = \Phi$ cannot admit R-symmetry hence cannot have an R-multiplet.

A more interesting case is that conservation of the R-symmetry current can happen classically but be violated by anomalies, hence preventing the existence of an R-multiplet.

Example: a $U(1)$ gauge theory with FI terms and generic superpotential only has an S-multiplet, no FZ-multiplet nor R-multiplet.

Example: $SU(N)$ super Yang–Mills has N vacua and admits domain walls between regions sitting in different vacua, so there cannot be an R-multiplet.

If there exists a shortening condition somewhere on the RG flow then it must be preserved along the RG flow. More precisely, in the extreme IR the multiplet may reduce further because some of the operators in it become redundant[2].

2.2 Coupling theories to backgrounds

References Seiberg–Festuccia `https://arxiv.org/abs/1105.0689`, Dumitrescu (review) `https://arxiv.org/abs/1608.02957`.

Some of this can be done in curved superspace, see Buchbinder–Kuzenko `http://inspirehep.net/record/485478`.

Let's say we want to put a non-supersymmetric theory on curved space. A convenient way is to couple to gravity (introducing diffeomorphism invariance), then to fix the metric to the desired one, and then send the Planck mass to infinity to decouple fluctuations of the metric. The resulting curved-space theory will be invariant under isometries of the manifold.

Nikita Nekrasov asks: the metric is now a coupling; is it renormalized? We'll come back to it.

Example: consider a theory with $U(1)$ symmetry namely $\partial^\mu J_\mu = 0$. To couple to a gauge field one adds $a_\mu J^\mu + O(a^2)$ to the action, where the $O(a^2)$ terms are seagull terms. Exercise: check how improvements of J change the Lagrangian.

Example: coupling the theory to gravity is done by $\mathcal{L} \to \mathcal{L} + \frac{1}{2}h^{\mu\nu}T_{\mu\nu} + O(h^2)$ where the metric is $g = \delta + h$ (close to the flat metric δ). Improving the stress-energy tensor $T_{\mu\nu} \to T_{\mu\nu} + (\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2)u$ adds to the Lagrangian a term $R(h)u$, where $R(h)$ is the linearized Ricci scalar.

Example: consider a theory with $U(1)$ flavour symmetry hence $\partial^\mu j_\mu = 0$.

[2] A redundant operator has trivial correlators at separated points.

This current belongs to a supermultiplet $D^2 J = \overline{D}^2 J = 0$ with components

$$J = J + i\theta j + i\overline{\theta}\overline{j} + \theta\sigma^\mu\overline{\theta} j_\mu + \cdots \tag{24}$$

This can couple to a gauge multiplet $(a_\mu, D, \lambda_\alpha, \overline{\lambda}_{\dot{\alpha}}$ by

$$\mathcal{L} \to \mathcal{L} + J^\mu a_\mu + JD + \text{fermions} + \text{seagull terms.} \tag{25}$$

We want to treat the gauge multiplet as a background, in other words set it to some value and don't make it dynamical. This will break supersymmetry unless $\delta(a_\mu, D, \lambda, \overline{\lambda}) = 0$. The first two are trivial because δa_μ and δD depend on the gaugino, which cannot be set to non-zero value (they are Grassmann parameters). The only non-trivial relation are $\delta\lambda = 0$, namely $i\zeta D + \sigma^{\mu\nu}\zeta F_{\mu\nu} = 0$ and $\delta\overline{\lambda} = 0$, which is similar. Note that we do not need this equation to hold for all spinors ζ if we only want to preserve some of the supersymmetry but not all.

Given the above examples, the strategy to place a supersymmetric theory on curved space is as follows. Couple the theory to some supergravity, then choose background values for the supergravity multiplet (including the metric) that preserve some supersymmetry. There are several versions of 4d $\mathcal{N} = 1$ supergravity.

- Old minimal supergravity has $12 + 12$ components, can be coupled to the FZ multiplet.
- New minimal supergravity has $12 + 12$ components, can be coupled to the R-multiplet, so it can be coupled to any 4d $\mathcal{N} = 1$ theory with $U(1)_R$ symmetry.
- 16/16 supergravity has $16 + 16$ components, can be coupled to any S-multiplet.

On-shell, the old minimal and new minimal supergravities are equivalent. The 16/16 supergravity is basically obtained by adding a chiral multiplet to new minimal supergravity.

The coupling to supergravity goes as follows (i labels various $U(1)$ symmetries)

$$\mathcal{L} \to \mathcal{L} + \frac{1}{2} h^{\mu\nu} T_{\mu\nu} + B^i J_i + \psi^{\mu\alpha} S_{\mu\alpha} + \overline{\psi}^{\mu\dot{\alpha}} S_{\mu\dot{\alpha}} + F^i j^i + \cdots + \text{seagull} \tag{26}$$

Note that we do not need the supersymmetric theory to actually have a Lagrangian description: even without a Lagrangian it makes sense to talk about the operators in the S-multiplet, and to couple these operators to supergravity.

Again, the choice of background $g_{\mu\nu}$ etc must be such that the variation of fermions (gravitinos) vanishes. We find

$$\nabla_\mu \zeta_\alpha = \mathcal{M}(g, B^i)_\alpha{}^\beta \zeta_\beta + \widetilde{\mathcal{M}}(g, B^i)_\alpha{}^{\dot{\beta}} \overline{\zeta}_{\overline{\beta}} \tag{27}$$

where $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are matrices that only depend on the supergravity multiplet, not on the multiplets in the supersymmetric theory. So we just have to repeat the work for old minimal, new minimal and 16/16 supergravity, not for each supersymmetric theory that comes about.

2.3 FZ multiplet and old minimal supergravity

Consider an FZ multiplet, namely $Y_\alpha = D_\alpha X$ so $\overline{D}_{\dot\alpha} X = 0$. The coupling is[3]

$$\frac{1}{2} h^{\mu\nu} T_{\mu\nu} - \frac{1}{2} b^\mu j_\mu - \frac{1}{4} M \overline{X} - \frac{1}{4} \overline{M} X + \cdots \tag{28}$$

and the supersymmetry equations are

$$\nabla_\mu \zeta = \frac{i}{6} M \sigma_\mu \overline{\zeta} + \frac{i}{3} b_\mu \zeta + \frac{i}{3} b^\nu \sigma_{\mu\nu} \zeta \tag{29}$$

$$\nabla_\mu \overline{\zeta} = \frac{i}{6} \overline{M} \sigma_\mu \zeta - \frac{i}{3} b_\mu \overline{\zeta} - \frac{i}{3} b^\nu \overline{\sigma}_{\mu\nu} \overline{\zeta} \tag{30}$$

As promised, these equations only depend By dimension analysis M, $\overline{M}$, b scale like $1/r$, so, in the UV, the theory approaches the original supersymmetry theory in flat space. This is good because we are trying to put that theory on curved space. It is not so good because that means we are missing some deformations of the theory that preserve supersymmetry but are non-trivial in flat space.

3 Lecture 3, July 18

Last time we explained how to put the theory on a curved space by coupling to supergravity, setting supergravity fields to background values and freezing the supergravity fluctuations by sending $M_P \to \infty$. Supersymmetry is preserved if and only if the variation of the gravitini vanishes. This is the generalized Killing spinor equation.

For the old minimal supergravity on S^4 with radius r, the equation (29) turns out to imply $b^\mu = 0$ and we have to solve

$$\nabla_\mu \zeta = \frac{i}{6} M \sigma_\mu \overline{\zeta} \tag{31}$$

$$\nabla_\mu \overline{\zeta} = \frac{i}{6} \overline{M} \sigma_\mu \zeta \tag{32}$$

The solution turns out to be $M = \overline{M} = 3i/r$, which spoils reflection positivity. In the Lagrangian, M couples to X (that we saw earlier), which is in general non-zero. However, in a superconformal field theory we know that the FZ multiplet reduces to the superconformal multiplet, namely $X = 0$ (after improvement). This restores reflection positivity.

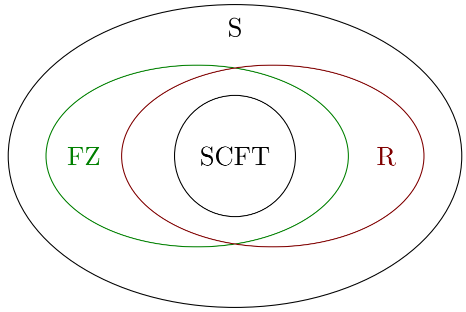
For a non-conformal field theory we find a superalgebra $\mathfrak{osp}(1|4)$ on S^4. For a conformal field theory it is $\mathfrak{su}^*(4|1)$, the same as flat space (we typically hear more often about the Lorentzian superconformal algebra $\mathfrak{su}(2,2|1)$).

Note: there exist theories, for instance a free chiral field with mass, in which the S-multiplet can be improved to be an FZ multiplet or an R multiplet, but the improvements are different.

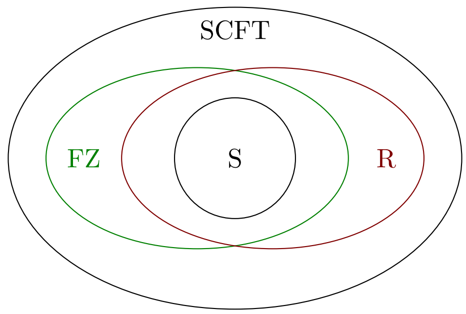
Slogan: the more special a theory is, the more backgrounds it can be put on.

[3] In this section only (because Guido realized how inconvenient using multiple colors is) supergravity multiplets are in a different color.

Theories with specializations of S-multiplets



Manifolds on which such theories can be put



In particular, even though it looks like 16/16 supergravity contains more fields to play with, we know from these general considerations that it must have fewer solutions than new minimal or old minimal supergravity, which themselves have fewer solutions than conformal supergravity.

3.1 Supergravity backgrounds in general

Actually we use conventions of `https://arxiv.org/abs/1407.2598` (Closset–Dumitrescu–Festuccia–Komargodski), not of references mentioned earlier.

We focus on the case of 4d $\mathcal{N} = 1$ theories with a $U(1)_R$ symmetry, namely which admit R-symmetry multiplets hence must be coupled to new minimal supergravity. The multiplets are

$$\text{R-multiplet:}\ (T_{\mu\nu}, S_{\mu\alpha}, \overline{S}_{\dot\alpha\mu}, j_\mu, F_{\mu\nu}) \tag{33}$$

$$\text{new minimal supergravity multiplet:}\ (g_{\mu\nu}, \psi_{\mu\alpha}, \overline{\psi}_{\mu\dot\alpha}, A_\mu, B_{\mu\nu}) \tag{34}$$

Instead of the 2-form gauge field $B_{\mu\nu}$ (whose gauge transformations are $B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu\omega_\nu - \partial_\nu\omega_\mu$), we use the dual of the field strength, so

$$V_\mu = \frac{1}{2}\epsilon_\mu{}^{\nu\rho\sigma}\partial_\nu B_{\rho\lambda}. \tag{35}$$

This is trivially conserved: $\partial_\mu V_\mu = 0$.

The generalized Killing spinor equations are[4]

$$(\nabla_\mu - iA_\mu)\zeta = \frac{-i}{2}V^\nu \sigma_\mu \widetilde{\sigma}_\nu \zeta \tag{36}$$

$$(\nabla_\mu + iA_\mu)\widetilde{\zeta} = \frac{i}{2}V^\nu \widetilde{\sigma}_\mu \sigma_\nu \widetilde{\zeta} \tag{37}$$

Note the presence of R-symmetry: A_μ is a R-symmetry gauge field, and ζ and $\widetilde{\zeta}$ are left-handed and right-handed spinors with charge $+1$ and -1 respectively under R-symmetry. A consequence of the R-symmetry, and of the fact that scalars in the supergravity multiplet have no R-charge, is that ζ and $\widetilde{\zeta}$ do not appear in each other's equations since they have different R-charges. In fact, V^μ and A_μ do not need to satisfy the supergravity equations.

Assume that we have a solution. (We will come back later to building solutions.)

The supersymmetry transformations that are preserved have the following algebra: $\{\delta_\zeta, \delta_{\zeta'}\} = 0$,

$$\{\delta_\zeta, \delta_{\widetilde{\zeta}}\}\Phi = 2i\mathcal{L}'_\kappa \Phi \tag{38}$$

where $\kappa^\mu = \zeta\sigma^\mu\widetilde{\sigma}$ and

$$\mathcal{L}'_\kappa \Phi = \mathcal{L}_\kappa \Phi - irK^\mu(A_\mu + (3/2)V_\mu)\Phi \tag{39}$$

where r is the R-charge of the field Φ.

The Lagrangian is

$$\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \cdots \tag{40}$$

where $\mathcal{L}_0$ is the minimal coupling, $\mathcal{L}_1$ is the linear coupling $-j^\mu A_\mu + \frac{1}{4}B_{\mu\nu}C^{\mu\nu}$ and $\mathcal{L}_2$ and higher consist of seagull terms, for instance couplings to V^2, curvature R etc, which scale like $1/\ell^2$ in terms of the typical size ℓ of the manifold.

Consider a chiral multiplet (ϕ, ψ, F) (where ϕ is a complex scalar and F an auxiliary field) with R-charge r. We have supersymmetry variations

$$\delta_\zeta \phi = \sqrt{2}\zeta\psi \quad \text{like in flat space} \tag{41}$$

$$\delta_\zeta \psi = \sqrt{2}\zeta F + \sqrt{2}i\sigma_\mu \widetilde{\zeta} D^\mu \phi \tag{42}$$

$$\delta_\zeta F = \sqrt{2}iD_\mu(\widetilde{\zeta}\widetilde{\sigma}^\mu \psi) \tag{43}$$

where the derivative is covariantized with respect to R-symmetry, namely $D^\mu\phi$ contains a term $-ir(A_\mu + (3/2)V_\mu)\phi$, and importantly $D_\mu\widetilde{\zeta} \neq 0$.

The Lagrangian is

$$\mathcal{L} = (D_\mu + iV_\mu)\widetilde{\phi}(D_\mu + iV_\mu)\phi + i\widetilde{\psi}\widetilde{\sigma}^\mu\left(D_\mu - \frac{i}{2}V_\mu\right)\psi - F\widetilde{F} - \frac{r}{4}R\widetilde{\phi}\phi + \left(\frac{3}{2}r - 1\right)V^\mu V_\mu \widetilde{\phi}\phi. \tag{44}$$

Note that the structure is as announced, with $1/\ell$ and $1/\ell^2$ terms. Note that the R-charge r appears explicitly.

[4] As usual in Euclidean signature ζ and $\widetilde{\zeta}$ are not related by complex conjugation.

Consider a free chiral multiplet with R-charge equal to 2/3, so that one can turn on a superpotential. That theory is an SCFT (at least classically) so we expect to be able to couple the theory to conformal supergravity. We can see that $\frac{r}{4}R\widetilde{\phi}\phi$ gives the conformal coupling $R\widetilde{\phi}\phi/6$, and see that V_μ drops out, leaving A_μ as the only gauge field.

3.2 Example: $S^3 \times \mathbb{R}$

This cylinder is locally conformally flat, so we can put superconformal theories easily on it. In other words it is a solution of conformal supergravity. Is it also a solution of new minimal supergravity, namely can we put 4d $\mathcal{N}=1$ theories with $U(1)_R$ symmetry on this?

Isometries are $SU(2)_L \times SU(2)_R \times \mathbb{R}$. Careful, these two $SU(2)$ have nothing to do with the ones for which we usually introduce dotted and undotted indices. We parametrize $\mathbb{R}$ by τ. We take the Ansatz $V = v\mathrm{d}\tau$ and $A = a\mathrm{d}\tau$. The Killing vector equations become

$$\left(\partial_\tau - ia - \frac{i}{2}v\right)\zeta = 0, \qquad \left(\partial_\tau + ia + \frac{i}{2}v\right)\widetilde{\zeta} = 0, \tag{45}$$

$$\vec{\nabla}\zeta = -\frac{v}{2}\vec{\sigma}\zeta, \qquad \vec{\nabla}\widetilde{\zeta} = -\frac{v}{2}\vec{\sigma}\widetilde{\zeta}. \tag{46}$$

Exercise: find solutions. This gives $v = \mp i/\ell$. The $v = -i/\ell$ case leads to ζ and $\widetilde{\zeta}$ that are invariant under $SU(2)_R$. If we choose $a = 0$ then $[H, Q_\zeta] = \frac{1}{2}Q$. To avoid this dependence of Q_ζ on time we use $a = i/2\ell$, which leads to $[H, Q_\zeta] = 0$. In particular compactifying the time direction preserves supersymmetry in that case.

In contrast, if we had worked with old minimal supergravity we would have found a supersymmetric background on $S^3 \times \mathbb{R}$, but not on $S^3 \times S^1$.

The Romelsberger index (partition function on $S^3 \times S^1$) can thus only be defined for theories with $U(1)$ R-symmetry.

We can compute $\{Q_a, Q_b\} = \{\overline{Q}^a, \overline{Q}^b\} = 0$ and

$$\{\overline{Q}^a, Q_b\} = \left(H + \frac{R}{\ell}\right)\delta^a_b + \frac{2}{\ell}J^a_b \tag{47}$$

where $J_{(ab)}$ are generators of $SU(2)_L$. In addition, supercharges have R-charges ± 1, namely $[R, Q_a] = -Q_a$ and $[R, \overline{Q}_a] = \overline{Q}_a$. We also have $[H, Q_a] = 0$, $[H, \overline{Q}_a] = 0$. Denoting $J_3 = J_{12}$ and $J_+ = iJ_{11}$ and $J_- = iJ_{22}$, we find the usual $[J_3, J_{\pm 1}] = \pm J_\pm$ and $[J_+, J_-] = 2J_3$ and $[J_+, Q_2] = -iQ_1$, $[J_3, Q_1] = Q_1$, $[J_+, Q_1] = 0$ etc.

This algebra is $\mathfrak{su}(2|1) \times \mathfrak{u}(1) \times \mathfrak{su}(2)_R$ where $\mathfrak{su}(2|1)$ is generated by J_3, $J_\pm$, Q, $\overline{Q}$, $R + \ell H$, and $\mathfrak{u}(1)$ is generated by H, and $\mathfrak{su}(2)_R$ does not act on any supercharge.

Let us now set $\ell = 1$.

Let us understand which representation of the superalgebra are unitary (with $\overline{Q}^a = (Q_a)^\dagger$). We can decompose into representations of $\mathfrak{su}(2) \times \mathfrak{u}(1)$. Start from a state annihilated by $\overline{Q}_{1,2}$ and J_+. Of course $\mathfrak{su}(2)$ generates a tower of states. Denoting by $(j)_r$ that tower of state with highest spin $j \in \frac{1}{2}\mathbb{Z}$ and R-charge r. Acting with a supercharge gives

$$\begin{array}{ccccc} & & (j+1/2)^h_{r-1} & & \\ & \nearrow & & \searrow & \\ (j)^h_r & & & & (j)^h_{r-2} \\ & \searrow & & \nearrow & \\ & & (j-1/2)^h_{r-1} & & \end{array} \tag{48}$$

where the multiplet $(j-1/2)^h_{r-1}$ is only present for $j \geq 1/2$. The unitarity condition is that norms of all these states are positive:

- $\Big\| Q_1|j,r\rangle \Big\|^2 \geq 0$ implies TODO
- $\Big\| Q_2|j,r\rangle + (i/\sqrt{2j})Q_1|j-1,r\rangle \Big\|^2 \geq 0$ (the state is chosen to be annihilated by J_+) implies TODO when $j \geq 1/2$;
- $\Big\| Q_1 Q_2|j,r\rangle \Big\|^2 \geq 0$ implies TODO so we don't learn anything except for $j=0$.

Long multiplets have $h \geq 2j+2-r$

- for $j \geq 1/2$ they are called L_j and contain $(j)^h_r + (j \pm 1/2)^h_{r-1} + (j)^h_{r-2}$;
- for $j=0$ they contain $(0)^h_r + (1/2)^h_{r-1} + (0)^h_{r-2}$.

Short multiplets (namely saturating the bound so some states have zero norm and the representation should be quotiented by these states) are

- for $h = 2j+2-r$; they are called S_j and contain $(j)^{2j+2-r}_r + (j+1/2)^{2j+2-r}_{r-1}$;
- singleton $j=0$, $h=-r$ contains $(0)^{-r}_r$ only and is called $\widehat{S}$.

When varying parameters in the theory, long multiplets can split into short multiplets when the bound becomes saturated. In detail,

$$L_j \xrightarrow{h \to 2j+2-r} S_j + S_{j-1/2}, \qquad j \geq 1/2 \tag{49}$$

$$L_0 \xrightarrow{h \to 2-r} S_0 + \widehat{S}, \qquad j = 0. \tag{50}$$

Exercise: check there is no typo.

Let us try to define the most general index that is invariant under recombinations. Here, $n[S^h_j]$ denotes the number of short multiplets of type S^h_j and so on.

$$I = \sum_h e^{-\beta h} \left(\sum_{j,h} \big(\alpha_j n[S^h_j] + \gamma n[\widehat{S}^h] \big) \right) \tag{51}$$

Invariance under recombination implies that $\alpha_j = -\alpha_{j-1/2}$ and $\alpha_{1/2} = -\gamma$ so we find

$$I = \mathrm{Tr}\, e^{-\beta H}(-1)^{2J_3} = \mathrm{Tr}\Big(e^{-\beta H}(-1)^F\Big). \tag{52}$$

In fact we can introduce fugacities similar to β for any other conserved charge that commutes with the superalgebra, for instance the J_3 generator of $\mathfrak{su}(2)_{\mathrm{right}}$.

A nice way to write the index with all fugacities turned on, roughly changing variables $e^{-\beta} \to pq$ and add fugacity p/q for J_3^{right}, is

$$\mathrm{Tr}_{\mathcal{H}_{\mathrm{on}\ S^3}}\Big((-1)^F p^{J_3^{\mathrm{left}}+J_3^{\mathrm{right}}-R/2+1} q^{J_3^{\mathrm{left}}-J_3^{\mathrm{right}}-R/2+1} u^{Q_{\mathrm{flavour}}}\Big). \tag{53}$$

This reduces to the previous expression when $p = q = e^{-\beta/\ell}$ and u is a flavour fugacity and Q_{flavour} is a flavour charge.

We can convert this index to a partition function on $S^3 \times S^1$, where u is a holonomy around S^1 of a background gauge field for the given flavour symmetry. Since the background gauge field is dynamical there is nothing wrong with taking u to be complex.

Note that h is fixed by j and r, and in particular it does not depend on the scale. This means that we can compute the index in the free theory and that tells us something about the infrared.

In addition, the supersymmetry algebra is a subalgebra of the superconformal algebra: $\mathfrak{su}(2|1)_{\mathrm{left}} \times \mathfrak{u}(1) \times \mathfrak{su}(2)_{\mathrm{right}} \subset \mathfrak{su}^*(4|1)$.

See `https://arxiv.org/abs/hep-th/0510060` and `https://arxiv.org/abs/hep-th/0707.3702` by Romelsberger (the second paper is more readable).

The index of a free chiral multiplet of charge r is

$$I = \prod_{m,n\geq 0} \frac{1-(pq)^{-r/2}p^{m+1}q^{m+1}}{1-(pq)^{r/2}p^m q^n}. \tag{54}$$

In the exercises you find a few special cases.

From a 4d $\mathcal{N} = 1$ theory on $S^3 \times S^1$, we can reduce along

- the Hopf fiber, which leads to a 3d $\mathcal{N} = 2$ theory on $S^2 \times S^1$;
- the S^1 time direction, which leads to a 3d $\mathcal{N} = 2$ theory of S^3;
- other reductions are possible, where we glue the cylinder after a $\mathfrak{su}(2)_{\mathrm{right}}$ rotation. This leads to pretty general squashed S^3 partition functions that preserve 4 supercharges.

3.3 Back to classifying backgrounds for 4d $\mathcal{N} = 1$ with $U(1)$ R-symmetry

Given the $U(1)$ R-symmetry the theory is coupled to new minimal supergravity so the generalized Killing spinor equations are

$$(\nabla_\mu - iA_\mu)\zeta = \frac{-i}{2}V^\nu \sigma_\mu \overline{\sigma}_\nu \zeta. \tag{55}$$

In Euclidean space-time we define $(\zeta_\alpha)^* = (\zeta^\dagger)^\alpha$ such that $\zeta^\dagger\zeta = |\zeta_1|^2 + |\zeta_2|^2$ and $(\zeta^\alpha)^* = -(\zeta^\dagger)_\alpha$

While there are manifolds that do not admit spinors (non-spin manifolds), all 4d orientable manifolds are $\mathrm{Spin}_\mathbb{C}$ namely admits charged spinors. The spinors will be in a $L \otimes \mathfrak{su}(2)_+$ bundle. If $\zeta = 0$ at some point then the generalized Killing spinor equations set $\zeta = 0$ everywhere. We deduce that $\zeta \neq 0$ everywhere for solutions. In fact, the set of all solutions must be linearly independent at every point (hence there can be at most two solutions because the bundle is two-dimensional).

This non-vanishing seems in contradiction with the fact that for 4d $\mathcal{N} = 2$ theories we find spinors that vanish at one pole. However, remember that the generalized Killing spinor equations in that case (or in old minimal supergravity) mix the spinors ζ and $\widetilde{\zeta}$ so the analogous non-vanishing statement is that the two spinors cannot both vanish at the same point.

Suppose we have a solution. Then

$$J^\mu_\nu = \frac{-2i}{|\zeta|^2}\zeta^\dagger\sigma^\mu{}_\nu\zeta \tag{56}$$

is a good tensor: it is invariant under $\mathrm{Spin}_\mathbb{C}$ acting on ζ and $\zeta^\dagger$. We can compute

$$J^\mu{}_\nu J^\nu{}_\rho = -\delta^\mu_\rho, \qquad g_{\mu\nu}J^\mu{}_\rho J^\nu{}_\lambda = g_{\rho\lambda} \tag{57}$$

which means there is an almost[5] complex structure J, and that it is compatible with the metric. This already rules out S^4 because that manifold is not almost complex.

The almost complex structure is in fact a complex structure, namely the Nijenhuis tensor vanishes; this is a tedious calculation

$$N^\mu{}_{\nu\rho} = J^\lambda{}_\nu\nabla_\lambda J^\mu{}_\rho - J^\lambda{}_\rho\nabla_\lambda J^\mu{}_\nu - J^\mu{}_\lambda\nabla_\nu J^\lambda{}_\rho + J^\mu{}_\lambda\nabla_\rho J^\lambda{}_\nu = 0. \tag{58}$$

Ther is a shortcut: check that the Lie bracket of holomorphic vectors is holomorphic. A holmorphic vector is X such that $X^\mu\widetilde{\sigma}_\mu\zeta = 0$.

We learn that any supercharge ζ leads to a complex structure on the manifold M with a metric compatible with holmorphy. Namely we can cover M in patches with complex coordinates z_I and holomorphic transition functions and where the metric is $\mathrm{d}s^2 = g_{i\bar{i}}\mathrm{d}z^i\mathrm{d}z^{\bar{i}}$.

What about the converse: given a nice complex manifold M with a complex metric, build generalized Killing spinors?

Consider the simplified form $(\nabla_\mu - iA_\mu)\zeta = 0$ (namely $V = 0$); it is (only) possible to cancel the holonomy of the Levi–Civita connection ∇_μ using a $U(1)$ gauge field if the holonomy of ∇_μ is in $\mathfrak{u}(1)_{\mathrm{left}} \times \mathfrak{su}(2)_- \subset \mathfrak{su}(2)_+ \times \mathfrak{su}(2)_-$, namely the manifold is Kähler.

From the vanishing of $N^\mu{}_{\nu\rho}$ we can deduce $\nabla_\mu J^\mu{}_\nu = \big(V_\nu + \overline{V}_\nu\big) + i\big(V_\mu - \overline{V}_\nu\big)J^\mu{}_\nu$, which can be inverted to $V_\mu = \frac{1}{2}\nabla_\nu J^\nu{}_\mu + U_\mu$ for an arbitrary conserved

[5] This means that the structure might not integrate to complex coordinates even locally.

U_μ to ensure $\nabla^\mu U_\mu = 0$ and in fact $U_{\bar{i}} = 0$. Note that V_μ only needs to be zero for a manifold that is not Kähler (but is Hermitian).

The next step is to build the so-called *Chern connection* compatible with $g_{\mu\nu}$ and $J^\mu{}_\nu$. Start from the Levi–Civita connection $\omega_{\mu\nu\rho}$ and shift it by a (con?)torsion tensor:

$$\omega^{\text{Chern}}_{\mu\nu\rho} = \omega_{\mu\nu\rho} - \frac{1}{2} J_\mu{}^\lambda \Big(\nabla_\lambda J_{\nu\rho} + \nabla_\nu J_{\rho\lambda} + \nabla_\rho J_{\lambda\nu} \Big). \tag{59}$$

The key is that this Chern connection has holonomy in $\mathfrak{u}(1) \times \mathfrak{su}(2)$ and we rewrite the generalized spinor equation in terms of that connection and using the explicit V^μ (up to the ambiguity U)

$$\Big(\nabla^{\text{Chern}}_\mu - i A^{\text{Chern}}_\mu \Big) \zeta = 0 \tag{60}$$

where $A^{\text{Chern}}_\mu = A_\mu + \frac{1}{4}(2\delta^\nu_\mu - iJ^\nu_\mu)\nabla_\rho J^\rho_\nu$. Now we are left with cancelling a $\mathfrak{u}(1) \times \mathfrak{su}(2)$ connection by a $\mathfrak{u}(1)$ gauge field. Thankfully ζ is not charged under the $\mathfrak{su}(2)$.

3.3.1 More explicitly

We define $P_{\mu\nu} = \zeta\sigma_{\mu\nu}\zeta$ is antisymmetric and holomorphic in each index and lives in the line bundle $L^2 \times \Lambda^{(2,0)}$ where $\Lambda^{(2,0)}$ is the canonical line bundle (we are on a 2 complex dimensional manifold). On the other hand, $L \sim K^{-1/2}$ where K is the canonical line bundle (so L is not defined but L^2 is). We can then let $p \coloneqq P_{12}$ and consider $s = pg^{-1/4}$ where $g = \det(\text{metric})$. Under a holomorphic coordinate change we get

$$s'(z') = s(z) \left(\frac{\det\big(\partial z'/\partial z\big)}{\det\big(\partial \bar{z}'/\partial \bar{z}\big)} \right)^{1/2}. \tag{61}$$

This can be undone by a $U(1)$ R-symmetry transformation. Under holomorphic coordinate changes plus $U(1)$ R-symmetry, s is a scalar.

Very explicitly,

$$e^1 = \sqrt{2 g_{1\bar{1}}} \mathrm{d}z^1 + \sqrt{\frac{2}{g_{1\bar{1}}}} g_{2\bar{1}} \mathrm{d}z^2 \tag{62}$$

$$e^2 = \sqrt{\frac{2}{g_{1\bar{1}}}} g^{1/4} \mathrm{d}z^2 \tag{63}$$

$$\mathrm{d}s^2 = e^1 e^{\bar{1}} + e^2 e^{\bar{2}} \tag{64}$$

$$\zeta = \frac{\sqrt{s}}{2} \begin{pmatrix} 1 & 0 \end{pmatrix} \tag{65}$$

and

$$A_i^{\text{Chern}} = \frac{-i}{8}\partial_i \log g - \frac{i}{2}\partial_i \log s \tag{66}$$

$$A_{\bar{i}}^{\text{Chern}} = \frac{i}{8}\partial_{\bar{i}} \log g - \frac{i}{2}\partial_{\bar{i}} \log s. \tag{67}$$

Conclusion: in a manifold with a self-dual complex structure we can build at least one supercharge.

3.3.2 Ask for two ζ

3.3.3 Ask for ζ and $\widetilde{\zeta}$

Then $K^\mu = \zeta\sigma^\mu\widetilde{\zeta}$ is a complex Killing vector with $K^\mu K_\mu = 0$ but K everywhere nonzero and K is holomorphic with respect to the two complex forms J and $\widetilde{J}$ built from ζ and $\widetilde{\zeta}$.

We decompose

$$J_{\mu\nu} = Q_{\mu\nu} + \frac{1}{2}\epsilon_{\mu\nu\rho\lambda}\psi^{\rho\lambda} \tag{68}$$

with $Q_{\mu\nu} = \frac{i}{|K|^2}(K_\mu \overline{K}_\nu - K_\nu \overline{K}_\mu)$. Two cases:

- $[K^\mu, \overline{K}^\nu] \neq 0$ then we get a non-trivial isometry algebra and can work out the only solutions $S^3 \times \mathbb{R}$ or $S^3 \times S^1$ which we already know about.
- $[K^\mu, \overline{K}^\nu] = 0$ then we have a nowhere vanishing torus fibration with

$$\mathrm{d}s^2 = \Omega(z,\bar{z})^2 \left(\big(\mathrm{d}w + h(z,\bar{z})\mathrm{d}z\big)\big(\mathrm{d}\overline{w} + \overline{h}(z,\bar{z})\mathrm{d}\bar{z}\big) + c(z,\bar{z})\mathrm{d}z\mathrm{d}\bar{z} \right) \tag{69}$$

  with $\Omega^2(z,\bar{z}) = 2|K|^2$ and $K = \partial_w$.

4 Lecture 5, July 20

- One supercharge Q corresponding to a $\zeta \implies M$ is complex (actually, Hermitian?).
- Two supercharges of the same chirality $\implies$ flat T^4, round $S^3 \times S^1$ or quotients thereof, K3 surface.
- Two supercharges of opposite chirality $\implies$ two complex structures, and $\zeta\sigma^\mu\widetilde{\zeta}$ is holomorphic with respect to both complex structures.

Now let us find out what background gauge fields $(a_\mu, D, \dots)$ preserve this supercharge. The condition (for one supercharge) is

$$i\zeta D + \sigma^{\mu\nu}\zeta f_{\mu\nu} = 0 \tag{70}$$

which implies that $f_{\bar{i}\bar{j}} = 0$ and $D = -J^{\mu\nu} f_{\mu\nu}$.

4.1 Hopf surfaces

Hopf surfaces are complex surfaces (real dimension 4) diffeomorphic to $S^3 \times S^1$ that can be constructed as quotients of $\mathbb{C}^2 \setminus \{(0,0)\}$ by $\mathbb{Z}$.

- Primary Hopf surfaces $\mathcal{M}_{p,q}$ are defined by identifying $(w,z) \sim (pw, qz)$ for some fixed $0 < |p| \leq |q| < 1$; the complex parameters p and q are complex structure moduli.
- Other Hopf surfaces are defined by identifying $(w,z) \sim (q^n w + \lambda z^n, qz)$ for some $n \in \mathbb{Z}$ and $0 < |q| < 1$ and $\lambda \in \mathbb{C} \setminus \{(0,0)\}$; the λ can be rescaled away but q is a genuine complex structure modulus.

In the first case, let $p = \exp(-\beta_p + i\theta_p)$ and $q = \exp(-\beta_q + i\theta_q)$ (where we choose representatives θ_p and θ_q modulo 2π arbitrarily) and parametrize

$$w = e^{(-\beta_p + i\theta_p)x} \cos\frac{\theta}{2} e^{i\varphi}, \qquad z = e^{(-\beta_q + i\theta_q)x} \cos\frac{\theta}{2} e^{i\chi}. \tag{71}$$

The identification is $x \sim x + 1$ so x parametrizes a circle. The geometry at fixed x is a squashed sphere

$$e^{2\beta_p x}|w|^2 + e^{2\beta_q x}|z|^2 = 1. \tag{72}$$

One possible metric consistent with the holomorphic structure is

$$\mathrm{d}s^2 = \ell^2\left(\sqrt{\frac{\beta_q}{\beta_p}} e^{2\beta_p x} \mathrm{d}w \mathrm{d}\overline{w} + \sqrt{\frac{\beta_p}{\beta_q}} e^{2\beta_q x} \mathrm{d}z \mathrm{d}\overline{z}\right) \tag{73}$$

On a primary Hopf surface $\mathcal{M}_{p,q}$ we can preserve two supercharges. It turns out that the partition function

$$Z(\mathcal{M}_{p,q}) \sim I(q,p) \tag{74}$$

is essentially equal to the index that we defined earlier. Flavour fugacities correspond to turning on background gauge fields. In particular, the partition function depends holomorphically on p and q.

As we will see, the partition function depends holomorphically on the complex structure and does not depend on the metric.

4.2 Dependence on parameters

See Klare–Tomasiello–Zaffaroni `https://arxiv.org/abs/1203.1062` and Dumitrescu–Festuccia–Seiberg `https://arxiv.org/abs/1205.1115`.

In principle the partition function could depend on

- the choice of metric;
- the choice of complex structure $J^\mu{}_\nu$;

- the other supergravity backgrounds $V_\mu = \nabla^\nu J_{\nu\mu} + U_\mu$;
- the background gauge fields.

The partition function does not depend on many of these parameters. The proper way to prove this is to compute variations with respect to the complex structure etc and check that some of these are Q-exact. Instead, we will analyse the problem only near flat space, in linearized supergravity.

We consider theories with $U(1)$ R-symmetry, that couple to new minimal supergravity. The variation of the Lagrangian away from flat space and zero background currents is

$$\delta\mathcal{L} = \frac{-1}{2}\delta g^{\mu\nu}T_{\mu\nu} + \delta A_\mu^R j_R^\mu + \delta V^\mu \mathcal{A}_\mu + \left(\delta a^\mu j_\mu^{\text{flavour}} + \delta D J^{\text{flavour}}\right) \tag{75}$$

where the terms in parentheses come from background currents (and we omitted fermionic terms?). Then

$$f_{\overline{i}\overline{j}} = 0, \qquad D = -2i\left(f_{w\overline{w}} + f_{z\overline{z}}\right) \tag{76}$$

which implies $a_{\overline{i}} = \overline{\partial}_{\overline{i}}\lambda$ and $\overline{\partial}_{[\overline{i}}\delta a_{\overline{j}]} = 0$.

The flavour current (we drop the superscript) has components J, j_α, $j_{\dot\alpha}$, j_μ and obeys $\partial^\mu j_\mu = 0$. The variations are

$$\{Q_\zeta, J\} = i\zeta j, \qquad \{Q_\zeta, j^{\dot\alpha}\} = -i(\widetilde{\sigma}^\mu\zeta)^{\dot\alpha}\left(j_\mu - i\partial_\mu J\right) \tag{77}$$

$$\{Q_\zeta, j_\alpha\} = 0, \qquad \{Q, j_\mu\} = -2\zeta\sigma_{\mu\nu}\partial^\nu j. \tag{78}$$

Using the complex structure we can check that $j_{\overline{i}} - i\overline{\partial}_{\overline{i}}J$ are Q-exact, and that they are the only Q-closed bosonic operators we can write. Then focussing on the variation of the Lagrangian due to background gauge fields we compute

$$\delta\mathcal{L} = \delta a^\mu j_\mu + \delta D J \tag{79}$$

$$= 2\delta a_w\left(j_{\overline{w}} - i\overline{\partial}_{\overline{w}}J\right) + 2\delta a_z\left(j_{\overline{z}} - i\overline{\partial}_{\overline{z}}J\right) + 2\delta a_{\overline{w}}\left(j_w - i\partial_w J\right) + 2\delta a_{\overline{z}}\left(j_z - i\partial_z J\right) + \text{total derivatives} \tag{80}$$

where we used $D = -2i\left(f_{w\overline{w}} + f_{z\overline{z}}\right)$ and integrated by parts to put everything in terms of δa. Now the first two terms are Q-exact so the partition function cannot depend on a_w and a_z. On the other hand the other two $\left(j_w - i\partial_w J\right)$ and $\left(j_z - i\partial_z J\right)$ are not Q-closed, so how can $Q\delta\mathcal{L}$ vanish? Well,

$$Q\delta\mathcal{L} = 2\overline{\partial}_{[\overline{i}}\delta a_{\overline{j}]}\zeta\sigma^{\overline{i}\overline{j}}j + \text{total derivatives.} \tag{81}$$

What about gauge-invariance? The Lagrangian is not invariant under background gauge transformations, but the partition function is. We already know that δa_w and δa_z do not affect the partition function, so we just have to check that $\delta a_{\overline{i}} = \overline{\partial}_{\overline{i}}\epsilon$ does not change the partition function. Exercise: by using current conservation check that the Lagrangian varies by

$$2\epsilon\partial_w\left(j_{\overline{w}} - i\overline{\partial}_{\overline{w}}J\right) + 2\epsilon\partial_z\left(j_{\overline{z}} - i\overline{\partial}_{\overline{z}}J\right) + \text{total derivative.} \tag{82}$$

Back to the geometry. Varying the equation $J^\mu{}_\nu J^\nu{}_\rho = -\delta^\mu_\rho$ gives

$$\delta J^\mu{}_\nu J^\nu{}_\rho + J^\mu{}_\nu \delta J^\nu{}_\rho = 0 \tag{83}$$

from which we learn that $\delta J^i{}_j = 0$ and $\delta J^i_{\bar{j}}$ is unconstrained. Varying the vanishing of the Nijenhuis tensor gives that $\theta^i = \delta J^i{}_{\bar{j}} \mathrm{d}\bar{z}^{\bar{j}}$ obeys $\bar{\partial}\theta^i = 0$ (here $\bar{\partial}$ is "half" of the exterior derivative d) so $\theta^i = \bar{\partial}\epsilon^i$ locally (because $\bar{\partial}^2 = 0$, just like $\mathrm{d}^2 = 0$). We are thus interested in how the Lagrangian varies under changes by ϵ^μ.

How must the metric vary in order to remain holomorphic? Vary $g_{\mu\nu} J^\mu{}_\rho J^\nu{}_\lambda = g_{\rho\lambda}$. We learn that $\delta g_{i\bar{j}}$ is unconstrained while

$$\delta g_{ij} = \frac{i}{2}\big(g_{i\bar{k}} \delta J^k{}_j + g_{j\bar{k}} \delta J^{\bar{k}}{}_i\big) \tag{84}$$

Then we can use formulas given in previous lectures to compute δA^R_μ and δV^μ and plug this back into (75) and integrate by parts. Altogether we find that

$$\delta\mathcal{L} = Q_\zeta\big(\cdots\big) + \delta J^i{}_{\bar{i}} \theta^{\bar{i}}{}_i. \tag{85}$$

We learn that

- the partition function does not depend on $\delta g_{i\bar{j}}$, hence does not depend on the hermitian metric;
- it depends on complex structure moduli, locally holomorphically (there could be singularities).

All that we said so far is disregarding anomalies!

4.3 3d $\mathcal{N} = 2$

We now dimensionally reduce to 3d $\mathcal{N} = 2$ supersymmetric theories with $U(1)$ R-symmetry. Now the R-multiplet contains: $j^{(R)}_\mu$ (R-symmetry current), $j^{(Z)}_\mu$ (central charge current), $T_{\mu\nu}$ (stress tensor), J from which we can build the string current $\epsilon_{\mu\nu\rho}\partial^\rho J$, and fermionic $S_{\mu\alpha}$ and $\bar{S}_{\mu\dot{\alpha}}$. This couples to supergravity fields

$$\text{supergravity multiplet}\colon A^{(R)}_\mu, C_\mu, g_{\mu\nu}, H, \psi_{\mu\alpha}, \bar{\psi}_{\mu\dot{\alpha}}. \tag{86}$$

Alternatively we can use the conserved current $V^\mu = \#\epsilon^{\mu\nu\rho}\partial_\nu C_\rho$ (where # is some coefficient I couldn't read). We find a similar structure to 4d: instead of complex manifolds we get manifolds with a transverse holomorphic fibration. The partition function only depends on that fibration. Mathematicians have classified such fibrations.

There are many ways of defining a squashed S^3, either squashing the metric, or keeping the metric round but tuning other supergravity fields, etc. It turns out that the partition function only depends on a single parameter b. Consider now starting from $b = 1$ (round sphere) and varying b. Then

$$\left.\frac{\delta\mathcal{L}}{\delta b^2}\right|_{b=1} = \left(A^\mu - \frac{3}{2}V^\mu\right) j^{(R)}_\mu + C^\mu j^{(Z)}_\mu . \tag{87}$$

Then

$$\left.\frac{\delta Z}{\delta b^2}\right|_{b=1} = \int \mathrm{d}^3x\sqrt{g}\int \mathrm{d}^3y\sqrt{g}\, v^\mu v^\nu \langle j^{(R)}_\mu(x) j^{(R)}_\nu(y)\rangle + \text{contact terms} \tag{88}$$

It can be shown that the contact terms can only affect the imaginary part of the partition function. For a general superconformal field theory we can write

$$\langle j^{(R)}_\mu(x) j^{(R)}_\nu(y)\rangle = \frac{\tau_{RR}}{16\pi^2}(\delta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)\frac{1}{|x|^2} \tag{89}$$

The single parameter τ_{RR} also controls the stress-tensor two-point function. By a conformal transformation we can map such two-point functions to the plane, so through the localization calculation we learn the flat-space two-point function of the stress tensor.