4d $\mathcal{N} = 2$ localization

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July 16–20, 2018

Nobody (even the typist) proof-read these notes, so there may be obvious mistakes: tell BLF.

Abstract

4d $\mathcal{N} = 2$ supersymmetries. These are lecture notes for the 2018 IHÉS summer school on *Supersymmetric localization and exact results.*

These lecture notes assume familiarity with supersymmetry at the level of the first few chapters of the book by Wess and Bagger.

1 Lecture 1, July 16

We work in Euclidean signature. A 4d $\mathcal{N} = 2$ supersymmetric theory is a QFT invariant under $\mathcal{N} = 2$ super-Poincaré symmetry. The generators of the supersymmetry algebra are P_m ($m = 1, \dots, 4$) translations, M_{mn} (antisymmetric) rotations, supercharges $Q_{I\alpha}$ and $\widetilde{Q}^{I\dot{\alpha}}$ for $I = 1, 2$ labelling the two supersymmetries, and finally two central charges Z and $\widetilde{Z}$. (Anti)commutators are

$$\{Q_{J\alpha}, \widetilde{Q}^{I\dot{\alpha}}\} = \delta^I_J (\sigma^m)_\alpha{}^{\dot{\alpha}} P_m \tag{1}$$

$$\{Q_{I\alpha}, Q_{J\beta}\} = \epsilon_{IJ}\epsilon_{\alpha\beta} Z \tag{2}$$

$$[Z, \text{anything}] = 0. \tag{3}$$

The super-Poincaré algebra has $SU(2)_R \times U(1)_r$ symmetry when $Z = \widetilde{Z} = 0$ but a non-zero $(Z, \widetilde{Z})$ breaks $U(1)_r$.

Massless matter multiplets:

- Vector multiplet $(A_m, \phi, \widetilde{\phi}, \lambda_{I\alpha}, \widetilde{\lambda}^{I\dot{\alpha}}, D_{IJ})$ where all of the components are in the adjoint representation of the gauge group. In terms of 4d $\mathcal{N} = 1$ multiplets this decomposes into a vector multiplet and an adjoint chiral multiplet.
- Hypermultiplet $(q_{IA}, \psi_{A\alpha}, \widetilde{\psi}^{\dot{\alpha}}_A)$ where A is a flavour index. For n flavours this has $USp(2n)$ flavour symmetry.

Consider a single hypermultiplet and call its scalars Q and $\widetilde{Q}$. The four real scalars have $\mathfrak{so}(4) = \mathfrak{su}(2)_R \times \mathfrak{su}(2)_{\text{flavour}}$ symmetry

$$\begin{matrix} Q & \widetilde{Q} \\ -\widetilde{Q}^* & Q^* \end{matrix} \tag{4}$$

where $SU(2)_R$ mixes the rows while $USp(2)_{\text{flavour}}$ mixes the columns.

Flavour symmetry of N hypermultiplets transforming in representation R of the gauge group G:

- If R is a complex representation then the flavour symmetry is $U(N)$.
- If R is a real representation then the flavour symmetry is $USp(2N)$.
- If R is pseudoreal then the flavour symmetry is $SO(2N)$.

Consider a (constant) supercharge $\delta = \xi^{I\alpha} Q_{I\alpha} + \widetilde{\xi}_{I\dot{\alpha}} \widetilde{Q}^{I\dot{\alpha}}$. For instance $\delta A_m = i\xi^I \sigma_m \widetilde{\lambda}_I - i\widetilde{\xi}^I \widetilde{\sigma}_m \lambda_I$. The *commuting* spinor parameters ξ^I are just there to keep track of what supercharge we are looking at.

The Yang–Mills action is

$$\mathcal{L}_{\text{Yang–Mills}} = \frac{1}{2} F_{mn} F^{mn} - 4 D_m \widetilde{\phi} D^m \phi - \frac{1}{2} D_{IJ} D^{IJ} + 4[\phi, \widetilde{\phi}]^2 + \cdots \tag{5}$$

and we could write the hypermultiplet action too. Supersymmetry invariance implies the existence of a conserved current: when applying the Noether procedure (namely considering a position-dependent supersymmetry variation),

$$\delta \mathcal{L} = \partial_m k^m + \partial_m \xi_I J^{Im} + \partial_\mu \widetilde{\xi}_I \widetilde{J}^{Im} \tag{6}$$

where J and $\widetilde{J}$ are conserved currents (the $S_{\mu\alpha}$ of Guido Festuccia's talk). By adding a few terms of the form $\sigma^m D_m \widetilde{\xi}_I \phi$ to the supersymmetry transformations we can ensure that J obeys a further nice property that $\widetilde{\sigma}^m J^I_m = 0 = \sigma^m \widetilde{J}^I_m$.

So we find an invariance of the action under δ provided that

$$\partial_m \xi_I = -i \sigma_m \widetilde{\xi}'_I \tag{7}$$

$$\partial_m \widetilde{\xi}_I = -i \widetilde{\sigma}_m \xi'_I \tag{8}$$

where ξ' and $\widetilde{\xi}'$ are whatever they need to be. Solving this we find

$$\xi_I = \hat{\xi}^{(s)}_I + (-i) x^m \sigma_m \hat{\widetilde{\xi}}^{(c)}_I \tag{9}$$

$$\widetilde{\xi}_I = \hat{\widetilde{\xi}}^{(s)}_I + (-i) x^m \widetilde{\sigma}_m \hat{\xi}^{(c)}_I \tag{10}$$

where hatted spinors are constants and (s) and (c) stand for "supersymmetry" and "superconformal".

What is now the algebra of these generalized supersymmetry variations? Namely, what is $\left(\delta_{\xi,\tilde{\xi}}\right)^2$? (For Poincaré supersymmetry only the first term below would remain.)

$$\left(\delta_{\xi,\tilde{\xi}}\right)^2 = i\mathcal{L}_v + \text{Scale}(w) + U(1)_r(\theta) + SU(2)_R(\theta_{IJ}) + \text{gauge transformation} \tag{11}$$

where $v = 2\widetilde{\xi}^I\widetilde{\sigma}^m\xi_I$. Interestingly, $\partial_m v_n + \partial_n v_m = \frac{1}{2}\delta_{mn}\partial^k v_k$, which is the conformal Killing vector equation, namely v generates conformal transformation, namely $\mathcal{L}_v g \sim g$ (it rescales the metric). Plugging in $v = 2\widetilde{\xi}^I\widetilde{\sigma}^m\xi_I$ the explicit spinors we found earlier (that depend linearly on the position x), we find an expression quadratic in position:

$$v^m = \underbrace{a^m}_{\text{translation}} + \underbrace{\lambda^m{}_n x^n}_{\text{rotation}} + \underbrace{\lambda_D}_{\text{dilation}} x^m + \underbrace{k^l}_{\text{special conformal}}(x^2\delta^m_l - 2x^m x_l) \tag{12}$$

Exercise: work out the whole superconformal algebra

$$[D, P_m] = P_m \tag{13}$$
$$[D, K_m] = -K_m \tag{14}$$
$$[K^m, P_n] = \delta^m_n D + M^m{}_n \tag{15}$$
$$\{Q_{\alpha I}, \widetilde{Q}^{\dot{\alpha} J}\} = \delta^J_I P_\alpha{}^{\dot{\alpha}} \tag{16}$$
$$\{S_{\alpha I}, \widetilde{S}^{\dot{\alpha} J}\} = \delta^J_I K_\alpha{}^{\dot{\alpha}} \tag{17}$$
$$\{Q, S\} = D + M + R + r \quad \text{(schematically)} \tag{18}$$

We have learned that the 4d $\mathcal{N} = 2$ theory is supercoformal invariant, *classically*. Quantumly we need to make sure that the beta function vanishes, which happens for instance in 4d $\mathcal{N} = 4$, or in 4d $\mathcal{N} = 2$ theories with appropriate matter content.

1.1 Weyl vs conformal transformations

Weyl transformations are just local rescalings of the metric and fields $g_{\mu\nu}(x) \to \Omega(x)^2 g_{\mu\nu}(x)$ and $\varphi(x) \to \Omega(x)^{-\Delta_\varphi}\varphi(x)$.

Conformal transformations are more restrictive: they are diffeomorphisms $g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x)$ such that $g' = \Omega^2 g$ can be brought back to the original metric by a Weyl transformation.

In fact a conformally-invariant unitary theory turns out to be Weyl-invariant (see a paper by Luti etc).

We are interested in Weyl-covariant objects.

- The conformal Killing spinor equations $\nabla_m\xi_I = -i\sigma_m\widetilde{\xi}'_I$ (where $\partial_m\xi_I + \frac{1}{4}\omega_m{}^{rs}\sigma_{rs}\xi_I$ are invariant under $\xi \to \Omega^{1/2}\xi$ and $g \to \Omega^2 g$.

- The covariantized supersymmetry transformation rules (with extra terms such as $\sigma^m D_m \widetilde{\xi}_I \phi$) are Weyl covariant.
- Another covariant object is the Laplacian $\Box - R/6$, where R is the Ricci scalar.

To put the theory on S^4 of radius r, notice that this manifold is locally conformally flat:

$$\mathrm{d}s^2_{S^4_r} = \frac{1}{\big(1 + x^2/(4r^2)\big)^2} (\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 + \mathrm{d}x_4^2) \tag{19}$$

where $x^2 = \sum_i (x_i)^2$. By a conformal transformation from flat space we can put the 4d $\mathcal{N} = 2$ supersymmetric theory on S^4.

The superconformal algebra in signature $(+,+,+,-)$ is $\mathfrak{su}(2,2|\mathcal{N})$, with bosonic components $\mathfrak{su}(2,2) \simeq \mathfrak{so}(4,2)$ and $\mathfrak{su}(\mathcal{N}) \times \mathfrak{u}(1)$. In Euclidean signature the correct superalgebra is $\mathfrak{su}^*(4|2)$.

In flat space it is useful to consider the algebra under which massive (namely non-conformal) theories are invariant, namely the super-Poincaré algebra. This is half of all supercharges in the superconformal algebra. Doing the same exercise on S^4 we find the supersymmetry algebra $\mathfrak{osp}(2|4)$, containing the $\mathfrak{sp}(4) = \mathfrak{so}(5)$ isometries of S^4. This subalgebra can be described in the language of conformal Killing spinors by the equation

$$\nabla_m \xi_I = -i\sigma_m \widetilde{\xi}'_I, \qquad \widetilde{\xi}'_I = -iS_{\mu\nu}\sigma^{\mu\nu}\xi_I \tag{20}$$

for some arbitrary $S_{\mu\nu}$. Then we find explicitly

$$\xi_I = \frac{1}{\sqrt{\cdots}} \Big(\hat{\xi}_I^{(s)} - i\sigma^m x_m \hat{\widetilde{\xi}}_I^{(c)} \Big) \tag{21}$$

$$\widetilde{\xi}_I = \frac{1}{\sqrt{\cdots}} \Big(\hat{\widetilde{\xi}}_I^{(s)} - i\sigma^m x_m \hat{\xi}_I^{(c)} \Big) \tag{22}$$

with $\hat{\widetilde{\xi}}^{(c)} \sim \frac{1}{2r} \hat{\widetilde{\xi}}^{(s)}$.

Beyond putting the theory on S^4 we can in fact put it on

$$S^4_b \colon \left\{ \frac{x_0^2}{r^2} + \frac{x_1^2 + x_2^2}{\ell^2} + \frac{x_3^2 + x_4^2}{\tilde{\ell}^2} = 1 \right\} \qquad \text{and we set } b = \sqrt{\ell/\tilde{\ell}}. \tag{23}$$

It is called the squashed sphere.

1.2 Localization on S^4

See Benini's lecture tomorrow for the supersymmetric localization argument. Provided $Q[D\phi]$ and $QS = 0$ and $Q\mathcal{O} = 0$ and $\int Q^2 V = 0$, we can apply supersymmetric localization:

$$I_t := \int [D\phi] \mathcal{O} e^{-S[\phi] - t\int QV} \tag{24}$$

has a vanishing t-derivative. Now if $QV|_{\text{bosonic}} \geq 0$ one can take $t \to +\infty$. It introduces an infinitely steep potential and we eventually find

$$\int [D\phi] \mathcal{O} e^{-S[\phi]} = \sum\!\!\!\!\!\!\int_{\{\phi_0\}=\text{zeros of } QV|_{\text{bos}}} \left(\mathcal{O} e^{-S}\right)|_{\phi_0} Z_{\text{1-loop}}[\phi_0] \tag{25}$$

Two choices: Q and V. Three things to determine: localization locus $\{\phi_0\}$, value of observables and classical action for ϕ_0, one-loop determinant $Z_{\text{1-loop}}$.

Choose Q such that $Q^2 = bJ_{12} + b^{-1}J_{34} + (b + b^{-1})R$. In other words choose ξ and $\widetilde{\xi}$. It turns out ξ vanishes at the North pole and $\widetilde{\xi}$ at the South pole.

Choose the canonical V

$$V = \sum_{\text{fermions } \psi} (Q\psi)^\dagger \psi \tag{26}$$

Exercise: check $Q^2 V = 0$.

Warning: before starting we need to realize the supersymmetry Q off-shell, otherwise $Q^2 V$ will only vanish up to the equations of motion.

Step 1. Let us now find zeros of $QV|_{\text{bosonic}}$. This is $Q\psi = 0$ for all fermions ψ, subject to reality conditions. The conclusion is that for the hypermultiplet all hypermultiplet fields must vanish. For the vector multiplet, smooth solutions have: $F_{mn} = 0$ (so in some choice of gauge $A_m = 0$), $\phi = \widetilde{\phi} = \frac{-i}{2} a_0$ is constant, and $D_{IJ} = -ia_0 w_{IJ}$ where $w_{IJ} = -4\xi_I \sigma^{mn} \xi_J S_{mn} / (\xi^K \xi_K)$. Everything is controlled by the same constant a_0. On the other hand, at the North pole ξ vanishes so the constraint on F is a bit weaker, namely $F^+_{\mu\nu} = 0$ while at the South pole $F^-_{\mu\nu} = 0$. These equations are the instanton and anti-instanton equations respectively.

Step 2. The hypermultiplet action contributes nothing. The vector multiplet action (5) gives

$$\frac{1}{g^2_{\text{YM}}} \int \mathrm{d}^4 x \sqrt{g} \mathcal{L}_{\text{Yang–Mills}} = \frac{8\pi^2}{g^2_{\text{YM}}} \operatorname{Tr} \hat{a}_0^2 \tag{27}$$

where $\hat{a}_0 = \sqrt{\ell \tilde{\ell}} a_0$.

Step 3. Then we need to write the contribution from quadratic fluctuations. One way on the round S^4 is to decompose into spherical harmonics. Alternatively one can pair up bosonic and fermionic modes whose contributions cancel. A more conceptual way is to use an index theorem; see Zabzine's lectures next week.

At the end of the day,

$$Z^{\text{VM}}_{\text{1-loop}} = \prod_{\text{roots } \alpha > 0} \Upsilon_b(i\alpha(\hat{a}_0)) \Upsilon_b(-i\alpha(\hat{a}_0)) \tag{28}$$

$$Z^{\text{HM}}_{\text{1-loop}} = \prod_{\text{weights } w \text{ of } R} \Upsilon_b(iw(\hat{a}_0) + Q/2) \tag{29}$$

with $Q = b + b^{-1}$. Here

$$\Upsilon_b(x) = \prod_{m,n \geq 0} (mb + nb^{-1} + x)((m+1)b + (n+1)b^{-1} - x) \tag{30}$$

regularized.

2 Lecture 2, July 17

Recap from yesterday. We built 4d $\mathcal{N}=2$ theories on S^4. With the full power of supergravity one can do the same on S^4_b, see `https://arxiv.org/abs/1206.6359`. The localizing supercharge Q obeys

$$Q^2 = bJ_{12} + b^{-1}J_{34} + (b+b^{-1})R. \tag{31}$$

We take $V = \sum_{\text{fermions } \psi} (Q\psi)^\dagger \psi$ and reduce to the locus $QV|_{\text{bosonic}} = 0$.

Smooth solutions have hypermultiplets set to zero and vector multiplets set to zero except the vector multiplet scalar $\phi = \widetilde{\phi} = \frac{-i}{2} a_0$ and auxiliary field $D_{IJ} = -i a_0 w_{IJ}$ where w_{IJ} is built from the generalized Killing spinors, see last lecture.

There are also singular solutions with $F^+ = 0$ but $F^- \neq 0$ at the North pole, and $F^- = 0$ but $F^+ \neq 0$ at the South pole.

We can compute the classical action and one-loop determinants.

$$\frac{1}{g^2_{\text{YM}}} \int \mathrm{d}^4x \sqrt{g} \mathcal{L}_{\text{Yang–Mills}} = \frac{8\pi^2}{g^2_{\text{YM}}} \operatorname{Tr} \hat{a}_0^2 \tag{32}$$

$$Z^{\text{VM}}_{\text{1-loop}} = \prod_{\text{roots } \alpha>0} \Upsilon_b(i\alpha(\hat{a}_0)) \Upsilon_b(-i\alpha(\hat{a}_0)) \tag{33}$$

$$Z^{\text{HM}}_{\text{1-loop}} = \prod_{\text{weights } w \text{ of } R} \Upsilon_b(iw(\hat{a}_0) + Q/2) \tag{34}$$

and combine them into

$$Z = \int \mathrm{d}\hat{a}_0 \, e^{\frac{-8\pi^2}{g^2_{\text{YM}}} \operatorname{Tr} \hat{a}_0^2} Z^{\text{vector multiplet}}_{\text{1-loop}}(\hat{a}_0) Z^{\text{hypermultiplet}}_{\text{1-loop}}(\hat{a}_0) \times Z_{\text{nonperturbative}}. \tag{35}$$

We still have to understand the nonperturbative contribution due to instantons and antiinstantons at the poles.

Masses are easily introduced by noticing that they are equivalent to background values for the scalar field in a nondynamical vector multiplet coupled to the given flavour symmetry.

What operators can we add?

- We can put chiral operators at the North pole (concretely $Tr\phi^k$) and the South pole (concretely $\operatorname{Tr} \widetilde{\phi}^k$). To perform the computation we just need to insert inside the integral (35) the value of these operators on the localization locus.
- Wilson loops, see Pestun's original paper;
- 't Hooft loops, see paper by Gomis–Okuda–Pestun;
- surface operators, either as disorder operator or defect operators.

2.1 Instantons

Our goal is to find the nonperturbative contribution from instantons at the poles. Infinitesimally close to the poles the sphere is close to flat space so it makes sense to ask the question in flat space.

Back to non-supersymmetric Yang–Mills in flat space.

$$\frac{1}{g_{\mathrm{YM}}^2}\int \mathrm{d}^4x\,\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) \tag{36}$$

Search for finite-action solutions solving the equation of motion. Note that

$$\int \mathrm{Tr}\,F\wedge\star F = \frac{1}{2}\int \mathrm{Tr}(F\pm\star F)\wedge(F\pm\star F)\mp\int \mathrm{Tr}\,F\wedge F \geq \mp\int \mathrm{Tr}\,F\wedge F \tag{37}$$

The inequality is saturated if $F=\pm\star F$. Solutions of this *first order* differential equation, called (anti)selfdual connections or (anti)instantons automatically obey the equations of motion $DF=\pm D\star F=0$ (by the Bianchi identity).

The lower bound is a topological charge. Gauge field configurations with $F_{\mu\nu}\to 0$ fast enough at infinity (to keep the action finite) must be pure gauge at infinity, which means that $A_m \xrightarrow{i\to\infty} g^{-1}\partial_m g$. The function $g\colon S^3_\infty\to G$ cannot necessarily be continuously deformed to a constant, and the obstruction is measured by $\pi_3(G)$, equal to $\mathbb{Z}$ for any simple gauge group G other than $U(1)$. The precise map is

$$\begin{cases}\pi_3(G) & \to \mathbb{Z}\\ g & \mapsto \frac{1}{8\pi^2}\int \mathrm{Tr}(F\wedge F).\end{cases} \tag{38}$$

We will typically denote the instanton number by $k=\frac{1}{8\pi^2}\int\mathrm{Tr}(F\wedge F)$.

Example: $k=1$ instanton for $SU(2)$ gauge group

$$A_\mu^{\mathrm{singular}}(x)=\frac{\rho^2(x-X)_\nu}{(x-X)^2\big((x-X)^2+\rho^2\big)}\widetilde{\eta}^i_{\mu\nu}(g\sigma^i g^{-1}) \tag{39}$$

where $\widetilde{\eta}$ is defined by $\sigma_{\mu\nu}=\frac{1}{2}(\sigma_\mu\widetilde{\sigma}_\nu-\sigma_\nu\widetilde{\sigma}_\mu)=\widetilde{\eta}^i_{\mu\nu}\tau_i$. This solution is characterized by 8 coordinates:

- X position of the instanton (4 coordinates);
- ρ size (1 coordinate);
- $g\in SU(2)$ global gauge (3 coordinates).

What about $SU(N)$ instantons? We can map the instanton above using any embedding $SU(2)\to SU(N)$. Changing the embedding corresponds to conjugating by $h\in SU(N)$, but many choices of h give the same instanton, namely conjugating by an element of $U(N-2)$ changes nothing and conjugating by an element of $SU(2)\subset SU(N)$ is equivalent to changing $g\in SU(2)$. Altogether, choices are parametrized by $SU(N)/S[U(N-2)\times U(2)]$. Together with the 8 coordinates found for $SU(2)$ we get

$$8+(N^2-1)-((N-2)^2+4-1)=4N. \tag{40}$$

For multiple instantons, it can be shown using an index theorem that

$$\dim \mathcal{M}_{k,SU(N)}=4kN. \tag{41}$$

The index theorem relates the dimension of the k-instanton moduli space with the number of zero-modes, namely normalizable solutions to linearized field equations for fluctuations.

To be more concrete, linearize around a solution A_μ^{cl}. We write $A_\mu = A_\mu^{\text{cl}}(X)+\delta A_\mu$, where X are coordinates on the moduli space of instantons, and get

$$D_\mu \delta A_\nu - D_\nu \delta A_\mu = \epsilon_{\mu\nu\rho\sigma} D^\rho \delta A^\sigma \tag{42}$$

but we want to restrict to solutions that are orthogonal to the space of gauge transformations, so

$$0=\int \mathrm{d}^4x \ \mathrm{Tr}\, \delta A_\mu D^\mu \Lambda \quad \Leftrightarrow \quad D_\mu \delta A^\mu = 0. \tag{43}$$

Solutions to these equations are in one-to-one correspondence with coordinates on the instanton moduli space. In one direction it is straightforward: the variation of A_μ^{cl} with respect to a coordinate α obeys the equations above. On the other hand, $\frac{\delta S^{\text{cl}}}{\delta A_\mu^{\text{cl}}}=0$ implies

$$0=\frac{\partial}{\partial x^\alpha}\frac{\delta S^{\text{cl}}}{\delta A_\mu^{\text{cl}}}=\int \frac{\delta^2 S^{\text{cl}}}{\delta A_\mu^{\text{cl}}(x)\delta A_\mu^{\text{cl}}(y)}\frac{\partial A_\mu^{\text{cl}}(y)}{\partial x^\alpha}\mathrm{d}^4 y. \tag{44}$$

The zero-modes also provide us with a metric:

$$g_{\alpha\beta}=\int \mathrm{d}^4x\, \mathrm{Tr}\big(\delta_\alpha A_\mu \delta_\beta A^\mu\big). \tag{45}$$

Adding Fermions. Just like we studied bosonic zero modes to find collective coordinates it makes sense to find fermionic zero modes.

The supersymmetry variation is

$$\delta\lambda_I = \frac{1}{2}\sigma^{mn}\xi_I F_{mn}+\cdots \tag{46}$$

$$\delta\widetilde{\lambda}_I = \frac{1}{2}\sigma^{mn}\widetilde{\xi}_I F_{mn}+\cdots \tag{47}$$

For an instanton background the $\widetilde{\xi}_I$ supersymmetry is broken but $\delta\lambda_I=0$ so ξ_I is preserved. Broken supersymmetries give rise to some of the kN fermionic zero-modes, but not all.

The fact that instantons preserve some supersymmetry implies that bosonic zero modes will be paired with fermionic zero-modes in multiplets of the preserved supersymmetry. Two questions: which supersymmetry? which supermultiplets?

So, let us look at fermionic zero modes

$$\widetilde{\not{D}}\lambda = 0, \qquad \not{D}\widetilde{\lambda} = 0 \tag{48}$$

where $\widetilde{\not{D}} = \widetilde{\sigma}^\mu D_\mu$. Compute

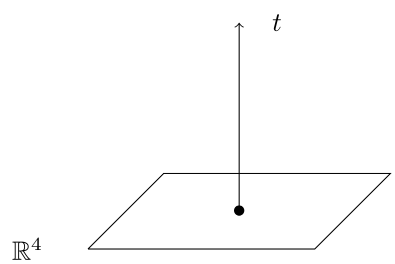
$$\widetilde{\not{D}}\not{D}\widetilde{\lambda} = \widetilde{\sigma}^\mu \sigma^\nu D_\mu D_\nu \widetilde{\lambda} = (\delta^{\mu\nu} + \sigma^{\mu\nu}) D_\mu D_\nu \widetilde{\lambda} = D^2\widetilde{\lambda} + \frac{1}{2}\sigma^{\mu\nu} F_{\mu\nu}\widetilde{\lambda}. \tag{49}$$

If we are considering instantons (so $k > 0$), then the last term vanishes. Since $D^2\widetilde{\lambda} = 0$ implies $\widetilde{\lambda} = 0$ we learn that $\not{D}\widetilde{\lambda} = 0$ has no nontrivial solution.

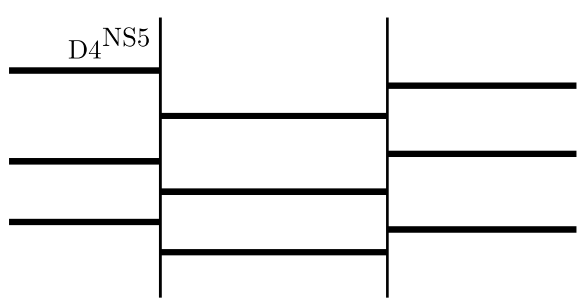
On the other hand $\widetilde{\not{D}}\lambda = 0$ has non-trivial solutions. In fact index theorems show that the number of zero-modes of $\widetilde{\not{D}}$ is kN if λ is in the fundamental representations. For other representations we find other countings (in particular for the adjoint it has to match with the bosonic zero modes).

2.2 ADHM matrix model

Lift to one dimension up, namely consider the 5d $\mathcal{N} = 1$ analogue of our 4d $\mathcal{N} = 2$ theories we are interested in. Then the analogue of instantons will be 1d objects that in the slow-varying approximation are described by a 1d supersymmetric quantum mechanics, whose supersymmetry is 2d $\mathcal{N} = (0, 4)$ reduced to 1d.



4d $\mathcal{N} = 2$ gauge theories can be realized in string theories as the world-volume theory on D4 branes stretching between NS5 branes.



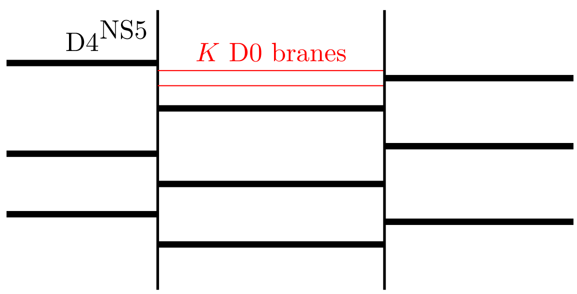
Let us unravel this a little bit.

- The world-volume theory on infinite D4 branes is 5d $\mathcal{N}=2$ super Yang–Mills.
- The world-volume theory on D4 branes stretching between NS5 branes is 4d $\mathcal{N}=2$ super Yang–Mills (so half as many supercharges as the infinite D4 branes had).
- The infinite D4 branes stretching out of the diagram give rise to flavour symmetries.

The matter content is read by looking at strings stretching between branes:

- strings stretching between the middle N D4 branes to each other gives rise to an adjoint hypermultiplet of the $SU(N)$ gauge group;
- strings stretching between the middle N D4 branes and the outer D4 branes gives rise to $N+N$ (one set for each side of the diagram) fundamental hypermultiplets of the $SU(N)$ gauge group

Now instantons are realized by D0 branes within the D4 branes and stretching between NS5 branes too. (Read paper by Michael Douglas "Branes within branes".)



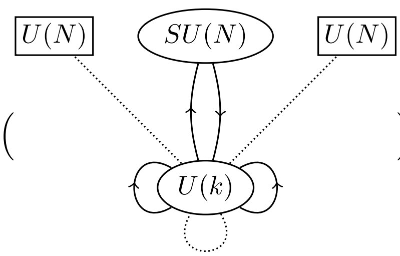
We can read off the field content of the 0d theory. It preserves 4 supercharges (half of the 4d $\mathcal{N}=2$), and multiplets coincide with dimensional reductions of 2d $\mathcal{N}=(0,4)$ multiplets:

- D0–D0 strings give a $U(k)$ vector multiplet and a $U(k)$ hypermultiplet, which in $\mathcal{N}=(0,2)$ language means a vector multiplet and Fermi multiplet, plus a pair of chiral multiplets
- D0–D4$_{\text{left}}$ strings give Fermi multiplets in the bifundamental representation of the $U(k)$ gauge group and the left $U(N)$ flavour symmetry group.
- D0–D4$_{\text{right}}$ strings give Fermi multiplets in the bifundamental representation of the $U(k)$ gauge group and the right $U(N)$ flavour symmetry group.

- D0–D4$_{\text{middle}}$ strings give Fermi multiplets in the hypermultiplet representation of the $U(k)$ gauge group and the 4d $SU(N)$ gauge group.

Instantons are described by D0 branes that lie inside the D4 branes. The possible configurations are parametrized by the Higgs branch of the 0d matrix model described above:

$$\mathcal{M}_{k,SU(N)\text{ with }2N\text{ flavours}} = \text{Higgs branch}\Big(\quad \Big) \tag{50}$$



where we used 2d $\mathcal{N} = (0, 2)$ quiver notations:

- dotted lines are Fermi multiplets,
- arrows are chiral multiplets,
- round nodes are gauge groups (the $SU(N)$ gauge group is a 4d gauge group, the $U(k)$ gauge group is a 0d gauge group),
- square nodes are flavour groups.

References:

- Tong's lecture notes;
- Vandoren and van Nieuwenhuizen;
- Douglas "Branes within branes" `https://arxiv.org/abs/hep-th/9512077` and "Gauge fields and D-branes" `https://arxiv.org/abs/hep-th/9604198`.

3 Lecture 3, July 18

Summary of yesterday: we solved $F^+ = 0$ in flat space in the topological sector with $\frac{1}{8\pi^2}\int F \wedge F = k \in \mathbb{Z}_{\geq 0}$. We found that there is a moduli space of dimension $4kN$. The local coordinates X on this moduli space are collective coordinates of the k instantons. Directions are in one-to-one correspondence with bosonic zero-modes $A_\mu = A_\mu^{\text{cl}}(X) + \delta A_\mu$. If the theory has fermions there are also fermionic zero-modes around the instanton background.

In a supersymmetric gauge theory, instanton backgrounds turn out to preserve half of the supersymmetry. The bosonic and fermionic zero-modes are paired by that supersymmetry. In 4d $\mathcal{N} = 2$ theories the residual 0d symmetry coincides with the dimensional reduction of 2d $\mathcal{N} = (0, 4)$ supersymmetry.

Conceptual jump: consider the same problem in a $(4+1)$d theory. On each time slice fix $\int F \wedge F = 8\pi^2 k$. Now instantons are 1d objects (namely particles), and we can study the dynamics of these instantons. How do we determine this 1d theory? In a supersymmetric theory this is going to be some 1d supersymmetric theory. We can determine it by considering D0 branes inside D4 branes as discussed yesterday. The outcome is (50).

3.1 Back to localizing on S^4

The 4d localizing supercharge obeys $Q^2 = bJ_{12} + b^{-1}J_{34} + (b+b^{-1})R + iM \cdot \text{flavour} + i\phi \cdot \text{gauge}$ where b describes the squashing of the sphere S^4_b. We know that instantons are solutions of the BPS equations, namely Q is preserved by the instanton background, so Q must be a supersymmetry of the instanton world-volume theory. We can rewrite

$$Q^2_{0\text{d}} = (b+b^{-1})J + (b-b^{-1})J_{\text{left}} + iM \cdot \text{flavour} + i\phi \cdot (\text{4d gauge}) + i\varphi_{0\text{d}} \cdot (\text{0d gauge}). \tag{51}$$

where $\varphi_{0\text{d}}$ is a scalar in the $U(k)$ vector multiplet of the 0d theory.

$\mathcal{N}=(0,2)$	FM	CM	CM	FM	VM	FM	CM	CM
J	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	1	$\frac{1}{2}$	$\frac{1}{2}$
J_{left}	0	0	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$
$U(k)$	k	$\overline{k}$	k	$\overline{k}$	adj	adj	adj	adj
$SU(N)$		N	$\overline{N}$					
$U(N)_{\text{left}}$	$\overline{N}$							
$U(N)_{\text{right}}$			N					

where empty entries are trivial representations and CM is chiral multiplet and FM is Fermi multiplet and VM is vector multiplet. Using 2d $\mathcal{N}=(0,2)$ localization, or 1d $\mathcal{N}=2$ localization, or 0d localization, we get that the k-instanton partition function is

$$Z_k = \int_{\text{JK}} \left(\prod_{I=1}^{k} \mathrm{d}\varphi_I \right) Z_{\text{D0–D0}}(\varphi) Z_{\text{D0–D4}_{\text{left}}}(\varphi, M) Z_{\text{D0–D4}_{\text{right}}}(\varphi, \widetilde{M}) Z_{\text{D0–D4}_{\text{gauge}}}(\varphi, \phi) \tag{52}$$

Explicitly

$$Z_{\text{D0–D0}} = \prod I, J \frac{(\varphi_{IJ})'(\varphi_{IJ} + b + b^{-1})}{(\varphi_{IJ} + b)(\varphi_{IJ} + b^{-1})} \qquad \text{from all the adjoints of } U(k) \tag{53}$$

$$Z_{\text{D0–D4}_{\text{left}}} = \prod_{I=1}^{k} \prod_{A=1}^{N} (\varphi_I - iM_A) \tag{54}$$

$$Z_{\text{D0–D4}_{\text{right}}} = \prod_{I=1}^{k} \prod_{A=1}^{N} (\varphi_I - i\widetilde{M}_A) \tag{55}$$

$$Z_{\text{D0–D4}_{\text{gauge}}} = \prod_{I=1}^{k} \prod_{A=1}^{N} \frac{1}{\left(\varphi_I - i\phi_A + \frac{1}{2}(b + b^{-1})\right)\left(-\varphi_I + i\phi_A + \frac{1}{2}(b + b^{-1})\right)}. \tag{56}$$

The one-loop determinant of each $\mathcal{N} = (0, 2)$ multiplet is either a linear factor (for Fermi multiplets and for vector multiplets because their field strength lies in a Fermi multiplet) namely a fermionic contribution, or the inverse of a linear factor (for chiral multiplets) namely a bosonic contribution.

3.2 The final result

$$\begin{aligned} Z_{S^4_b} = \int \mathrm{d}\phi \, \exp\left(\frac{-8\pi^2 \operatorname{Tr} \phi^2}{g^2_{\text{YM}}}\right) Z^{\text{hypermultiplet}}_{\text{1-loop}}(\phi, M, \tilde{M}) \\ \times Z^{\text{vector multiplet}}_{\text{1-loop}}(\phi) \left(\sum_k q^k Z_k\right) \left(\sum_k \overline{q}^k Z_k\right). \end{aligned} \tag{57}$$

Here $q = \exp(2\pi i\tau)$ and $\tau = \theta/(2\pi) + 4\pi i/g^2_{\text{YM}}$. Note that the q^k comes from the contribution of the instanton to the classical action:

$$\exp\left(\frac{-1}{2g^2_{\text{YM}}} \operatorname{Tr} \int F \wedge \star F + \frac{\theta}{8\pi^2} \operatorname{Tr} \int F \wedge F\right) = q^k \qquad \text{using } F = \star F. \tag{58}$$

Let us compute the k-dimensional integral by closing the k contours. The "JK" (Jeffrey–Kirwan) prescription essentially states we should select poles only of factors that came from negatively charged fields. This means that for each I one of the following equations holds:

- $\varphi_I = i\phi_A + \frac{1}{2}(b + b^{-1})$ for some 4d color $1 \leq A \leq N$ (from the 0d point of view this is a flavour);
- $\varphi_I = \varphi_J + b$ for any other 0d color $1 \leq J \leq k$;
- $\varphi_I = \varphi_J + b^{-1}$ for any other 0d color $1 \leq J \leq k$.

For any such choice of φ we can draw a diagram by drawing a box for each φ_I that is equal to $i\phi_A + \frac{1}{2}(b + b^{-1})$, then for each $\varphi_I = \varphi_J + b$ write a box for

φ_I to the left of the box for φ_J, and for each $\varphi_I = \varphi_J + 1/b$ write a box for φ_I to the bottom of the box for φ_J. Due to the factor $(\varphi_{IJ})'$ we will not end up with boxes on top of each other. Due to a subtle interaction between the $(\varphi_{IJ} + b + b^{-1})$ numerator and $\varphi_{IJ} + b$ and $\varphi_{IJ} + b^{-1}$ denominator, the diagram we get is a Young diagram.

We end up writing Z_k as a sum over k-box collections of N Young diagrams, which you can find in Nekrasov's original paper and elsewhere:

$$Z_{\text{Nekrasov}} = \sum_k q^k Z_k = \sum_{\vec{Y}} q^{|\vec{Y}|} Z_{\vec{Y}} \tag{59}$$

where $\vec{Y} = (Y_1, \ldots, Y_N)$ is a collection of N Young diagrams and $|\vec{Y}|$ is the total number of boxes.

3.3 Higgs branch localization

Recall that adding a Q-exact term to the action does not change the result, so we can use a different deformation action. Instead of the standard V we use

$$V = \sum_{\text{fermions } \psi} (Q\psi)^\dagger \psi + \operatorname{Tr}\Big(H^{IJ}\Big(\xi_{(I}\lambda_{J)} - \widetilde{\xi}_{(I}\widetilde{\lambda}_{J)}\Big)\Big). \tag{60}$$

Here H^{IJ} is some functional of bosonic fields that we will specify later and that is in the triplet representation of $SU(2)_R$, namely I and J are symmetrized. Here ξ and $\widetilde{\xi}$ describe the supercharge $Q_{\xi,\widetilde{\xi}}$ and λ and $\widetilde{\lambda}$ are gaugini. We have

$$QV|_{\text{bosonic}} = \sum_{\text{fermions } \psi} (Q\psi)^\dagger Q\psi + \operatorname{Tr}\Big(H^{IJ}\Big(\xi_{(I}Q\lambda_{J)} - \widetilde{\xi}_{(I}Q\widetilde{\lambda}_{J)}\Big)\Big) \tag{61}$$

Note that $Q\lambda_I = \cdots + D_{IJ}\xi^J$ and $Q\widetilde{\lambda}_I = \cdots + D_{IJ}\widetilde{\xi}^J$ are linear in the auxiliary field D_{IJ}, while these fields appear quadratically in the standard part of $QV|_{\text{bosonic}}$. We can perform the Gaussian integral over D_{IJ} and get as a critical point[1]

$$D_{IJ} = \frac{-1}{2} H_{IJ} - i\phi_1 w_{IJ} + \cdots \tag{62}$$

where the dots are a $1/t$ or $1/t^2$ term due to the fact that we are looking at the critical point of $tQV + S$ and not just of tQV. On round S^4 we have $w_{IJ} = \left(\begin{smallmatrix} 0 & 1/r \\ 1/r & 0 \end{smallmatrix}\right)$. Then a calculation shows that $QV|_{\text{bosonic}}$ is a sum of positive terms.

The vector multiplet localization equations are $D_\mu \phi_1 = 0$, $v^\mu D_\mu \phi_2 = 0$, $[\phi_1, \phi_2] = 0$, and

$$-2s(F^-_{\mu\nu} - 4\phi_2 S_{\mu\nu}) + 2\big(\kappa \wedge \mathrm{d}_A \phi_2\big)^-_{\mu\nu} = \frac{1}{2} H_{IJ} \Theta^{IJ}_{\mu\nu} \tag{63}$$

$$-2\widetilde{s}(F^+_{\mu\nu} + 4\phi_2 \widetilde{S}_{\mu\nu}) - 2\big(\kappa \wedge \mathrm{d}_A \phi_2\big)^+_{\mu\nu} = -\frac{1}{2} H_{IJ} \widetilde{\Theta}^{IJ}_{\mu\nu} \tag{64}$$

[1] Actually our contour is $\phi = (\phi_2 - i\phi_1)/2$ and $\widetilde{\phi} = (-\phi_2 - i\phi_1)/2$ with ϕ_1 and ϕ_2 real.

with $s = \xi^I \xi_I$ and $\widetilde{s} = \widetilde{\xi}_I \widetilde{\xi}^I$ and $\kappa_\mu = g_{\mu\nu} v^\nu$ where $v^\nu = \xi^I \sigma^\mu \widetilde{\xi}_I$.

The hypermultiplet localization equations are $\phi_1 q_I = 0$ and $F_{IA} = 0$ and

$$-2\sigma^\mu \widetilde{\xi}^I D_\mu q_I - \sigma^\mu D_\mu \widetilde{\xi}^I q_I - 2\xi^I \phi_2 q_I = 0 \tag{65}$$

$$-2\widetilde{\sigma}^\mu \xi^I D_\mu q_I - \widetilde{\sigma}^\mu D_\mu \xi^I q_I + 2\widetilde{\xi}^I \phi_2 q_I = 0. \tag{66}$$

Let us see what these equations tell us for various choices of H^{IJ}.

- Choose $H^{IJ} = 0$. Then we retrieve the Coulomb branch localization locus $0 = \phi_2 = A_\mu$ and $\phi_1 = a =$ constant and $D_{IJ} = -iaw_{IJ}$ (this is (62) for $H^{IJ} = 0$). At the North and South poles we get instantons because the coefficients s and $\widetilde{s}$ each vanish at one pole.
- In the case of a $U(1)$ gauge group, choose

  $$H_{IJ} = -\frac{\zeta^{\mathrm{FI}}}{\ell} w_{IJ} - i \begin{pmatrix} Q\widetilde{Q} & QQ^\dagger - \widetilde{Q}\widetilde{Q}^\dagger \\ QQ^\dagger - \widetilde{Q}\widetilde{Q}^\dagger & -\widetilde{Q}^\dagger Q^\dagger \end{pmatrix}_{IJ}. \tag{67}$$

  The second term (the matrix) is the moment map of some flavour symmetry. We shall split the constant FI parameter $\zeta^{\mathrm{FI}} = \zeta_{\mathrm{vec}} + \zeta_{\mathrm{SW}}$ (where SW stands for Seiberg–Witten).

  In that case we have solutions that can be dubbed "deformed Coulomb branch", where hypermultiplet scalars vanish and

  $$A = \frac{\zeta}{3r} K, \qquad \phi_2 = \frac{1}{6\ell} \zeta \cos\rho, \qquad \phi_1 = a, \qquad D_{IJ} = \left(\frac{1}{2}\frac{\zeta}{\ell} - ia\right) w_{IJ} \tag{68}$$

  where since the FI parameter is in a $U(1)$ factor, there can still be instantons and antiinstantons at the poles. Note that in the $\zeta \to 0$ limit we retrive the Coulomb branch.

  We also have solutions with non-zero hypermultiplets

  [...]

  At certain points of the Coulomb branch we have a vanishing mass $\phi_1 + m = 0$ in $(\phi_1 + m) q_I = 0$. We can solve the simpler equestions $\zeta_{\mathrm{SW}} = QQ^\dagger - \widetilde{Q}\widetilde{Q}^\dagger$ and $Q\widetilde{Q} = 0$. Depending on the sign of the ζ_{SW} exactly one of the two Q and $\widetilde{Q}$ takes a nonzero value.

  We actually find solutions as linear combinations of giving a nonzero vev to Q hypermultiplet scalars or $\widetilde{Q}$ hypermultiplet scalars, and of deformed Coulomb branch vacua (68), but the two have different ζ parameters, which get summed. This works for special values of ζ for (68).

Assume that the ϕ_1 is uniquely solved in terms of the masses M in $(\phi_1 + M) q_I = 0$, then $U(N)$ is broken to $U(1)^N$.

On the other hand, imposing $\zeta_{\mathrm{SW}} = Q^\dagger Q - \widetilde{Q}\widetilde{Q}^\dagger$ gives a vev to the hypermultiplet scalars, whih breaks $U(1)^N$ completely by the Higgs mechanism. We get discrete vacua.

A **third type** of configurations we can find is by relaxing $H^{\rm SW}_{IJ} \neq 0$ and without loss of generality $\zeta > 0$. These equations are not exactly solvable but we can study their aspect by considering various limits. On S^4_b define two squashed two-spheres $S^2_\varphi \colon \{x_3 = x_4 = 0\} \subset S^4_b$ and $S^2_\chi \colon \{x_3 = x_4 = 0\} \subset S^4_b$ intersecting at the North pole and South pole.

We define the winding $Q \sim e^{im\varphi} e^{in\chi}$ far away from the core. There exists a vortex like solution whose core is each of the S^2: on the two-plane orthogonal to that sphere it looks like a vortex with winding number n around one two sphere and m around the other. These are smoothly joined together at the poles. See `https://arxiv.org/abs/1508.07329`.

The vortex size is proportional to $\sqrt{n/\zeta}$ which goes to zero as $\zeta \to \infty$, giving singular vortices.

Even though for finite ζ we cannot do anything analytically, but since S^4 has a finite size we can bound the maximum number of vortices at finite ζ. BPS equations in fact imply a bound

$$mb + nb^{-1} + \frac{Q}{2} \leq \frac{\zeta Q}{12}. \tag{69}$$

Summary: we have parameters ζ that we can tune. For $\zeta = 0$ we just get the Coulomb branch locus. For any finite ζ we get deformed Coulomb branch vacua. When the bound (69) gets saturated for some integers n and m then we pick up further solutions with a number of vortices n and m. We get discrete vacua.

From Coulomb branch intuition expect point-like solutions at the North and South poles. Locally around poles we have the complex manifold $\mathbb{C}^2$ away from each poles. The almost complex structure $\widetilde{J}^\mu{}_\nu : TM \to TM$ is integrable (Nijenhuis tensor vanishes) but is not well-defined on S^4.

Define $\alpha = Q \in \Omega^{(0,0)}_{\widetilde{J}}$ one hypermultiplet scalar and $\beta = -\widetilde{s}^{-1}\widetilde{Q}^\dagger \widetilde{\Theta}_{11} \in \Omega^{(0,2)}_{\widetilde{J}}$.

Expand around the deformed Coulomb branch vacuum $A = A_{\rm vac} + a$ and $\phi_2 = (\phi_2)_{\rm vac} + \Delta\phi_2$. The equations are

$$0 = \overline{\partial}_a \alpha + \overline{\partial}{}^*_a \beta \tag{70}$$

$$0 = F^{(0,2)}_a - \frac{i}{2}\overline{\alpha}\beta \tag{71}$$

and

$$F^{\widetilde{J}}_a = \frac{-1}{4}\left[\frac{\zeta_{\rm SW}}{2} + \frac{2\Delta\phi_2}{\ell} - |\alpha|^2 + |\beta|^2\right]\widetilde{J} \tag{72}$$

$$0 = \Delta\phi_2 \alpha = [\Delta\phi_2 + 2\ell^{-1}]\beta. \tag{73}$$

The first two equations imply $\alpha = 0$ or $\beta = 0$ and a few constraints on derivatives. Signs imply $\alpha \neq 0$ and we deduce $\Delta\phi_2 = 0$, and the equations reduce to Seiberg–Witten monopole equations.

[...]

3.4 Comparing partition functions

Since H_{IJ} was introduced through a Q-exact term, the answer needs to be independent of H_{IJ}. Even though the backgrounds are not known analytically, the one-loop determinant can be computed using an index theorem, which only requires knowing the solution near fixed points of Q^2. We find

$$Z_{\text{1-loop}}^{\text{vector multiplet}} = \prod_{\alpha\neq 0} \Upsilon_b\big(i\alpha(\hat{a})\big), \tag{74}$$

$$Z_{\text{1-loop}}^{\text{hypermultiplet}} = \prod_{w\in\text{weights}(R)} \Upsilon_b\big(iw(\hat{a}) + iM + Q/2\big)^{-1}, \tag{75}$$

where $\hat{a}_{\text{NP}} = \hat{a}_{\text{SP}} = \hat{a} = \sqrt{\ell\tilde{\ell}}\big(2i(\phi\widetilde{s} + \widetilde{\phi}s) + 2iv^\mu A_\mu\big)$ gives the same value when evaluated at the North pole and at the South pole.

On the deformed Coulomb branch we get

$$S_{\text{cl}}^{\text{vector multiplet}} = \frac{8\pi^2\ell\tilde{\ell}}{\text{Tr}}\left(a + \frac{i\zeta Q}{12\sqrt{\ell\tilde{\ell}}}\right) \tag{76}$$

Evaluating $\hat{a}$ on the deformed Coulomb branch configuration is

$$\hat{a} = \sqrt{\ell\tilde{\ell}}\left(a + \frac{i\zeta Q}{12\sqrt{\ell\tilde{\ell}}}\right). \tag{77}$$

The picture that emerges is that ζ moves the Coulomb branch integral transversely to the contour. The contour theorem tells us that the integral does not change provided we pick up discrete contributions associated to the poles that cross the integration contour. These poles that move from one side to the other as ζ is varied precisely correspond to vortex-like configurations and Seiberg–Witten monopoles that are allowed by the bound.

Importantly, for this picture to work, the instanton partition function, evaluated at $a = -M + mb + n/b$, had better match the contribution of Seiberg–Witten monopoles on top of the vortices.

Sending $\zeta \to \infty$ kills the deformed Coulomb branch integral and leads just to a sum over all vortex–vortex configurations. The result is

$$Z = \sum_{\text{Higgs vacua } v} Z_{\text{cl}}^v Z_{\text{1-loop}}^{\prime v} Z_{\text{resummed}}^v \tag{78}$$

where

$$Z_{\text{resummed}}^v = \sum_{m,n\geq 0} Z_{\text{cl}}^{(m,n)} Z_{\text{pert}}^{v,(m,n)} |Z_{\text{SW}}^{(m,n)}|^2 \tag{79}$$

For $n = 0$ we should expect the summand to become the S^2 partition function of the m-vortex theory.

3.5 AGT correspondence

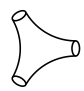
Claim: $Z_{S^4}^{\text{class S}} \leftrightarrow$2d Liouville (or Toda) CFT.

3.5.1 Theories of class S of type A_1

Gaiotto `https://arxiv.org/abs/0904.2715`.

The idea of class S theories is to associate a 4d $\mathcal{N} = 2$ theory to any Riemann surface.

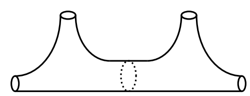
4 free hypermultiplets carry flavour symmetry $USp(8)$. Indeed, we have 16 real scalars, which have $SO(16)$ flavour symmetry group, but the $SU(2)_R$ symmetry has commutant $USp(8)$. This has a subgroup $SU(2)_a \times SU(2)_b \times SU(2)_c$. Let us draw a cute picture to describe 4 hypermultiplets:



(80)

Here each puncture represents an $SU(2)$ flavour symmetry.

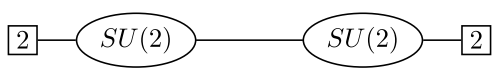
Next consider an $SU(2)$ gauge theory with $N_f = 4$ fundamental hypermultiplets (note that $\beta = 0$ so we have an SCFT. Since the fundamental of $SU(2)$ is pseudoreal the theory has $SO(8)$ flavour symmetry, but let us only make $\mathfrak{so}(4) \times \mathfrak{so}(4) = \mathfrak{su}(2)^4$ explicit. Then we draw a picture



(81)

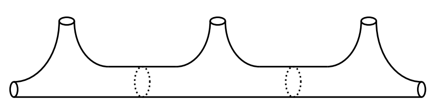
4 Lecture 5, July 16

Next, consider



(82)

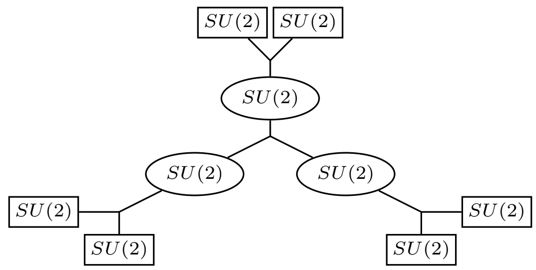
The 2 hypermultiplets on the left are in a pseudoreal representation of the $SU(2)$ gauge group hence carry $\mathfrak{so}(4) = \mathfrak{su}(2) \times \mathfrak{su}(2)$ flavour symmetry. The bifundamental hypermultiplet in the middle is in the $[2, 2]$ representation of the $SU(2) \times SU(2)$ gauge group, namely the vector representation of $\mathrm{Spin}(4) = SU(2) \times SU(2)$, so it it in a real representation and it carries $\mathfrak{usp}(2) = \mathfrak{su}(2)$ flavour symmetry. The 2 hypermultiplets on the right have $\mathfrak{su}(2)^2$ flavour symmetry. Altogether we have $\mathfrak{su}(2)^5$ flavour symmetry. Accordingly, the Riemann surface we draw is a sphere with five punctures:



(83)

The theory is built by taking three sets of 4 hypermultiplets and gauging diagonal $\mathfrak{su}(2)$ flavour symmetries. Each set of hypermultiplets is represented by a "pair of pants" (three-punctured sphere), and gauging is represented by gluing punctures together.

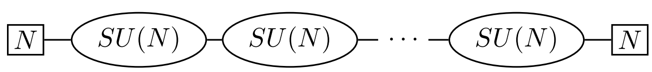
We can in fact do the same with generalized quivers such as



(84)

This corresponds to a six-punctured sphere obtained by gluing one three-punctured sphere to each puncture in a three-punctured sphere.

Some brief comments on higher-rank cases. Consider N^2 hypermultiplets with $N > 2$. This has $USp(2N^2)$ flavour symmetry, but we make $SU(N) \times SU(N) \times U(1)$ manifest. We depict such a theory of free hypermultiplets as a three-punctured sphere with punctures labeled by the corresponding flavour symmetry: there are two types of punctures, one for $SU(N)$ flavour symmetry (called "maximal puncture") and one for $U(1)$. By gauging together the $SU(N)$ flavour symmetries we can build more complicated Riemann surfaces. We get punctured spheres with two maximal punctures and some number of $U(1)$ punctures. The corresponding quiver gauge theory is



(85)

To get more general Riemann surfaces we need to know what a sphere with three maximal punctures corresponds to. It corresponds to an isolated non-Lagrangian SCFT called T_N that has $SU(N)^3$ flavour symmetry. For $N = 3$ this $SU(3)^3$ symmetry is enhanced to E_6 and the SCFT is actually the Minahan–Nemeschansky E_6 theory involved in Argyres–Seiberg duality.

4.1 Generalized S-duality

Let us go back to A_1 class S theories. The gauge coupling is encoded in the Riemann surface's complex structure as the parameter used for gluing three-punctured spheres: the coordinate z close to one puncture is identified with the coordinate w close to another puncture by $zw = \text{constant} = q$ where $q = e^{2\pi i \tau}$ where $\tau = \frac{4\pi i}{g_{\text{YM}}^2} + \frac{\theta}{2\pi}$.

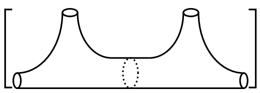
The weak coupling limit ($g_{\text{YM}} \to 0$) gives $\tau \to i\infty$ so $q \to 0$, meaning that the tube joining the two three-punctured spheres is very long.

When making the coupling stronger, we find that the punctures get close to each other in a different pattern, which can lead to a different weakly-coupled description. This is consistent with S-duality found by Seiberg and Witten in 4d $\mathcal{N} = 2$ $SU(2)$ with $N_f = 4$ (corresponding to the four-punctured sphere for us).

4.2 Partition function

We consider 4d $\mathcal{N}=2$ $SU(2)$ with 4 fundamental hypermultiplets, and we give masses $p_1, \dots, p_4$ by turning on backgrounds for vector multiplet scalars coupled to the four $SU(2)$ flavour symmetries. The partition function is

$$Z_{S^4}\left[\ \cdot\ \right] \tag{86}$$



$$= \int_{-\infty}^{\infty} \mathrm{d}P\, Z_{\text{1-loop}}[P, p_i] (q\overline{q})^{P^2} \left(\sum_{\vec{Y}} q^{|\vec{Y}|} \dots\right)\left(\sum_{\vec{Y}} \overline{q}^{|\vec{Y}|} \dots\right) \tag{87}$$

where

$$Z_{\text{1-loop}}[P, p_i] = \frac{\Upsilon(2iP)\Upsilon(-2iP)}{\prod_{j=1}^{4} \Upsilon\bigl(\frac{Q}{2} + iP + i\mu_j\bigr)\Upsilon\bigl(\frac{Q}{2} - iP + i\mu_j\bigr)} \tag{88}$$

with μ_j being sums and differences of the p_j.

4.3 Liouville CFT

Quick CFT facts. Local CFT data:

- spectrum (dimensions of operators);
- three-point couplings.

In general dimension, the Euclidean conformal group is $SO(d+1,1)$, generated by translations, rotations, dilations, special conformal transformations. In 2d, $SO(3,1) = SL(2,\mathbb{C})$ is roughly $SL(2) \times SL(2)$; it enhances to two Virasoro algebras, with generators L_n for $n \in \mathbb{Z}$ and similarly $\overline{L}_n$. Correlators are constrained by conformal invariance.

$$\langle \phi_1(x)\phi_2(y)\rangle = \frac{\delta_{\phi_1\phi_2}}{|x-y|^{2\Delta}} \tag{89}$$

$$\langle \phi_1(x)\phi_2(y)\phi_3(z)\rangle = \frac{\lambda_{123}}{|x-y|^{\Delta_1+\Delta_2-\Delta_3}|y-z|^{\Delta_1+\Delta_3-\Delta_2}|z-x|^{\Delta_2+\Delta_3-\Delta_1}} \tag{90}$$

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{1}{|x_1-x_2|^{2\Delta_\phi}|x_3-x_4|^{2\Delta_\phi}} f(u,v) \tag{91}$$

where $u = |x_1-x_2|^2|x_3-x_4|^2 \,/\, \bigl(|x_1-x_3|^2|x_2-x_4|^2\bigr)$ and $v = |x_1-x_4|^2|x_3-x_2|^2 \,/\, \bigl(|x_1-x_3|^2|x_2-x_4|^2\bigr)$. In 2d we can use the simpler cross-ratios $z = (z_1-z_2)(z_3-z_4) \,/\, \bigl((z_1-z_3)(z_2-z_4)\bigr)$ and $\overline{z}$.

The OPE expresses products of two operators close-by as a sum of operators at a point:

$$\phi_1(x)\phi_2(0) = \sum_{\mathcal{O}\text{ primary}} \lambda_{12\mathcal{O}} \bigl[C_{\mathcal{O}}(x,\partial_y)\mathcal{O}(y)\bigr]\big|_{y=0} \tag{92}$$

Using this twice in the four-point function we find

$$\begin{aligned}&\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle\\&=\sum_{\mathcal{O},\widetilde{\mathcal{O}}}\lambda_{\phi\phi\mathcal{O}}\lambda_{\widetilde{\mathcal{O}}\phi\phi}\underbrace{C_{\mathcal{O}}(x_1-x_2,\partial_y)C_{\widetilde{\mathcal{O}}}(x_1-x_2,\partial_{\widetilde{y}})\langle\mathcal{O}(y)\widetilde{\mathcal{O}}(\widetilde{y})\rangle\big|_{y=\widetilde{y}=0}}_{\text{conformal partial wave}}\end{aligned} \tag{93}$$

and the sum restricts to $\mathcal{O}=\widetilde{\mathcal{O}}$ since the two-point function vanishes otherwise. As promised, we expressed the 4-point function in terms of 3-point structure constants and conformal partial waves, which are completely fixed by conformal invariance.

Liouville CFT is a 2d CFT with central charge $c=1+6Q^2$ where $Q=b+b^{-1}$ and we recall that the central charge shows up in the stress-tensor OPE:

$$T(z)T(0)\sim\frac{c/2}{z^4}+\frac{2T(0)}{z^2}+\frac{\partial T(0)}{z}+\cdots \tag{94}$$

Liouville CFT has operators V_α labeled by $\alpha=Q/2+iP$ with P real and with $V_\alpha=R(\alpha)V_{Q-\alpha}$ for some coefficient $R(\alpha)$ called "reflection amplitude". This means that only $P\geq 0$ is physically relevant. The holomorphic and antiholomorphic dimensions of V_α are

$$h(\alpha)=\overline{h}(\alpha)=\alpha(Q-\alpha)=\frac{Q^2}{4}+P^2. \tag{95}$$

The OPE is

$$V_{\alpha_1}(z,\overline{z})V_{\alpha_2}(0)=\frac{1}{2}\int_{-\infty}^{+\infty}\mathrm{d}P\,C_{\alpha_1\alpha_2}^{Q/2+iP}\big(z\overline{z}\big)^{Q^2/4+P^2-\Delta_1-\Delta_2}\Big(V_{Q/2+iP}(0)+\underbrace{\cdots}_{\text{descendants}}\Big) \tag{96}$$

where "descendants" means "Virasoro descendants", obtained by acting with raising operators L_{-n}, $n>0$ in the mode decomposition of $T(z)$. The structure constants are $C_{\alpha_1\alpha_2}^{\alpha_3}=C(\alpha_1,\alpha_2,Q-\alpha_3)$ in terms of three-point functions. The three-point functions are given by the DOZZT formula

$$C(\alpha_1,\alpha_2,\alpha_3)=\Big(\pi\mu\gamma(b^2)b^{2-2b^2}\Big)^{(Q-\alpha)/b}\frac{\Upsilon'(0)\prod_{i=1}^3\Upsilon(2\alpha_i)}{\Upsilon(\alpha-Q)\prod_{i=1}^3\Upsilon(\alpha-2\alpha_i)} \tag{97}$$

where $\alpha=\alpha_1+\alpha_2+\alpha_3$.

The four-point functions are

$$\begin{aligned}&\langle V_{\alpha_1}(z,\overline{z})V_{\alpha_2}(0)V_{\alpha_3}(1)V_{\alpha_4}(\infty)\rangle\\&=\frac{1}{2}\int_{-\infty}^{+\infty}\mathrm{d}P\,C(\alpha_1,\alpha_2,Q/2+iP)C(Q/2-iP,\alpha_3,\alpha_4)\left|\mathcal{F}_p\begin{pmatrix}\alpha_2&\alpha_3\\\alpha_1&\alpha_4\end{pmatrix}\middle|\,z\right)\right|^2.\end{aligned} \tag{98}$$

Acknowledgements

We thank Vivek Saxena for spotting some typos.