3 Answers for exercise session 3, July 19

3.1 Exercises about Guido Festuccia's course

Exercise 3.1.

Exercise 3.2. What is the number of DOFs (physical components) in a closed k-form in d dimensions?

Answer. (Provided by Yale Fan.) First note that the Bianchi identity gives

$$(\#\text{ DOFs in closed } k\text{-form}) = (\#\text{ DOFs in } k\text{-form}) - (\#\text{ DOFs in exact } (k+1)\text{-form}).$$

Equivalently, we have

$$(\#\text{ DOFs in exact } (k+1)\text{-form}) = (\#\text{ DOFs in } k\text{-form}) - (\#\text{ DOFs in closed } k\text{-form})$$

simply by taking the exterior derivative of a k-form and noting that the resulting exact $(k+1)$-form is independent of shifts of the k-form by closed k-forms. For the purpose of counting DOFs, we note that closed forms are LOCALLY exact, so that we in fact have

$$(\#\text{ DOFs in closed } k\text{-form}) = (\#\text{ DOFs in } k\text{-form}) - (\#\text{ DOFs in closed } (k+1)\text{-form}).$$

Recursively applying this formula and noting that top forms are automatically closed gives

$$(\#\text{ DOFs in closed } k\text{-form}) = \sum_{j=k}^{d}(-1)^{j-k}(\#\text{ DOFs in } j\text{-form}) = \sum_{j=k}^{d}(-1)^{j-k}\binom{d}{j}$$

(this is "inclusion-exclusion"). An alternative (but really, equivalent) method of counting is as follows. Again, a closed k-form is locally exact: $F_k^c \sim dF_{k-1}$. But in making this identification, F_{k-1} is only defined up to shifts by a closed form F_{k-1}^c where $F_{k-1}^c \sim dF_{k-2}$, and so on (these are "gauge ambiguities"). Thus we can recursively apply the same formula, rewritten as

$$(\#\text{ DOFs in closed } k\text{-form}) = (\#\text{ DOFs in } (k-1)\text{-form}) - (\#\text{ DOFs in closed } (k-1)\text{-form}),$$

in the other direction. This gives:

$$(\#\text{ DOFs in closed } k\text{-form}) = -\sum_{j=0}^{k-1}(-1)^{j-k}(\#\text{ DOFs in } j\text{-form}) = -\sum_{j=0}^{k-1}(-1)^{j-k}\binom{d}{j}.$$

The two expressions for (# DOFs in closed k-form) are equal because $\sum_{j=0}^{d}(-1)^j\binom{d}{j} = 0$.

In fact these formulas simplify further thanks to $\sum_{k=0}^{m}(-1)^k\binom{n+1}{k} = (-1)^m\binom{n}{m}$. The number of physical degrees of freedom is then $\binom{d-1}{k-1}$ (zero for $k=0$). □

Exercise 3.3.

Exercise 3.4.

Exercise 3.5 (Special cases of the index)**.**

Answer. (Provided by Jonathan Schulz and co..)

- $r = 1$:

$$\prod_{m,n\geq 0}\frac{1-(pq)^{-\frac{1}{2}}p^{m+1}q^{n+1}}{1-(pq)^{\frac{1}{2}}p^m q^n} = \prod_{m,n\geq 0}\frac{1-(pq)^{\frac{1}{2}}p^m q^n}{1-(pq)^{\frac{1}{2}}p^m q^n} = \prod_{m,n\geq 0} 1 = 1$$

  For $r = 1$, the superpotential $W = m\Phi^2$ has R-charge 2 and thus preserves supersymmetry. This potential introduces a mass m for Φ, so Φ freezes out in the IR, thus the IR theory is the trivial SCFT, consistent with the index 1.

- r and $2-r$:

$$\left(\prod_{m,n\geq 0}\frac{1-(pq)^{-\frac{r}{2}}p^{m+1}q^{n+1}}{1-(pq)^{\frac{r}{2}}p^m q^n}\right)\left(\prod_{a,b\geq 0}\frac{1-(pq)^{-\frac{2-r}{2}}p^{a+1}q^{b+1}}{1-(pq)^{\frac{2-r}{2}}p^a q^b}\right)$$
$$=\left(\prod_{m,n\geq 0}\frac{1-(pq)^{-\frac{r}{2}}p^{m+1}q^{n+1}}{1-(pq)^{\frac{r}{2}}p^m q^n}\right)\left(\prod_{a,b\geq 0}\frac{1-(pq)^{\frac{r}{2}}p^a q^b}{1-(pq)^{-\frac{r}{2}}p^{a+1}q^{b+1}}\right)=1$$

  The superpotential $W=m\Phi_1\Phi_2$ has R-charge 2 and is thus valid. This introduces a mass term for both Φ_1 and Φ_2, so a theory with this superpotential also has a trivial SCFT in the IR.

- $r=2$:

$$\prod_{m,n\geq 0}\frac{1-(pq)^{-1}p^{m+1}q^{n+1}}{1-(pq)^1 p^m q^n}=\frac{1-1}{1-(pq)}\prod_{\substack{m,n\geq 0\\(m,n)\neq(0,0)}}\frac{1-(pq)^{-1}p^{m+1}q^{n+1}}{1-(pq)^1 p^m q^n}=0.$$

  The valid superpotential $W=\lambda\Phi$ leads to a scalar potential $V=|\frac{\partial W}{\partial\phi}|^2=|\lambda|^2$, so for $\lambda\neq 0$ supersymmetry is spontaneously broken. In this case, no BPS states exist, so the index is zero.

□

3.2 Exercises about Francesco Benini's course

Exercise 3.6.

Answer. (Provided by Jonathan Schulz and co..) We use the following vielbein on S_2:

$$ds^2=r^2(d\theta^2+\sin^2\theta d\phi^2)=\sum_a e^a e^a,\quad e^a\equiv e^a_\mu dx^\mu,$$

$$e^1_\theta=r,\quad e^1_\phi=0,\quad e^2_\theta=0,\quad e^2_\phi=r\cos\theta$$

The spin connection fulfills

$$de^a+\omega^{ab}\wedge e^b=0,$$

so $\omega^{12}=-\cos\theta d\phi$.

The covariant derivative on spinors

$$\nabla_\mu\varepsilon=\partial_\mu\varepsilon-\tfrac{1}{4}\omega^{ab}_\mu\gamma_a\gamma_b\varepsilon$$

simplifies in two dimensions to

$$\nabla_\mu\varepsilon=\partial_\mu\varepsilon-\tfrac{1}{4}\omega^{12}_\mu[\gamma_1,\gamma_2]\varepsilon.$$

Using the vielbein, the right hand side of the equation can be rewritten to

$$\frac{i}{2R}\gamma_\mu\varepsilon=\frac{i}{2R}e^a_\mu\gamma_a\varepsilon.$$

With the gamma matrices $\gamma_a=(\sigma^1,\sigma^2)$ (which now are just like in flat space, as we have a vielbein) we get the equations

$$\partial_\theta\varepsilon=\tfrac{i}{2}\sigma^1\varepsilon$$
$$\partial_\phi\varepsilon-\tfrac{i}{2}\cos(\theta)\sigma^3\varepsilon=\tfrac{i}{2}\sin(\theta)\sigma^2\varepsilon.$$

The ansatz

$$\varepsilon = \begin{pmatrix} \varepsilon^+ \\ \varepsilon^- \end{pmatrix}, \quad \varepsilon^+ + \varepsilon^- = e^{i\theta/2} f(\phi), \quad \varepsilon^+ - \varepsilon^- = e^{-i\theta/2} g(\phi)$$

solves the equation in θ and yields the equations

$$f' = \tfrac{i}{2} g, \quad g' = \tfrac{i}{2} f$$

in ϕ. These are solved by

$$f(\phi) = ae^{i\phi/2} + be^{-i\phi/2},$$
$$g(\phi) = ae^{i\phi/2} - be^{-i\phi/2}.$$

These solutions are anti-periodic and thus ε is a valid globally defined spinor on S_2 for any values of $a, b \in \mathbb{C}$. The computation in $\tilde{\varepsilon}$ is identical and yields two more complex parameters. □

Exercise 3.7.