3 Exercise session 3, July 19

3.1 Exercises about Guido Festuccia's course

Exercise 3.1. Show that

$$(\nabla_\mu - iA_\mu)\zeta = \frac{-i}{2} V^\nu \sigma_\mu \widetilde{\sigma}_\nu \zeta \tag{1}$$

$$(\nabla_\mu + iA_\mu)\widetilde{\zeta} = \frac{i}{2} V^\nu \widetilde{\sigma}_\mu \sigma_\nu \widetilde{\zeta} \tag{2}$$

implies that $\kappa^\mu = \zeta\sigma^\mu\widetilde{\zeta}$ is a Killing vector field. Note that this differs from earlier similar exercises because of the extra terms in the *generalized* Killing spinor equations.

Exercise 3.2. In d dimensions, compute in two ways how many physical components a closed k-form F has. First count the number of components, subtract the number of equations, add back the number of relations between these equations, and so on. Second write $F = \mathrm{d}A$ with A a $k-1$ form, but note that A has a gauge ambiguity, itself having a gauge ambiguity and so on. The two expressions are equal thanks to $\sum_j (-1)^j \binom{d}{j} = 0$.

Exercise 3.3. Check that the $S^3 \times S^1$ supersymmetry algebra $\mathfrak{su}(2|1)_{\text{left}} \times \mathfrak{u}(1) \times \mathfrak{su}(2)_{\text{right}}$ is a subset of $\mathfrak{su}^*(4|1)$ that is **a** maximal subset under the constraint that it only contains isometries of $S^3 \times S^1$ and not any conformal transformations.

Exercise 3.4. By decomposing a free chiral field of R-charge r on $S^3 \times S^1$ into spherical harmonics, work out that the index takes the form

$$I = \prod_{m,n\geq 0} \frac{1-(pq)^{-r/2}p^{m+1}q^{m+1}}{1-(pq)^{r/2}p^m q^n} \tag{3}$$

Exercise 3.5. Check that for a chiral multiplet Φ of R-charge r,

- for $r = 1$ the index reduces to 1, consistent with the possibility of adding a superpotential $W = \Phi^2$ that gives mass to Φ hence leads to a trivial theory in the IR;
- for two chirals Φ_1 and Φ_2 of R-charges r and $2 - r$, the product of their indices is also 1; give a similar interpretation in terms of superpotential;
- for $r = 2$ the index reduces to 0, suggesting that supersymmetry is broken; explain that with a superpotential.

Hint: superpotentials must have R-charge 2 to preserve R-symmetry, which on curved space is in the supersymmetry algebra.

3.2 Exercises about Francesco Benini's course

Exercise 3.6. Solve the conformal Killing spinor equations on the sphere S^2.

$$\nabla_\mu \epsilon = \frac{i}{2R}\gamma_\mu \epsilon \tag{4}$$

$$\nabla_\mu \widetilde{\epsilon} = \frac{i}{2R}\gamma_\mu \widetilde{\epsilon}. \tag{5}$$

Each of these equations should have 2 solutions.

The partition function of a 2d $\mathcal{N} = (2,2)$ theory with chiral multiplets in the representation $R = \bigoplus_I R_I$ (each with an R-charge q_I and a twisted mass μ_I) of the gauge group G on S^2 (of radius 1) is a sum over GNO-quantized fluxes (m has integer eigenvalues on any representation of

G) and an integral over the Cartan algebra $\mathfrak{t}$ of G:

$$Z_{S^2} = \frac{1}{|\mathrm{Weyl}(G)|} \sum_{\mathsf{m}} \int_{\mathfrak{t}} \frac{\mathrm{d}a}{(2\pi)^{\mathrm{rank}\, G}} Z_{\mathrm{cl}}(z, \bar{z}; a, \mathsf{m})\, Z_{\mathrm{gauge}}(a, \mathsf{m})\, Z_{\mathrm{matter}}(q, a + \mu, \mathsf{m}), \tag{6}$$

$$Z_{\mathrm{gauge}} = \prod_{\alpha > 0} (-1)^{\alpha(\mathsf{m})} \left[\alpha(a)^2 + \frac{\alpha(\mathsf{m})^2}{4} \right], \tag{7}$$

$$Z_{\mathrm{matter}} = \prod_{I,\rho} \frac{\Gamma\big(\frac{q_I}{2} - i\mu_I - i\rho(a) - \frac{\rho(\mathsf{m})}{2}\big)}{\Gamma\big(1 - \frac{q_I}{2} + i\mu_I + i\rho(a) - \frac{\rho(\mathsf{m})}{2}\big)}, \tag{8}$$

$$Z_{\mathrm{cl}} = \prod_{\ell} z_\ell^{\mathrm{Tr}_\ell(ia + \frac{\mathsf{m}}{2})}\, \bar{z}_\ell^{\mathrm{Tr}_\ell(ia - \frac{\mathsf{m}}{2})}. \tag{9}$$

The products range over positive roots α of G and over weights ρ of each representation R_I, while $z_\ell = e^{-2\pi\zeta_\ell + i\vartheta_\ell}$ is one combination of FI parameter and theta parameter for each $U(1)$ gauge factor, and Tr_ℓ is the projection onto the ℓ-th $U(1)$ factor: for $G = \prod_\ell U(N_\ell)$ these really are traces.

Exercise 3.7. Specialize to a theory with gauge group $U(1)$ and 4 chiral multiplets of charges $(+1, +1, -1, -1)$, or do the case of N_f charges $+1$ and N_f charges -1.

Depending on the sign of the FI parameter, close the a integration contour towards $\pm i\infty$ and use the residue formula. Recall that R-charges obey $0 < r < 2$. Recall that $\Gamma(x)$ has poles at $-k$ for $k \in \mathbb{Z}_{\geq 0}$. Perform the change of variables $k^{\pm} = k + |m|/2 \pm m/2 \geq 0$ and factorize the result into a finite sum of terms, where each term is (series in k^+ holomorphic in z) times (series in k^- holomorphic in $\overline{z}$) times the residue at $k^+ = k^- = 0$.

3.3 Exercises about Wolfger Peelaers' course