4d $\mathcal{N} = 1$ supersymmetry

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Nobody (even the typist) proof-read these notes, so there may be obvious mistakes: tell BLF.

Abstract

We discuss aspects of 4d $\mathcal{N} = 1$ localization. These are lecture notes for the 2018 IHÉS summer school on *Supersymmetric localization and exact results.*

These lecture notes assume familiarity with supersymmetry at the level of the first few chapters of the book by Wess and Bagger.

1 Lecture 1, July 18

1.1 First some 4d $\mathcal{N} = 1$ review

Supersymmetry algebra generated by Q_α, $\overline{Q}_{\dot\beta}$, P_m with $\{Q_\alpha, \overline{Q}_{\dot\beta}\} = 2\sigma^m_{\alpha\dot\alpha} P_m$ where $(\sigma^m)_{\alpha\dot\alpha}$ are given by $\sigma^0 = -1$ and σ^i are Pauli matrices for $1 \leq i \leq 3$, while $(\overline{\sigma}^m)^{\alpha\dot\alpha}$ given by $\overline{\sigma}^0 = -1$ and $\overline{\sigma}^i = -\sigma^i$.

1.1.1 Holomorphy

A chiral superfield is Φ such that $\overline{D}_{\dot\alpha}\Phi(x, \theta, \overline{\theta}) = 0$. In coordinates $(y^m, \theta, \overline{\theta})$ with $y^m = x^m + i\theta\sigma^m\overline{\theta}$, we can write $\overline{D}_{\dot\alpha} = -\partial/\partial\overline{\theta}^{\dot\alpha}$. Then

$$\Phi(y, \theta) = \phi(y) + \sqrt{2}\psi_\alpha\theta^\alpha + \theta\theta F(y). \tag{1}$$

The superpotential term $\int d^4y \int d^2\theta W(\Phi(y, \theta))$ is supersymmetry invariant provided W is a holomorphic function of the chiral multiplets. In the low-energy effective action, W_{eff} is still a holomorphic function. This makes symmetries be more constraining.[1]

Seiberg and collaborators had very powerful results using this holomorphy.

[1] The holomorphy idea works in any theory with at least 4 supercharges, such as 4d $\mathcal{N} = 1$ or 3d $\mathcal{N} = 2$ or 2d $\mathcal{N} = (2, 2)$.

1.1.2 Chiral ring

The lowest component ϕ of Φ obeys $\delta_{\bar\epsilon}\phi = [\bar\epsilon\overline{Q}, \phi] = 0$ because $\delta\phi = \epsilon\psi$ (no $\bar\epsilon$ term). Thus ϕ is $\delta_{\bar\epsilon}$-closed. Then $\partial_\mu\phi \sim [\delta_{\bar\epsilon}, \delta_\epsilon]\phi = \delta_{\bar\epsilon}\delta_\epsilon\phi$. Thus moving a chiral operator does not change its class in the cohomology of $\delta_{\bar\epsilon}$:

$$\phi(x) = e^{x^\mu\partial_\mu}\phi(0) = \phi(0) + \delta_{\bar\epsilon}(\cdots). \tag{2}$$

The cohomology of $\delta_{\bar\epsilon}$ is called the *chiral ring*. Since

$$\partial_\mu\phi_1\phi_2\cdots = \delta_{\bar\epsilon}(\cdots), \tag{3}$$

correlators of chiral operators do not depend on the positions of the chiral operators. By cluster decomposition,

$$\langle\phi_1(x_1)\phi_2(x_2)\cdots\rangle = \langle\phi_1(0)\rangle\langle\phi_2(0)\rangle\cdots \tag{4}$$

so the condensates $\langle\phi_1(0)\rangle$ are enough to compute all correlators. In theories with R-symmetry, the condensates all vanish except if ϕ has no R-charge. However, the R-charge and dimension are related for chiral operators so that means ϕ is the identity.

1.2 4d $\mathcal{N}=1$ SQCD

See `https://arxiv.org/abs/hep-th/9509066` (Intriligator–Seiberg). They consider a gauge theory with gauge group $G = SU(N_c)$, with N_f fundamental flavours $\big($one flavour is a pair of chiral multiplets, one transforming in the fundamental and one in the antifundamental representation of $SU(N_c)\big)$. The chiral superfields are:

- $W^\alpha = \lambda^\alpha + \cdots$ the gauge field strength, in the adjoint of $SU(N_c)$ and space-time spinor;
- Q^i the fundamental chirals;
- $\widetilde{Q}^i$ the antifundamental chirals.

We can only turn on masses by adding a term $Q\widetilde{Q}$ to the superpotential.

The action is (here $\tau = \theta/(2\pi) + i(4\pi)/g^2_{\mathrm{YM}}$)

$$\begin{aligned}\mathcal{L} \simeq \frac{-1}{8\pi}\,\mathrm{Tr}\bigg(\int \mathrm{d}^2\theta\tau W_\alpha W^\alpha\bigg) + \text{h.c.} \\ + \int \mathrm{d}^2\theta\mathrm{d}^2\overline{\theta}\big(Q^i e^V Q_i^\dagger + \widetilde{Q}_i e^V \widetilde{Q}^{\dagger i}\big) \\ + \int \mathrm{d}^2\theta\big(m_i^j Q^i\widetilde{Q}_j\big) + \text{h.c.}\end{aligned} \tag{5}$$

Gauge-invariant chiral superfields are

- "Meson" $M_i^j = Q_r^j\widetilde{Q}_i^r$ where $1 \le r \le N_c$;

- "Baryons" $B^{j_1\cdots j_{N_c}} = \epsilon^{r_1\cdots r_{N_c}} Q^{j_1}_{r_1}\cdots Q^{j_{N_c}}_{r_{N_c}}$ and antibaryons built from $\widetilde{Q}$;
- "Glueballs" $S \simeq \mathrm{Tr}(W_\alpha W^\alpha) \simeq Tr(\lambda_\alpha\lambda^\alpha) + \cdots$.

We will be interested in the expectation value of glueballs. This is a purely strong coupling phenomenon: even instantons cannot contribute.

The 1-loop β function, which is in fact exact, is

$$\beta = -\frac{3N_c - N_f}{16\pi^2} g^3_{\mathrm{YM}}. \tag{6}$$

For $N_f = 3N_c$ we get a CFT but let us actually focus on $\boxed{N_f < N_c}$. Then we can define the scale Λ (complex) by

$$\left(\frac{\Lambda}{\mu}\right)^{3N_c-N_f} = e^{2\pi i\tau_{\mathrm{eff}}}. \tag{7}$$

The effective potential

$$W_{\mathrm{eff}} = (N_c - N_f)\left(\frac{\Lambda^{3N_c-N_f}}{\det M}\right)^{1/(N_c-N_f)} + m^i_j M^j_i. \tag{8}$$

(where m is the masses and M the mesons) is fixed by symmetries, up to a numerical factor that is fixed by a one instanton calculation.

Integrating out M (for large masses) by $\partial W_{\mathrm{eff}}/\partial M^j_i = 0$ we find

$$M = m^{-1}\left(\Lambda^{3N_c-N_f}\det m\right)^{1/N_c}. \tag{9}$$

Note that the power $1/N_c$ has N_c branches, so we expect N_c vacua; a supersymmetric localization calculation of the Witten index confirms this. The effective twisted superpotential gives

$$W_{\mathrm{eff}} = N_c\left(\Lambda^{3N_c-N_f}\det m\right)^{1/N_c} = N_c\Lambda_0^3 \tag{10}$$

where $\Lambda_0^{3N_c} = \Lambda^{3N_c-N_f}\det m$ is the effective low-energy dynamical scale. From this we compute the gauge condensation:

- Make the gauge coupling τ into a chiral superfield $\tau(\theta) = \tau + \theta\psi_\tau + F_\tau\theta\theta$ that would have an expectation value $\langle\tau(0)\rangle = \tau$.
- Then that coupling being a chiral superfield it must appear holomorphically in the effective superpotential.
- Then
$$\langle \mathrm{Tr}\,\lambda\lambda\rangle \sim \partial\log Z/\partial F_\tau \sim \frac{\partial}{\partial F_\tau}\int \mathrm{d}^2\theta\, W_{\mathrm{eff}} \sim -16\pi^2\Lambda_0^3 \tag{11}$$
where we used that we know $\log Z$ in the low-energy theory, and we have to remember that Λ_0 is essentially an exponential of τ (this is why the power of Λ does not change).

Note that we can do the opposite: from the gluino condensate $\langle \mathrm{Tr}(\lambda\lambda)\rangle$ we can integrate it and get the effective superpotential and from that we get correlators of all chiral ring operators.

1.2.1 Seiberg duality

For the range $\frac{3}{2}N_c < N_f < 3N_c$ then at low energies the theory is claimed to flow to a nontrivial CFT. Moreover there is a "magnetic" dual theory with gauge group $SU(\widetilde{N}_c)$ with N_f flavours q and $\widetilde{q}$ and N_f^2 gauge singlets M_i^j which flows to the same infrared fixed point; here $\widetilde{N}_c = N_f - N_c$.

1.3 Localization

See other lectures for how supersymmetric localization works. We shall use the canonical choice $V \sim (\delta\lambda_i)^\dagger \lambda_i$.

1.3.1 4d $\mathcal{N} = 1$ theory on S^4

We can put 4d $\mathcal{N} = 1$ theories on S^4 in two ways:

- as Guido explained by building a supergravity background of an appropriate supergravity;
- by starting from 4d $\mathcal{N} = 2$ put on S^4 by Pestun, and decomposing multiplets into 4d $\mathcal{N} = 1$ multiplets.

The problem is that $V \sim (\delta\lambda_i)^\dagger \lambda_i$ does **not** satisfy $\delta V|_{\text{bosonic}} \geq 0$. So while the theory makes sense and its partition function is probably finite, we cannot do localization.

1.3.2 4d $\mathcal{N} = 1$ theory on Euclidean $S^3 \times S^1$

We normalize the radius of S^3 to be 1 and the radius of S^1 to be β.

Recall supersymmetry on Euclidean $\mathbb{R}^4$ (vector multiplet):

$$\delta A_\mu = \frac{i}{2}(\epsilon\sigma_\mu\overline{\lambda} - \overline{\epsilon}\overline{\sigma}_m\lambda) \tag{12}$$

$$\delta\lambda = \frac{1}{2}\sigma^{mn}\epsilon F_{mn} - \epsilon D \tag{13}$$

$$\delta\overline{\lambda} = \frac{1}{2}\overline{\sigma}^{mn}\overline{\epsilon}F_{mn} - \overline{\epsilon}D \tag{14}$$

$$\delta D = \frac{-i}{2}\epsilon\sigma^m D_m\overline{\lambda} - \frac{i}{2}\overline{\epsilon}\overline{\sigma}^m D_m\lambda \tag{15}$$

where $m = 1,2,3,4$ and $\sigma^m = (\sigma^1,\sigma^2,\sigma^3,1)$ and $\overline{\sigma}^m = (\sigma^m)^\dagger$ and $\sigma_{mn} = \frac{1}{2}(\sigma_m\overline{\sigma}_n - \sigma_n\overline{\sigma}_m)$ and $\overline{\sigma}_{mn} = \frac{1}{2}(\overline{\sigma}_m\sigma_n - \overline{\sigma}_n\sigma_m)$. Recall that bars do not mean complex conjugation.

To place the theory on $S^3 \times S^1$ just replace $D_m \to D_m - iqV_m$ with q the R-charge, such that ϵ and λ has R-charge $+1$ and $\overline{\epsilon}$ and $\overline{\lambda}$ has R-charge -1, and $V_m \mathrm{d}x^m = \frac{-i}{2}\mathrm{d}t$.

This procedure should be equivalent to Guido's approach with new minimal supergravity. Importantly the theory must have a non-anomalous R-symmetry. Note that SQCD's obvious R-symmetry is anomalous but it can be changed to become non-anomalous.

Preserved supersymmetries are labeled by generalized Killing spinors, such that

$$D_m \epsilon = \frac{-1}{2}\sigma_m \overline{\sigma}_4 \epsilon \tag{16}$$

$$D_m \overline{\epsilon} = \frac{-1}{2}\overline{\sigma}_m \sigma_4 \overline{\epsilon}. \tag{17}$$

The Yang–Mills action on $S^1 \times S^3$ is

$$\mathcal{L}_{\mathrm{YM}} = \mathrm{Tr}\left(\frac{1}{g_{\mathrm{YM}}^2}\left(\frac{1}{2}F_{mn}F^{mn} + D^2 + i\overline{\lambda}\overline{\sigma}^m D_m \lambda + \frac{i\theta}{16\pi^2}F\wedge F\right) + \text{fermions}\right). \tag{18}$$

again we define $\tau = \theta/(2\pi) + 4\pi i/g_{\mathrm{YM}}^2$.

There are actually two choices for $(\epsilon, \overline{\epsilon})$. Then we take the standard V.

- The first choice has $\epsilon \neq 0$ and $\overline{\epsilon} \neq 0$ everywhere. Then, defining $F^\pm = (F \pm \star F)/2$ we have

$$(\delta\lambda)^\dagger \delta\lambda \sim F^+_{mn}F^{+mn} + D^2 + \text{fermions} \tag{19}$$

$$(\delta\overline{\lambda})^\dagger \delta\overline{\lambda} \sim F^-_{mn}F^{-mn} + D^2 + \text{fermions}. \tag{20}$$

  When coefficients are the same we get $V \sim |F|^2 + D^2$ while if coefficients are different we get an additional $F \wedge F$ contribution. The saddle-point configuration is trivial: $F_{mn} = D = 0$ (more precisely there are holonomies around the time circle). The classical action is zero and we are left with a free theory, for which we can simply do a mode expansion in S^3 spherical harmonics. We cannot compute any local observables, only the partition function, which in the case of SCFT is the superconformal index.

- The second choice has $\epsilon = 0$ and $\overline{\epsilon}$ nonvanishing. Now $\mathrm{Tr}(\lambda\lambda)$ is Q-closed. The deformation term is

$$\delta V|_{\text{bosonic}} \sim \mathrm{Tr}((F^-)^2 + D^2) \tag{21}$$

  so the saddle point is an instanton $F^- = 0$ (anti-self-dual condition). For this choice of $(\epsilon, \overline{\epsilon})$ the classical action is

$$\mathcal{L}_{\mathrm{YM}}|_{F^-=0, D=0, \overline{\lambda}=0} = \mathrm{Tr}\left(\frac{1}{g_{\mathrm{YM}}^2}(F^+)^2 + \frac{i\theta}{16\pi^2}(F^+)^2\right) = \frac{i\tau}{8\pi}\,\mathrm{Tr}(F \star F) \tag{22}$$

  usual factor, gauge-coupling-dependent.

2 Lecture 2, July 20

2.1 Superconformal index of 4d $\mathcal{N}=1$ theories on $S^3\times S^1$

See Rastelli–Razamat `https://arxiv.org/abs/1608.02965` and Hosomichi `https://arxiv.org/abs/1412.7128`.

4d $\mathcal{N}=1$ on $S^3\times S^1$ has $SU(2)_L\times SU(2)_R\times U(1)$ isometry; we denote generators of the Cartan by J_L^3, J_R^3, D. The Killing spinors obey $\gamma_3\epsilon=-\epsilon$ and $\gamma_3\bar\epsilon=-\bar\epsilon$. We have

$$R(\delta_\epsilon)=-1,\quad J_R^3(\delta_\epsilon)=\frac{1}{2},\quad R(\delta_{\bar\epsilon})=1,\quad J_R^3(\delta_{\bar\epsilon})=\frac{-1}{2}. \tag{23}$$

The partition function is equal to the index:

$$Z=\int_{S^3\times S^1}D\phi\, e^{-S(\phi)}=\mathrm{Tr}_{S^3}\Big[(-1)^F e^{-\beta(D-\frac{1}{2}R)} \tag{24}$$

where $(-1)^F$ comes from the fermion boundary conditions and $D-\frac{1}{2}R$ comes from the twist $D_t=\partial_t-\frac{1}{2}q$ where q is the R-charge.

Further twists $\partial_t\to\partial_t-i\xi(2J_R^3+R)-2i\eta J_L^3-im$ with m associated to a flavour symmetry. These twists preserve supersymmetry because $[Q,2J_R^3+R]=0$ and $[Q,J_L^3]=0$. These twists give

$$Z=\mathrm{Tr}_{S^3}\Big[(-1)^F q^{D-R/2}x^{2J_R^3+R}y^{2J_L^3}e^{iw\beta}\Big] \tag{25}$$

with $q=e^\beta$, $x=e^{i\beta\xi}$, $y=e^{i\beta\eta}$.

[...]

This is called the superconformal index.

Only states that obey $0=S|\alpha\rangle=Q|\alpha\rangle$ (hence $H|\alpha\rangle=0$) contribute. By a change of variables $x=x'/q$, $x'=(pp')^{1/2}$, $y=(p/p')^{1/2}$ we find the superconformal index

$$I=Z=\mathrm{Tr}_{S^3}\Big[(-1)^F p^{J_R^3+J_L^3+R/2}(p')^{J_R^3-J_L^3-R/2}e^{i\beta m}e^{-\beta H}\Big] \tag{26}$$

Since only states with $H=0$ contribute we can omit the last factor.

Because of localization we can set the gauge coupling to zero and get a free theory. Then the index is not too hard to compute. Let $A_\tau=a$; the Gaussian integral gives

$$\frac{\det_\lambda(D_\tau-ia+i\not{D}_{S^3})\det_c'(\det_\tau-ia)}{\det_A((-iD_\tau-ia)^2+\Delta_{S^3}^{\text{vector}})^{1/2}} \tag{27}$$

The full partition function is

$$Z\simeq\int\prod_{i=1}^r \mathrm{d}\hat a_i\prod_{\alpha>0}4\sin^2\frac{\alpha\hat a}{2}I_{\text{vec}}I_{\text{chiral}} \tag{28}$$

where r is the rank of the gauge group and $\hat{a} = \beta a$ and where

$$I_{\text{vec}} = \left(\prod_{n\geq 1}(1-p^n)(1-p'^n)\right)^r \prod_{\alpha\gtrless 0}\prod_{n\geq 1}(1-p^n e^{\alpha\hat{a}})(1-p'^n e^{\alpha\hat{a}}) \tag{29}$$

$$I_{\text{chiral}} = \prod_{w\in\text{weights}(R)} \Gamma(e^{iw\hat{a}}(pp')^{1/2}, p, p') \tag{30}$$

in terms of the elliptic Gamma functions $\Gamma(x, p, p')$.

These very complicated integrals, specialized to two Seiberg-dual gauge theories, turn out to give the same result through complicated mathematical identities.

2.2 Gaugino condensation

See Terashima `https://arxiv.org/abs/1410.3630` and `https://arxiv.org/abs/1509.02916` and Davies et al `https://arxiv.org/abs/hep-th/9905015`.

$\epsilon = 0$, $\bar{\epsilon} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i(-\chi+\phi+\theta)/2} \\ e^{i(-\chi+\phi+\theta)/2}\end{pmatrix}$

For chiral multiplets, $\delta V|_{\text{bosonic}} = |D_m\phi|^2 + r^2|\phi|^2 + |F|^2$ so saddle points are $\phi = F = 0$.

Note that we retrieve holomorphy without needing to talk about superfields.

Consider the Q-exact term

$$\mathcal{L}_{\overline{w}} = \delta\left(\frac{\partial\overline{W}(\overline{\phi})}{\partial\overline{\phi}^i}\overline{\eta}\overline{\psi}^i\right) = \frac{\partial\overline{W}}{\partial\overline{\phi}^i}\overline{F} + \text{fermions} \tag{31}$$

where we used $\overline{\eta} = U\times(0,1)$ with $U = (\bar{\epsilon}i\sigma^2(\bar{\epsilon})^*$ and $\bar{\epsilon} = U\times(1,0)$ so $\overline{\eta\epsilon} = 1$. So parameters in the antiholomorphic superpotential cannot show up in the partition function or correlators of holomorphic operators.

Consider 4d $\mathcal{N} = 1$ super Yang–Mills with gauge group $G = SU(N_c)$ on $\mathbb{R}^3\times S^1$. We want to determine $\langle\lambda\lambda\rangle$. We add

$$\delta V \sim \tau\left(\int (F^-)^2 + D^2 + \cdots\right) \tag{32}$$

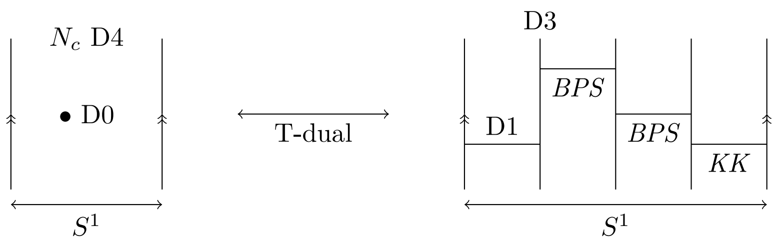
to make the theory weakly coupled and localize onto anti-self-dual "instantons". Configurations on $\mathbb{R}^3\times S^1$ are classified by

- the instanton charge $Q \sim \int F\wedge F$;
- the Wilson loop $\langle\phi\rangle = \lim_{|x|\to\infty}\int_0^\beta \mathrm{d}x_0\, A_0$, which generically breaks G to $U(1)^r$;
- monopole charge for $U(1)^r$.

To get a non-zero value for $\langle\lambda\lambda\rangle$ we need two zero modes, but an instanton has $2N_c$.

Antiselfdual configurations with 2 zero modes are "fundamental monopoles". There are r of them, T-dual (in the sense that the scalar ϕ is converted to A_0) of BPS monopoles, "fractional instantons".

The brane picture (in a sufficiently supersymmetric setting) is that the $SU(N_c)$ theory is realized by N_c D4 branes compactified on S^1, and an instanton is realized by a D0 brane.



(33)

The T-dual to the stack of N_c D4 branes is N_c D3 branes localized at points along S^1 (the positions are eigenvalues of $\langle\phi\rangle$), and the D0 brane becomes a D1 brane wrapping S^1, which splits into N_c D1 branes stretching between neighboring D3 branes. Each of these D1 brane segments is consistent on its own and describes a fundamental monopole: $N_c - 1$ BPS monopoles and one KK monopole.

The solution for a single monopole can be given explicitly. Here α_i ranges over simple roots, so $1 \leq i \leq r$. We define $v = \frac{1}{\beta}\langle\phi\rangle\alpha_i$.

- BPS monopole. Magnetic charge α_i, instanton charge $Q = \frac{1}{2\pi}\alpha_i \cdot \langle\phi\rangle$, the classical action is $-i\tau\alpha_i\langle\phi\rangle$
- The KK monopole has monopole charge $\alpha_0 = -\sum_{i=1}^{r}\alpha_i$, and has $Q = q + \frac{1}{2\pi}\alpha_0\langle\phi\rangle$ and $S = -2\pi i\tau - i\tau\alpha_0\langle\phi\rangle$.

We will call these $r+1$ monopoles "fundamental monopoles".

Now $S^3 \times \mathbb{R}$ is non-compact so we need to find vacua of the $t \to \infty$ limit to impose at infinity in space.

We shall send $t \to \infty$ and get a weakly-coupled limit.

Let us consider the effective action for the massless fields. We should get $U(1)$ vector multiplets with zero KK momentum, which is equivalent to 3d $\mathcal{N} = 2$ $U(1)^r$ vetor multiplets.

The Wilson loop ϕ and dual photon σ combine into $Z = i(\tau\phi + \sigma)$ and can be repackaged into 3d $\mathcal{N} = 2$ chiral superfield χ. The action becomes

$$S \to \frac{1}{4\pi\beta}\int \mathrm{d}^3x\, \chi^\dagger\chi|_{\theta\theta\overline{\theta}\overline{\theta}} \tag{34}$$

with no potential.

To find the vacua we need the scalar potential, which comes from the superpotential, which we can deduce from the fermion bi-linear. To compute the fermion bilinear we need configurations with two fermionic zero modes, namely fundamental monopoles.

The path integral measure of zero modes is

$$\int \mathrm{d}\mathcal{M}^{(j)}_{?} = \frac{\mu^2}{g^2}\frac{?}{2\pi}e^{-S_i}\int \mathrm{d}^3x\mathrm{d}\Omega\mathrm{d}^2\xi \tag{35}$$

where μ is the cutoff scale, $g = g(\mu)$ the gauge coupling, which is defined at $t = 0$, and $S_i = -2\pi i\delta_{i,0} - \alpha_i\langle Z\rangle$.

The correlator

$$\langle\lambda_\alpha(x)\otimes\lambda_\beta(0)\rangle \longrightarrow \sum_{j=0}^{r}\frac{2^?\pi^2\mu^3\beta}{g^2}\alpha_j\otimes\alpha_j e^{-S_j}\times\int \mathrm{d}^3a\, S_F(x-a)_\alpha{}^\gamma\times S_F(a)_{\beta\gamma}. \tag{36}$$

(where S_F are fermionic propagators?) To give rise to this gaugino two-point function, the low-energy effective action must contain a term $(\cdots)\lambda\lambda$, so this two-point function tells us what the $(\cdots)$ should be, hence tells us the superpotential. We deduce

$$W(\chi) = \frac{\mu^3\beta}{g^2}\left(\sum_{j=1}^{r}e^{\alpha_j\chi} + e^{2\pi i\tau+\alpha_0\chi}\right) + \text{derivatives of }\chi. \tag{37}$$

The vacuum $\partial W/\partial\chi = 0$ is $\chi = \frac{2\pi i\tau}{c_2}\sum_{j=1}^{r}\varpi_i$ where $\alpha_j\varpi_i = \delta_{ij}$ with c_2 the ... number of the gauge group.

Then

$$\langle W\rangle = \frac{\mu^3\beta}{g^2}c_2 e^{2\pi i\tau/c_2} = \beta c_2\Lambda^3 \tag{38}$$

where $\Lambda^3 = (\mu^3/g^2)\exp\bigl(2\pi i\tau(\mu)/c_2\bigr)$ and

$$\left\langle\frac{\mathrm{Tr}(\lambda\lambda)}{16\pi^2}\right\rangle = \Lambda^3 e^{2\pi i s/c_2} \tag{39}$$

for $0 \leq s < c_2$.

The vacua are discrete, so they cannot mix when τ is varied. The result above then applies to $\tau = 0$, which is the original theory.

Could we do $\mathbb{R}^4$*?* This same localization technique does not work for $\mathbb{R}^4$ because there is always a range of energy scales $< \Lambda' = \Lambda e^{-t}$ for which the theory is strongly coupled. On the other hand, on $\mathbb{R}^3\times S^1$ we can choose t large enough to make the theory weakly coupled.

Consider now 4d $\mathcal{N} = 1$ on $\mathbb{R}^3\times S^1$ with $W_v = 2\pi i\tau_0 S + F(S, S_i)$. With some work we can show that

$$\langle S\rangle \tag{40}$$

is given by $\partial W_S/\partial S$ for the VY superpotential

$$W_S = -c_2 S\left(\log\frac{S}{\Lambda^3} - 1\right) + F(S). \tag{41}$$

We can do similar calculations with matter. See Dijkgraaf–Vafa superpotential.