1 Exercise session 1, July 16

1.1 Exercises about Guido Festuccia's course

Exercise 1.-1 (Symmetric Energy Momentum Tensor)**.** [1] A local, translation invariant field theory has a conserved energy momentum tensor $\hat{T}_{\mu\nu}$. It is not necessarily symmetric.

$$\partial^\nu T_{\mu\nu} = 0 \ , \qquad P_\mu = \int d^3x T_\mu{}^0 \tag{1}$$

- Check that $T_{\mu\nu}$ can be improved as follows

$$T_{\mu\nu} \to T_{\mu\nu} + \partial^\rho I_{\mu\nu\rho} \ , \qquad I_{\mu\nu\rho} = -I_{\mu\rho\nu} \tag{2}$$

If the theory is Lorentz invariant there exists a real conserved current $j_{\mu\nu\rho} = -j_{\nu\mu\rho}$ giving the Lorentz generators

$$\partial^\rho j_{\mu\nu\rho} = 0 \ , \qquad J_{\mu\nu} = \int d^3x j_{\mu\nu}{}^0 \ . \tag{3}$$

They satisfy the algebra $[P_\mu, J_{\nu\rho}] = -i(\eta_{\mu\nu}P_\rho - \eta_{\mu\rho}P_\nu)$.

- Show that $j_{\mu\nu\rho}$ is then given by

$$j_{\mu\nu\rho} = x_\mu T_{\nu\rho} - x_\nu T_{\mu\rho} + s_{\mu\nu\rho} \ , \qquad s_{\mu\nu\rho} = -s_{\nu\mu\rho} \ , \tag{4}$$

  where $s_{\mu\nu\rho}$ is a local operator that does not *explicitly* depend on x.

- Show that using a linear combination of $s_{\mu\nu\rho}$ you can define an improvement $I_{\mu\nu\rho}$ that makes $T_{\mu\nu}$ symmetric.
- Show that in terms of the symmetric energy momentum tensor $j_{\mu\nu\rho} = x_\mu T_{\nu\rho} - x_\nu T_{\mu\rho}$.

Exercise 1.0 (BPS String)**.** Consider the conserved string current $C_{\mu\nu} \sim \epsilon_{\mu\nu\rho\lambda}F^{\rho\lambda}$ where $F_{\mu\nu} = -F_{\nu\mu}$, $\partial_{[\mu}F_{\nu\rho]} = 0$.

- Suppose the corresponding charge is carried by a string-like object lying along the 3 axis show that the corresponding string charge by unit lenght is

$$\frac{Z_3}{L} = \pm TL \ , \qquad Z_{0,1,2} = 0 \tag{5}$$

  for some T.

- In the rest frame of the string write down the susy algebra

$$\{\bar{Q}_{\dot\alpha}, Q_\alpha\} = 2\sigma^\mu_{\alpha\dot\alpha}(P_\mu + Z_\mu) \ , \quad \{Q_\alpha, Q_\beta\} = 0 \tag{6}$$

  and obtain that $\frac{M}{L} \geq T$ where M is the mass of the string.

- Check that if the bound is saturated the string object preserves two supercharges.

Exercise 1.1 (Strings in sQED)**.** The Lagrangian for sQED with an FI term in Wess-Zumino gauge is

$$\mathcal{L} = \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + D_\mu\bar\phi_+ D^\mu\phi_+ + D_\mu\bar\phi_- D^\mu\phi_- - \frac{e^2}{2}(\bar\phi_+\phi_+ - \bar\phi_-\phi_- - \xi)^2 + \text{fermions} \ . \tag{7}$$

Here ξ is the FI parameter which we take to be positive $\xi > 0$. The scalar fields $\phi_\pm$ have opposite charges so that $D_\mu\phi_\pm = (\partial_\mu \mp iA_\mu)\phi_\pm$.

Consider a static string-like field configuration where the only fields that are turned on are A_1 , A_2 , ϕ_+ and let these fields depend only on x_1, x_2.

[1] Numbering chosen for consistency with earlier versions of the exercise sheet.

- Check that you can rewrite the energy functional for the theory as

$$E = \int dx_1 dx_2 \left[\left| \frac{1}{\sqrt{2}e^2} F_{12} + \frac{e}{\sqrt{2}} (\bar{\phi}_+ \phi_+ - \xi) \right|^2 + (D_1\phi_+ + iD_2\phi_+)^* (D_1\phi_+ + iD_2\phi_+) \right] + Q_{\text{top}} \tag{8}$$

check that Q_{top} is a topological contribution.

- Show that the energy functional can be saturated by the following ansatz $x_1 + ix_2 = re^{i\alpha}$

$$\phi_+ = \sqrt{\xi} n(r) e^{i\alpha} \ , \qquad A_i = a(r) \frac{\partial \alpha}{\partial x^i} \tag{9}$$

with the boundary conditions $n(0) = a(0) = 0$ and $n(\infty) = a(\infty) = 1$.

- Compute the string charge for this configuration using the string current

$$C_{\mu\nu} = \frac{1}{4} \xi \epsilon_{\mu\nu\rho\lambda} F^{\rho\lambda} \tag{10}$$

The configurations found above preserve two supercharges (if you still have energy you can check it).

Exercise 1.2. Consider a Wess–Zumino model, namely chiral multiplets with values in some manifold endowed (on each coordinate patch) with a Kähler potential $K(\Phi, \overline{\Phi})$ and superpotential $W(\Phi)$. When changing coordinate patch, $K \to K + \Lambda(\Phi) + \overline{\Lambda}(\overline{\Phi})$ and $W \to W +$ constant. Derive the equation of motion $\overline{D}^2 \partial_i K = 4 \partial_i W$ by varying $\Phi \to \Phi + \delta\Phi$ and noting that for C chiral, the vanishing of $\int \mathrm{d}^4x \int \mathrm{d}^2\theta C \delta\Phi$ for all chiral $\delta\Phi$ implies $C = 0$. Then check that

$$\mathbb{S}_{\alpha\dot{\alpha}} = 2g_{i\bar{j}} D_\alpha \Phi^i \overline{D}_{\dot{\alpha}} \overline{\Phi}^{\bar{j}} \qquad \chi_\alpha = \overline{D}^2 D_\alpha K \qquad Y_\alpha = 4 D_\alpha W \tag{11}$$

obey (on-shell) the constraints

$$\begin{gathered} \overline{D}^{\dot{\alpha}} \mathbb{S}_{\alpha\dot{\alpha}} = \chi_\alpha + Y_\alpha, \quad \overline{D}_{\dot{\alpha}} \chi_\alpha = 0, \quad D^\alpha \chi_\alpha = \overline{D}_{\dot{\alpha}} \overline{\chi}^{\dot{\alpha}}, \\ D_\alpha Y_\beta + D_\beta Y_\alpha = 0, \quad \overline{D}^2 Y_\alpha = 0. \end{gathered} \tag{12}$$

Exercise 1.3. Given superfields that obey (12) and the component decomposition

$$\begin{aligned} \mathbb{S}_\mu = j_\mu - i\theta\left(S_\mu - \frac{i}{\sqrt{2}}\sigma_\mu \psi\right) + i\overline{\theta}\left(\overline{S}_\mu - \frac{i}{\sqrt{2}}\overline{\sigma}_\mu \psi\right) + \frac{i}{2}\theta^2 \overline{Y}_\mu - \frac{i}{2}\overline{\theta}^2 Y_\mu \\ + \theta\sigma^\nu\overline{\theta}\left(2T_{\nu\mu} - \eta_{\nu\mu} A - \frac{1}{8}\epsilon_{\nu\mu\rho\sigma} F^{\rho\sigma} - \frac{1}{2}\epsilon_{\nu\mu\rho\sigma}\partial^\rho j^\sigma\right) + \dots \end{aligned} \tag{13}$$

show that $T_{\mu\nu}$ is conserved, $S_{\mu\alpha}$ is conserved, $F_{\mu\nu}$ is a closed 2-form and Y_μ is a closed 1-form.

1.2 Exercises about Francesco Benini's course

Exercise 1.4. Compute $\int_{S^2} e^{ic\cos\theta} \mathrm{dVol}(S^2)$.

Exercise 1.5. Show that $\{\mathrm{d}, i_V\} = \mathcal{L}_V$ hence $\mathrm{d}_V = \mathrm{d} - i_V$ squares to $-\mathcal{L}_V$.

Exercise 1.6. Let $\eta = g(V, \ )$ and

$$\Theta_V = \eta \wedge \left(\frac{-1}{|V|^2} \left(1 + \frac{\mathrm{d}\eta}{|V|^2} + \dots + \frac{(\mathrm{d}\eta)^{\dim M/2}}{|V|^{\dim M}} \right) \right). \tag{14}$$

Check that $\mathrm{d}_V \Theta_V = 1$ and deduce that **on** $\boldsymbol{M \setminus M_V}$, $\mathrm{d}_V \alpha = 0$ implies $\alpha = \mathrm{d}_V \beta$.

Exercise 1.7 (Very hard?)**.** Let A be a Hermitian matrix with eigenvalues $\lambda_1(A), \dots, \lambda_n(A)$ and likewise B. Compute the Harish-Chandra–Itzykson–Zuber integral

$$\begin{aligned} &\int_{U(N)} \exp(t \operatorname{Tr}(AUBU^\dagger)) \mathrm{d}U \\ &= \left(\prod_{i=1}^{n-1} i! \right) \frac{\det\big(\exp(t\lambda_i(A)\lambda_j(B))_{1 \le i,j \le n}\big)}{t^{(n^2-n)/2} \prod\limits_{1 \le i < j \le n} (\lambda_j(A) - \lambda_i(A)) \prod\limits_{1 \le i < j \le n} (\lambda_j(B) - \lambda_i(B))} \end{aligned}$$

See `https://terrytao.wordpress.com/2013/02/08/`

1.3 Exercises about Wolfger Peelaers' course

Exercise 1.8. Show that k hypermultiplets in the fundamental representation of $SU(N)$ for $N > 2$ have $U(k)$ flavour symmetry. What happens for $N = 2$?

Exercise 1.9. Show that $\nabla_m \xi = -i\sigma_m \widetilde{\xi}'$ is Weyl covariant with $\xi \to \Omega^{1/2}\xi$ (recall that $\nabla_m \xi = \partial_m \xi + \frac{1}{4}\omega_m{}^{ab}\sigma_{ab}\xi$)

Exercise 1.10 (Somewhat technical, only do some)**.** Verify that transformation rules are Weyl covariant.

Exercise 1.11 (Easy)**.** The scale parameter and $U(1)_r$ parameter are given by

$$w \sim \xi^I \xi'_I + \widetilde{\xi}_I \widetilde{\xi}'^I \tag{15}$$

$$r \sim \xi^I \xi'_I - \widetilde{\xi}_I \widetilde{\xi}'^I \tag{16}$$

find/verify the condition for $w = r = 0$.

1.4 On spinors

Here, Γ_a, $1 \le a \le d$ are Gamma matrices in d-dimensions. They obey $\{\Gamma_a, \Gamma_b\} = 2g_{ab}$ where g_{ab} is the metric, and they generate a 2^d-dimensional Clifford algebra. As a vector space this algebra is spanned by the antisymmetrized $\Gamma_{a_1 \dots a_k} = \Gamma_{[a_1 \dots a_k]}$ for all $0 \le k \le d$ and all indices a_i.

Exercise 1.12. Find all Killing spinors and conformal Killing spinors in flat space. Same question on S^n.

Exercise 1.13. Let ψ, χ be two Killing spinors or two conformal Killing spinors. Derive a differential equation obeyed by spinor bilinears $\chi\Gamma_{a_1 \dots a_k}\psi$. Are they Killing vectors, conformal Killing vectors?

Exercise 1.14. Let $h_{ab} = \mathrm{diag}(1, \dots, 1, -1, \dots, -1)$ have s '+1' and t '−1' (so $d = s + t$). Show that the Clifford algebra is isomorphic to a matrix algebra $M_{2^\#}(\bullet)$ (for some number #) with

$s - t \bmod 8$	0	1	2	3	4	5	6	7
$\bullet$ is	$\mathbb{R}$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}$	$\mathbb{C}$

In odd dimensions, show that the complexification of the Clifford algebra is a direct sum of two algebras in which $\Gamma_1 \dots \Gamma_d = \pm 1$ or $\pm i$. What is its minimal faithful real representation? What is its minimal faithful complex representation? This is the spinor representation.

Exercise 1.15. The generator of rotations $M_{ab} \in \mathfrak{so}(s,t)$ acts as $\frac{1}{4}\Gamma_a \Gamma_b$ on representations of the Clifford algebra. How does the spinor representation of the Clifford algebra decompose into complex/real representations of $\mathfrak{so}(s,t)$?

This table lists for each d the complex dimension of the minimal complex spinor (Dirac or Weyl), then for each (d,t) the real dimension of the minimal real spinor (Majorana, Majorana–Weyl, symplectic, symplectic Majorana–Weyl). Did whomever typed it make a mistake?

d		$t \equiv 0$		1		2		3 mod 4	
1	(D 2)	M	1	M	1				
2	(W 2)	M^-	2	MW	1	M^+	2		
3	(D 4)	s	4	M	2	M	2	s	4
4	(W 4)	sW	4	M^+	4	MW	2	M^-	4
5	(D 8)	s	8	s	8	M	4	M	4
6	(W 8)	M^+	8	sW	8	M^-	8	MW	4
7	(D 16)	M	8	s	16	s	16	M	8
8	(W16)	MW	8	M^-	16	sW	16	M^+	16
9	(D 32)	M	16	M	16	s	32	s	32
10	(W32)	M^-	32	MW	16	M^+	32	sW	32
11	(D 64)	s	64	M	32	M	32	s	64
12	(W64)	sW	64	M^+	64	MW	32	M^-	64

Exercise 1.16. The conformal algebra in $d = s + t$ dimensional flat space is $\mathfrak{so}(s+1, t+1)$. The superconformal algebra contains the conformal algebra, an R-symmetry algebra, and some fermionic generators that transform in a spinor representation of the conformal algebra. Using the following table of (complex) simple Lie superalgebras, show that the (complexified) superconformal algebras exist for $d \leq 6$ but not $d \geq 7$. In this table, $m, n \geq 1$ and we do not list purely bosonic Lie algebras. The factor $\mathbb{C}$ of $\mathfrak{sl}(m|n)$ must be removed if $m = n$.

	Bosonic algebra	Fermionic generators in representation
$\mathfrak{sl}(m\vert n, \mathbb{C})$	$\mathfrak{sl}(m, \mathbb{C}) \oplus \mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{C}$	$(m, \overline{n}) \oplus (\overline{m}, n)$
$\mathfrak{osp}(m\vert 2n, \mathbb{C})$	$\mathfrak{so}(m, \mathbb{C}) \oplus \mathfrak{sp}(2n, \mathbb{C})$	$(m, 2n)$
$\mathfrak{D}(2, 1, \alpha, \mathbb{C})$	$\mathfrak{sl}(2, \mathbb{C})^3$	$(2, 2, 2)$
$\mathfrak{F}(4, \mathbb{C})$	$\mathfrak{so}(7, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$	$(8, 2)$
$\mathfrak{G}(3, \mathbb{C})$	$\mathfrak{G}_2 \oplus \mathfrak{sl}(2, \mathbb{C})$	$(7, 2)$
$\mathfrak{P}(m, \mathbb{C})$	$\mathfrak{sl}(m+1, \mathbb{C})$	$\text{sym} \oplus \overline{\text{antisym}}$
$\mathfrak{Q}(m, \mathbb{C})$	$\mathfrak{sl}(m+1, \mathbb{C})$	adjoint

As a refinement, find the superconformal algebra in signature (s, t) in the table of real forms. Here, $m, n \geq 1$, $0 \leq p \leq m/2$, $0 \leq q \leq n/2$. The forms $\mathfrak{su}^*$, $\mathfrak{osp}^*$, $\mathfrak{Q}^*$ only exist for even rank; $\mathfrak{sl}'$ only if $m = n$.

Real form	Bosonic algebra
$\mathfrak{su}(m-p, p\vert n-q, q)$	$\mathfrak{su}(m-p, p) \oplus \mathfrak{su}(n-q, q) \oplus \mathfrak{u}(1)^\ddagger$
$\mathfrak{sl}(m\vert n)$	$\mathfrak{sl}(m, \mathbb{R}) \oplus \mathfrak{sl}(n, \mathbb{R}) \oplus \mathfrak{so}(1,1)^\ddagger$
$\mathfrak{sl}'(n\vert n)$ $(m = n)$	$\mathfrak{sl}(n, \mathbb{C})$
$\mathfrak{su}^*(m\vert n)$ $(m, n$ even$)$	$\mathfrak{su}^*(m) \oplus \mathfrak{su}^*(n) \oplus \mathfrak{so}(1,1)^\ddagger$
$\mathfrak{osp}(m-p, p\vert 2n)$	$\mathfrak{so}(m-p, p) \oplus \mathfrak{sp}(2n, \mathbb{R})$
$\mathfrak{osp}^*(m\vert 2n-2q, 2q)$ $(m$ even$)$	$\mathfrak{so}^*(m) \oplus \mathfrak{usp}(2n-2q, 2q)$
$\mathfrak{D}^p(2, 1, \alpha)$ §	$\mathfrak{so}(4-p, p) \oplus \mathfrak{sl}(2, \mathbb{R})$ $(p = 0, 1, 2)$
$\mathfrak{F}^p(4)$ for $p = 0, 3$	$\mathfrak{so}(7-p, p) \oplus \mathfrak{sl}(2, \mathbb{R})$
$\mathfrak{F}^p(4)$ for $p = 1, 2$	$\mathfrak{so}(7-p, p) \oplus \mathfrak{su}(2)$
$\mathfrak{G}_s(3)$ for $s = -14, 2$	$\mathfrak{G}_{2(s)} \oplus \mathfrak{sl}(2, \mathbb{R})$
$\mathfrak{P}(m)$	$\mathfrak{sl}(m+1, \mathbb{R})$
$\mathfrak{UQ}(m-p, p)$	$\mathfrak{su}(m+1-p, p)$
$\mathfrak{Q}(m)$	$\mathfrak{sl}(m+1, \mathbb{R})$
$\mathfrak{Q}^*(m)$ $(m$ odd$)$	$\mathfrak{su}^*(m+1)$

1.5 Special functions

Exercise 1.17. The Stirling formula states that $\log \Gamma(x) = x \log x - x - \frac{1}{2} \log x + O(1)$. Which of these coefficients can be fixed using $\Gamma(x+1) = x\Gamma(x)$?

Exercise 1.18. The Barnes double gamma function $\Gamma_b(x)$ is such that $\Gamma_b(x+b)/\Gamma_b(x) = \sqrt{2\pi} b^{xb-1/2}/\Gamma(xb)$ and $\Gamma_b = \Gamma_{1/b}$. If b is real, the function Γ_b is analytic away from $\{x \leq 0\} \subset \mathbb{R}$. Find its poles and their order. Find the large-x expansion of $\log \Gamma_b$.

This is related to the Upsilon function by $\Upsilon(x) = 1/\big(\Gamma_b(x)\Gamma_b(b + b^{-1} - x)\big)$. Check that the zeros of Υ are consistent with the product formula Wolfger gave during the lecture.