# 1 Answers for exercise session 1, July 16

## 1.1 Exercises about Guido Festuccia's course

**Exercise 1.-1** (Symmetric Energy Momentum Tensor).

**Exercise 1.0** (BPS String).

**Exercise 1.1** (Strings in sQED).

**Exercise 1.2.** Show that $\mathbb{S}_{\alpha\dot\alpha} = 2g_{i\bar j} D_\alpha \Phi^i \overline{D}_{\dot\alpha} \overline{\Phi}^{\bar j}$, $\chi_\alpha = \overline{D}^2 D_\alpha K$, $Y_\alpha = 4D_\alpha W$ in a Wess–Zumino model obey $\overline{D}^{\dot\alpha} \mathbb{S}_{\alpha\dot\alpha} = \chi_\alpha + Y_\alpha$, $\overline{D}_{\dot\alpha}\chi_\alpha = 0$, $D^\alpha \chi_\alpha = \overline{D}_{\dot\alpha}\overline{\chi}^{\dot\alpha}$, $D_\alpha Y_\beta + D_\beta Y_\alpha = 0$, $\overline{D}^2 Y_\alpha = 0$.

*Answer.* Notice first that for $C$ chiral, $\overline{D}^{\dot\alpha} D_\alpha C = \{\overline{D}^{\dot\alpha}, D_\alpha\} C$ is chiral since the translation $\{\overline{D}^{\dot\alpha}, D_\alpha\}$ commutes with $\overline{D}^{\dot\beta}$. We deduce that $\overline{D}^2 Y_\alpha = 0$. The antisymmetry $D_\alpha Y_\beta + D_\beta Y_\alpha = 0$ derives from $\{D_\alpha, D_\beta\} = 0$. Next, $\chi$ is chiral because $\overline{D}^3 = 0$. For the other relation $D^\alpha \chi_\alpha = \overline{D}_{\dot\alpha}\overline{\chi}^{\dot\alpha}$ we simply check (remember that $D^2 = D^\alpha D_\alpha$ while $\overline{D}^2 = \overline{D}_{\dot\alpha}\overline{D}^{\dot\alpha}$)

$$D^\alpha \overline{D}^2 D_\alpha - \overline{D}_{\dot\alpha} D^2 \overline{D}^{\dot\alpha} = \{D^\alpha, \overline{D}_{\dot\alpha}\}\overline{D}^{\dot\alpha} D_\alpha - \overline{D}_{\dot\alpha} D^\alpha \{\overline{D}^{\dot\alpha}, D_\alpha\} = \Big[\{D^\alpha, \overline{D}_{\dot\alpha}\}, \overline{D}^{\dot\alpha} D_\alpha\Big] = 0. \quad (1)$$

For the final constraint, first notice that $2D_\alpha \Phi^i \overline{D}_{\dot\alpha} \partial_i K = 2g_{i\bar j} D_\alpha \Phi^i \overline{D}_{\dot\alpha} \overline{\Phi}^{\bar j} = \mathbb{S}_{\alpha\dot\alpha}$ since $\overline{D}_{\dot\alpha}\Phi^j = 0$. We deduce (remember signs in Leibniz' rule with anticommuting derivatives, and remember $\overline{D}^{\dot\alpha}\overline{D}_{\dot\alpha} = -\overline{D}^2$)

$$\overline{D}^{\dot\alpha}\mathbb{S}_{\alpha\dot\alpha} - \chi_\alpha = 2\overline{D}^{\dot\alpha}\big(D_\alpha \Phi^i \overline{D}_{\dot\alpha}\partial_i K\big) + \overline{D}^{\dot\alpha}\overline{D}_{\dot\alpha}\big(D_\alpha \Phi \partial_i K\big) = D_\alpha \Phi^i \overline{D}^2 \partial_i K - \overline{D}^2 D_\alpha \Phi \partial_i K. \quad (2)$$

The last term vanishes since $\overline{D}^{\dot\alpha} D_\alpha \Phi$ is chiral. Using the equation of motion $\overline{D}^2 \partial_i K = 4\partial_i W$, the first term is $Y_\alpha$ as desired. It could be tempting to define $Y_\alpha = D_\alpha \Phi^i \overline{D}^2 \partial_i K$ so that we wouldn't need the equations of motion. However, $D_\alpha Y_\beta + D_\beta Y_\alpha = 0$ would then only hold on-shell. The need for the equations of motion is of course not surprising given the next exercise which shows that the conservation of $T_{\mu\nu}$ is encoded into the constraints on $\mathbb{S}$, $\chi$, $Y$. □

**Exercise 1.3.**

## 1.2 Exercises about Francesco Benini's course

**Exercise 1.4.** Compute $\int_{S^2} e^{ic\cos\theta} \mathrm{dVol}(S^2)$.

*Answer.* (Provided by Pieter Bomans.)

We consider the round 2-sphere with unit radius and metric

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2 \quad (3)$$

with volume form $\omega = \sin\theta \mathrm{d}\theta \wedge \mathrm{d}\phi$

Method 1: Direct integration simply gives

$$I(c) = \int_0^{2\pi} \mathrm{d}\phi \int_0^\pi \mathrm{d}\theta \sin\theta \mathrm{e}^{ic\cos\theta} = 2\pi \int_{-1}^1 \mathrm{d}z \mathrm{e}^{icz} = \frac{2\pi i}{c}(-\mathrm{e}^{ic} + \mathrm{e}^{-ic}) = 4\pi \frac{\sin c}{c} \quad (4)$$

We see that only the stationary points of $\cos\theta$ contribute to the integral. In this case the stationary phase approximation is exact.

Method 2: Using Duistermaat-Heckman localisation formula.

Let's take the area form on the sphere as the symplectic form. It is closed and non-degenerate. The vector field generating rotations around the $z$-axis is given by $V = \partial/\partial\phi$. The Hamiltonian associated to $V$ is determined by the equation

$$i_V \omega + \mathrm{d}H = 0 \quad (5)$$

which implies that $H = \cos\theta = z$ up to constant shifts. The integral we want to compute

$$I(c) = \int_{S^2} \omega \mathrm{e}^{icz} \tag{6}$$

is indeed of the form needed to apply the Duistermaat-Heckman integral and is given by the Duistermaan-Heckman formula

$$I(c) = \left(\frac{2\pi}{ic}\right)^l \sum_p \frac{\mathrm{e}^{icH(x_p)}}{\lambda_1^p \cdots \lambda_2^p} \tag{7}$$

There are two fixed points of the $U(1)$ action, namely the North and South pole with values for the Hamiltonian $z_{NP} = 1$ and $z_{SP} = -1$ and indices $\lambda_{NP} = 1$ and $\lambda_{SP} = -1$. The localisation formula thus gives

$$I(c) = \frac{2\pi}{ic}\left(\frac{\mathrm{e}^{icz_{NP}}}{\lambda_{NP}} + \frac{\mathrm{e}^{icz_{SP}}}{\lambda_{SP}}\right) = 4\pi\frac{\sin c}{c} \tag{8}$$

confirming the result above.

Method 3: Using Atiyah-Bott-Berline-Vergne localisation formula. The localisation formula of Duistermaat and Heckman is a special case of the more general localisation formula

$$\int_{\mathcal{M}} \alpha(c) = \left(\frac{2\pi}{ic}\right)^l \sum_p \frac{\alpha_0(c)(x_p)}{\lambda_1^p \cdots \lambda_l^p} \tag{9}$$

where $\alpha(c)$ is an equivariant cocycle. We can use this formula by making the integrand into an equivariantly closed polyform

$$\alpha(c) = \mathrm{e}^{ic\cos\theta}\left(\omega - \frac{1}{ic}\right) \tag{10}$$

and then apply the formula to get again the same answer. □

**Exercise 1.5.** Show that $\{\mathrm{d}, i_V\} = \mathcal{L}_V$ hence $\mathrm{d}_V = \mathrm{d} - i_V$ squares to $-\mathcal{L}_V$.

*Answer.* (Provided by Pieter Bomans.) We start by showing that this holds on functions and then argue that this identity extends to $p$-forms for any $p$[1]. Indeed acting with the lie derivative on a function $f$ gives

$$\mathcal{L}_V f := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \phi_t^* f = df(V). \tag{11}$$

The space of $-1$ forms is empty so $i_V f = 0$. Thus $(\mathrm{d}i_V + i_V\mathrm{d})f = i_V\mathrm{d}f = df(V)$. Since the space of all $p$-forms is generated by functions and the operation d ze can show that Cartan's formula holds for any $p$-form. Since both $\mathcal{L}_V$ and $(\mathrm{d}i_V + i_V\mathrm{d})$ commute with d and both are degree 0 derivations, i.e. they map $p$-forms to $p$-forms and

$$D(ab) = (Da)b + a(Db), \qquad \text{where } D = \mathcal{L}_V \text{ or } (\mathrm{d}i_V + i_V\mathrm{d}), \tag{12}$$

they are equal on all forms. Now it is easy to see that

$$d_V^2 = d^2 - \{d, i_V\} + i_V^2 = -\mathcal{L}_V\,. \tag{13}$$

□

**Exercise 1.6.** Let $\eta = g(V, \ )$ and

$$\Theta_V = \eta \wedge \left(\frac{-1}{|V|^2}\left(1 + \frac{\mathrm{d}\eta}{|V|^2} + \cdots + \frac{(\mathrm{d}\eta)^{\dim M/2}}{|V|^{\dim M}}\right)\right). \tag{14}$$

Check that $\mathrm{d}_V\Theta_V = 1$ and deduce that **on $\boldsymbol{M \setminus M_V}$**, $\mathrm{d}_V\alpha = 0$ implies $\alpha = \mathrm{d}_V\beta$.

[1] Another nice proof using homotopy can be found on `https://math.stackexchange.com/questions/1480545/proving-cartans-magic-formula-using-homotopy`

*Answer.* (Provided by Pieter Bomans.) we know that $\mathrm{d}_V\eta = -|V|^2 + \mathrm{d}\eta$ is invertible on $\mathcal{M}\ \mathcal{M}_V$. We can write $\Theta_V = \eta(\mathrm{d}_V\eta)^{-1})$. Then by construction of $\Theta_V$ we have $\mathrm{d}_V\Theta_V = \mathrm{d}_V\eta(\mathrm{d}_V\eta)^{-1} = 1$ and is well defined on $\mathcal{M}\ \mathcal{M}_V$. Now we can write any equivariantly closed polyform *alpha* as

$$\alpha = 1 \cdot \alpha = (\mathrm{d}_V\Theta_V)\alpha = \mathrm{d}_V(\Theta_V\alpha)\,. \tag{15}$$

This shows that $V$ equivariantly closed forms on $\mathcal{M}$ are equivariantly exact on $\mathcal{M}\ \mathcal{M}_V$. By Stokes' theorem we see that the integral $\int_\mathcal{M}\alpha$ does not recieve contributions from $\mathcal{M}\ \mathcal{M}_V$ and thus localises to the fixed points. □

**Exercise 1.7.** Compute $\int_{U(N)}\exp(t\,\mathrm{Tr}(AUBU^\dagger))\mathrm{d}U$.

*Answer.* First diagonalize $A = V^\dagger\tilde{A}V$ and $B = W\tilde{B}W^\dagger$ and notice that $\mathrm{Tr}(AUBU^\dagger) = \mathrm{Tr}(V^\dagger\tilde{A}VUW\tilde{B}W^\dagger U^\dagger) = \mathrm{Tr}(\tilde{A}\tilde{U}\tilde{B}\tilde{U}^\dagger)$ with $\tilde{U} = VUW$ a unitary matrix; we are just doing a change of coordinates in $U(N)$. Drop all tildes.

It can be argued (using the Duistermaat–Heckmann equivariant localization formula on symplectic manifolds) that the saddle-point approximation of the integral gives the correct result. Hopefully a later version of this answer sheet will explain this point carefully.

The saddle-point equations are obtained by varying $U \to U + \delta U$ and noting that $U^\dagger = U^{-1} \to U^{-1} - U^{-1}(\delta U)U^{-1}$. We find $[A, UBU^{-1}] = 0$. For generic diagonal $A$ this implies that $UBU^{-1}$ is diagonal. Since this matrix is conjugate to $B$ it has the same eigenvalues in some order, in other words it is equal to $\sigma B\sigma^{-1}$ where $\sigma$ is a permutation matrix. We learn that $\sigma^{-1}U$ and $B$ commute, hence (again for generic diagonal $B$) we find that $\sigma^{-1}U$ is a diagonal unitary matrix. Altogether saddle-points are $U^{(0)} = \sigma D$ for $\sigma \in S_N \subset U(N)$ and $D \in U(1)^N \subset U(N)$.

Now parametrize a neighborhood of $U^{(0)}$ by $U = \sigma D\exp(t^{-1/2}V)$ with $V$ antihermitian, so $U^\dagger = U^{-1} = \exp(-t^{-1/2}V)D^{-1}\sigma^{-1} + O(t^{-3/2})$. The action is

$$t\,\mathrm{Tr}(AUBU^{-1}) = t\,\mathrm{Tr}(A^{(\sigma)}(1 + t^{-1/2}V + t^{-1}V^2/2)B(1 - t^{-1/2}V + t^{-1}V^2/2)) + O(t^{-1/2}) \tag{16}$$

$$= t\,\mathrm{Tr}(A^{(\sigma)}B) - \frac{1}{2}\,\mathrm{Tr}([V, A^{(\sigma)}][V, B]) + O(t^{-1/2}). \tag{17}$$

Here we defined the diagonal matrix $A^{(\sigma)} = \sigma^{-1}A\sigma$, which has the same eigenvalues of $A$ but permuted according to $\sigma$. Being diagonal, it commutes with $D$ and with $B$. We used this and cyclicity of the trace to simplify expressions. In particular the $t^{1/2}$ term vanishes since $U^{(0)}$ is a saddle-point. For the $O(1)$ term we used $[A^{(\sigma)}, B] = 0$.

For the saddle-point approximation we thus need to compute the Gaussian integral

$$\int \mathrm{d}V\,\exp\biggl(\frac{-1}{2}\,\mathrm{Tr}\Bigl([V, A^{(\sigma)}][V, B]\Bigr)\biggr). \tag{18}$$

Remember that $\sigma^{-1}DV$ is a tangent vector to $U(N)$ at the saddle-point $\sigma^{-1}D$, so $V$ is an anti-Hermitian matrix. The Gaussian integral is only over the $V$ with all diagonal entries vanishing, because we only want to integrate over directions transverse to the localization locus (which is parametrized by $D$ and the discrete $\sigma$). The Gaussian integral gives a determinant to the power $-1/2$. How do we compute this determinant?

We complexify the space on which the operator acts and compute the determinant of the operator acting on the complex vector space of all complex matrices with vanishing diagonal. (This does not square the determinant for the same reason that the determinant of an explicit real matrix does not depend on whether we treat it as being a real matrix or a complex matrix whose components happen to be real.) Complexifying helps us find very simple eigenvectors: $V$ with a single non-zero entry in position $(i, j)$. The eigenvalue is $(A_{\sigma(i)} - A_{\sigma(j)})(B_i - B_j)$.

Taking the product over all $i \neq j$ then taking the square-root gives the result

$$\int_{U(N)} \mathrm{d}U\,\exp\Bigl(t\,\mathrm{Tr}(AUBU^\dagger)\Bigr) = c_N \sum_{\sigma\in S_N} \frac{t^{-(n^2-n)/2}\exp\Bigl(t\,\mathrm{Tr}(A^{(\sigma)}B)\Bigr)}{\Delta(A^{(\sigma)})\Delta(B)} \tag{19}$$

with $\Delta(B) = \prod_{i<j}(\lambda_i(B) - \lambda_j(B))$, a power of $t$ that came from the measure when we scaled $V$, a numerical coefficient $c_N$ that comes from the Gaussian integral and the integral over $D$ (on

which nothing depends). This is put in the desired form by using $\Delta(A^{(\sigma)}) = (-1)^\sigma \Delta(A)$ and summing over permutations, which precisely implements the determinant in the final formula

$$\int_{U(N)} \mathrm{d}U \, \exp\Bigl(t \operatorname{Tr}(AUBU^\dagger)\Bigr) = c_N \frac{t^{-(n^2-n)/2} \det \exp\Bigl(t\lambda_i(A)\lambda_j(B)\Bigr)_{1\leq i,j\leq N}}{\Delta(A)\Delta(B)}. \tag{20}$$

It would be interesting to find the coefficient $c_N$ by being more careful about coefficients, for instance about the volume of $U(N)$. □

## 1.3 Exercises about Wolfger Peelaers' course

**Exercise 1.8.**

**Exercise 1.9.**

**Exercise 1.10.**

**Exercise 1.11.**

## 1.4 On spinors

**Exercise 1.12.**

**Exercise 1.13.**

**Exercise 1.14.**

**Exercise 1.15.**

**Exercise 1.16.**

## 1.5 Special functions

**Exercise 1.17.** Try to get $\log\Gamma(x) = x\log x - x - \frac{1}{2}\log x + O(1)$ using $\Gamma(x+1) = x\Gamma(x)$.

*Answer.* Assume we know $\log\Gamma(x) = \alpha x\log x + \beta x + \gamma\log x + O(1)$ for $x\to\infty$, and apply this in $\Gamma(x+1) = x\Gamma(x)$. We find

$$\begin{aligned} 0 &= \log\Gamma(x+1) - \log\Gamma(x) - \log x && (21)\\ &= \alpha\Bigl((x+1)\log(x+1) - x\log x\Bigr) + \beta + \gamma\Bigl(\log(x+1) - \log x\Bigr) - \log x + O(1) && (22)\\ &= \alpha\Bigl(\log x + 1\Bigr) + \beta - \log x + O(1) && (23) \end{aligned}$$

so $\alpha = 1$ and $\beta = -1$ but it seems we cannot determine $\gamma = -1/2$ in this way. □

**Exercise 1.18.** The Barnes double gamma function $\Gamma_b(x)$ is such that $\Gamma_b(x+b)/\Gamma_b(x) = \sqrt{2\pi}b^{xb-1/2}/\Gamma(xb)$ and $\Gamma_b = \Gamma_{1/b}$. If $b>0$, the function $\Gamma_b$ is analytic away from $\{x\leq 0\}\subset\mathbb{R}$. Find its poles and their order. Find the large-$x$ expansion of $\log\Gamma_b$.

This is related to the Upsilon function by $\Upsilon(x) = 1/\bigl(\Gamma_b(x)\Gamma_b(b+b^{-1}-x)\bigr)$. Check that the zeros of $\Upsilon$ are consistent with the product formula Wolfger gave during the lecture.

*Answer.* The condition that $\Gamma_b = \Gamma_{1/b}$ implies that there are two shift relations:

$$\Gamma_b(x+b)/\Gamma_b(x) = \sqrt{2\pi}b^{xb-1/2}/\Gamma(xb), \qquad \Gamma_b(x+1/b)/\Gamma_b(x) = \sqrt{2\pi}b^{-x/b+1/2}/\Gamma(x/b). \tag{24}$$

Using the Stirling expansion we know

$$\log\Gamma_b(x+b)/\log\Gamma_b(x) = -\log\Gamma(xb) + (b\log b)x + O(1) = -bx\log x + bx + \frac{1}{2}\log x + O(1). \tag{25}$$

This is of order $x \log x$ so we search for an expansion

$$\log \Gamma_b(x) = \alpha x^2 \log x + \beta x^2 + \gamma x \log x + \delta x + \epsilon \log x + O(1). \tag{26}$$

We deduce

$$\log \Gamma_b(x+b) - \log \Gamma_b(x) = 2\alpha b x \log x + (\alpha + 2\beta) b x + (\gamma + b\alpha) b \log x + O(1). \tag{27}$$

Comparing to (25) we find $\alpha = -1/2$, $\beta = 3/4$, $\gamma = \frac{1}{2}(b + 1/b)$. This is invariant under $b \to 1/b$, consistently with $\Gamma_b = \Gamma_{1/b}$.

The poles of the Gamma function can be found by inverting the shift relation to $\Gamma(x) = \Gamma(x+1)/x$ and noticing that this blows up for $x = 0$ or for $x + 1$ equal to a pole of Gamma. We apply the same strategy and find that $\Gamma_b$ has simple poles at $-mb - nb^{-1}$ for all integers $m, n \geq 0$, and is analytic away from those poles. We deduce that the $\Upsilon$ function has two sets of zeros: at $-mb - nb^{-1}$ and $(1+m)b + (1+n)b^{-1}$ for $m, n \geq 0$. Each of these zeros is a zero of one of the linear factors in the regularized product description of the Upsilon function. $\square$