NOTES ON THE INTERACTION OF SOLITARY WAVES FOR NLS

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Abstract. In these notes, we review various results on the interaction of solitary waves for the nonlinear Schrödinger equation with power nonlinearity. After discussing briefly the well known question of stability of single solitary waves, we present a short proof of existence of multi-solitary waves in the case of weak interactions following [37]. Then, we give a sketch of the method from [47, 63] concerning two cases of strong interactions. In the sub-critical and super-critical cases, this method shows the existence of multi-solitary waves with logarithmic distance in time, extending a result previously known only in the integrable case. In the mass critical case, a new class of multi-solitary waves blowing up in infinite time at a logarithmic rate is obtained. By the pseudo-conformal transform, this yields the first example of solution blowing up in finite time faster than the pseudo-conformal rate and concentrating several bubbles at a point. These special behaviours are due to strong interactions and require refined computations.

1. Introduction

We consider the nonlinear Schrödinger equation

$$i\partial_t u + \Delta u + |u|^{p-1}u = 0, \quad t \in \mathbb{R},\ x \in \mathbb{R}^d, \tag{1}$$

with $p > 1$ for $d = 1, 2$ and $1 < p < \frac{d+2}{d-2}$ for $d \geq 3$. This condition on p corresponds to $\dot{H}^1$ sub-criticality, while the value $p = 1 + \frac{4}{d}$ corresponds to L^2 criticality. The special case $d = 1$ and $p = 3$ corresponds to the integrable cubic 1D NLS

$$i\partial_t u + \partial_x^2 u + |u|^2 u = 0, \quad t \in \mathbb{R},\ x \in \mathbb{R}. \tag{2}$$

Recall that equation (1) is locally well-posed in H^1: for any $u_0 \in H^1$, there exist $T^\star > 0$ and a unique maximal solution $u \in \mathcal{C}([0, T^\star), H^1)$ of (1) with $u(0) = u_0$. Moreover, $T^\star < +\infty$ implies $\lim_{t\uparrow T^\star} \|\nabla u(t)\|_{L^2} = +\infty$. In these notes, by solution of (1), we mean H^1 solution as above. By density and continuous dependence of the solution with respect to the initial data, one justifies formal computations that require using higher order Sobolev regularity. For example, the mass, momentum and energy of the solution are conserved, for all $t \in [0, T^\star)$,

$$\int |u(t,x)|^2 dx = \int |u_0(x)|^2 dx, \quad \Im \int \nabla u(t,x)\, \overline{u}(t,x) dx = \Im \int \nabla u_0(x)\, \overline{u}_0(x) dx,$$

$$E(u(t)) = \frac{1}{2}\int |\nabla u(t,x)|^2 dx - \frac{1}{p+1}\int |u(t,x)|^{p+1} dx = E(u_0).$$

We also recall the symmetries of the equation

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- Scaling, translation and phase invariances: if $u(t,x)$ is solution of (1) then, for any $\lambda > 0$, $\sigma \in \mathbb{R}^d$ and $\gamma \in \mathbb{R}$, the function $w(t,x) = \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda(x-\sigma))e^{i\gamma}$ is also solution of (1).
- Galilean invariance: if $u(t,x)$ is solution of (1) then, for any $\beta \in \mathbb{R}^d$, the function
$$w(t,x) = u(t, x-\beta t)e^{i\left(\frac{1}{2}(\beta\cdot x)-\frac{1}{4}|\beta|^2 t\right)} \quad \text{is also solution of (1).}$$

From the energy and mass conservations and the following Gagliardo-Nirenberg inequality
$$\forall v \in H^1(\mathbb{R}^d), \qquad \int |v|^{p+1} \leq C \left(\int |\nabla v|^2\right)^{\frac{d}{4}(p-1)} \left(\int |v|^2\right)^{1+\frac{1}{4}(2-d)(p-1)}$$
it follows that for $1 < p < 1 + \frac{4}{d}$, any H^1 solution of (1) is global and bounded in H^1.

References. For the local Cauchy problem in H^1, we refer to [25] and to general monographies on the nonlinear Schrödinger equation: [72, 5, 73, 22], where the physical relevance of equation (1) is also discussed.

For the integrable case, see the original paper [77] and the monography [20].

2. Solitary waves

2.1. **Existence and uniqueness of ground states.** We consider solitary waves of (1), i.e. solutions of the form $u(t,x) = e^{it}Q(x)$ where $Q \in H^1(\mathbb{R}^d)$ is solution of
$$\Delta Q + |Q|^{p-1}Q = Q. \tag{3}$$
It is well-known that for any $d \geq 1$, $1 < p < \frac{d+2}{d-2}$ (if $d \geq 3$), there exists a unique positive radial solution of (3), solution of a minimization problem. This solution, denoted by Q, is called the *ground state*, and the corresponding solution $u(t,x) = e^{it}Q(x)$ of (1) is the ground state standing wave, often called *soliton*. For any $\lambda > 0$, we denote
$$Q_\lambda(x) = \lambda^{-\frac{2}{p-1}} Q(\lambda^{-1}x), \quad \Delta Q_\lambda + Q_\lambda^p = \lambda^{-2} Q_\lambda.$$
By the symmetries of equation (1), for any $\lambda > 0$, $\beta \in \mathbb{R}^d$, $\sigma \in \mathbb{R}^d$ and $\gamma \in \mathbb{R}$,
$$R(t,x) = Q_\lambda(x - \sigma - \beta t)e^{i\Gamma(t,x)}, \quad \Gamma(t,x) = \frac{1}{2}(\beta\cdot x) - \frac{1}{4}|\beta|^2 t + \lambda^{-2}t + \gamma,$$
is a solitary wave of (1), moving on the line $x = \sigma + \beta t$. These solitary waves are orbitally stable with respect to the perturbation of the initial data in H^1 if and only if $1 < p < 1 + \frac{4}{d}$.

Theorem 1. (i) Stability in the sub-critical case. *Let $1 < p < 1 + \frac{4}{d}$. There exist $C > 0$ and $\delta > 0$ such that for any $u_0 \in H^1$, if $\|u_0 - Q\|_{H^1} \leq \delta$, then the corresponding solution u of* (1) *satisfies, for all $t \in \mathbb{R}$, $\|u(t) - e^{i\gamma(t)}Q(\cdot - \sigma(t))\|_{H^1} \leq C\|u_0 - Q\|_{H^1}$, for some continous functions σ and γ.*

(ii) Strong instability in the critical and super-critical cases. *Let $1 + \frac{4}{d} \leq p < \frac{d+2}{d-2}$. There exists a sequence of initial data $(u_{0,n})$ converging to Q in H^1 as $n \to +\infty$ such that the corresponding solutions u_n of* (1) *blow up in finite time.*

For $d = 1$, the ground state Q is explicit and equation (3) does not have any other solution in H^1 (up to translation and phase invariance). For $d \geq 2$, equation (3) has other solutions than the ground state, called bound states (see *e.g.* [5]), but we will not consider such solutions in these notes.

2.2. **Notation.** We consider a smooth real-valued cut-off function $0 \le \chi \le 1$ on $\mathbb{R}$ such that $\chi \equiv 1$ on $[-1,1]$ and $\chi \equiv 0$ on $(-\infty,-2] \cap [2,+\infty)$. Denote by $\langle f, g\rangle = \Re(\int f\bar{g})$ the L^2 scalar product of two complex valued functions $f, g \in L^2$. Let $\Sigma = H^1 \cap L^2(|x|^2 dx)$. Denote by $\mathcal{Y}$ the set of smooth functions f such that

$$\text{for all } p \in \mathbb{N}, \text{ there exists } q \in \mathbb{N} \text{ such that, for all } x \in \mathbb{R}^2,\ |f^{(p)}(x)| \lesssim |x|^q e^{-|x|}.$$

Let the operator $\Lambda = \frac{2}{p-1} + x \cdot \nabla$.

The linearization of equation (1) and of the energy functional E around $e^{it}Q$ involves the following Schrödinger operators

$$L_+ = -\Delta + 1 - pQ^{p-1}, \qquad L_- = -\Delta + 1 - Q^{p-1}, \quad \mathcal{L}f = iL_+(\Re f) - L_-(\Im f).$$

We recall the following relations

$$L_- Q = 0, \quad L_+(\Lambda Q) = -2Q, \quad L_+(\nabla Q) = 0, \quad L_-(xQ) = -2\nabla Q \tag{4}$$

that can be checked by direct computations using the equation $\Delta Q_\lambda + Q_\lambda^p = \lambda^{-2} Q_\lambda$. Remark that $\Lambda Q = -\frac{d}{d\lambda} Q_\lambda|_{\lambda=1}$, $\langle \Lambda Q, Q\rangle = \frac{4+d-dp}{2(p-1)}\langle Q, Q\rangle$ and so $\langle \Lambda Q, Q\rangle = 0$ in the mass critical case.

2.3. **Linearized theory in the mass sub-critical case.** In the case $1 < p < 1 + \frac{4}{d}$, we recall the following coercivity property under orthogonality conditions, for $\mu > 0$, for all $\eta \in H^1$,

$$\langle L_+ \Re\eta, \Re\eta\rangle + \langle L_- \Im\eta, \Im\eta\rangle \ge \mu \|\eta\|_{H^1}^2 - \frac{1}{\mu}\left(\langle \eta, Q\rangle^2 + |\langle \eta, xQ\rangle|^2 + \langle \eta, i\Lambda Q\rangle^2\right). \tag{5}$$

2.4. **Linearized theory in the mass critical case.** In the critical case $p = 1 + \frac{4}{d}$, there are additional relevant relations. We check that $L_-(|x|^2 Q) = -4\Lambda Q$ and we denote by $\rho \in \mathcal{Y}$ the unique radial H^1 function satisfying $L_+\rho = \frac{1}{4}|x|^2 Q$. Since $\langle \Lambda Q, Q\rangle = 0$, property (5) does not hold. A modified coercivity property writes, for $\mu > 0$, for all $\eta \in H^1$,

$$\langle L_+ \Re\eta, \Re\eta\rangle + \langle L_- \Im\eta, \Im\eta\rangle \ge \mu \|\eta\|_{H^1}^2 - \frac{1}{\mu}\left(\langle \eta, Q\rangle^2 + \langle \eta, |x|^2 Q\rangle^2 + |\langle \eta, xQ\rangle|^2 + \langle \eta, i\rho\rangle^2\right).$$

2.5. **Linearized theory in the mass super-critical case.** In the case $1 + \frac{d}{4} < p < \frac{d+2}{d-2}$, there exist two real opposite eigenvalues $-e_0$ and $e_0 > 0$ of the operator $\mathcal{L}$. Denote by $Y^\pm \in \mathcal{Y}$ the (L^2 normalized) eigenfunctions associated to $\pm e_0$. Again, property (5) does not hold, but a useful coercivity property can be expressed in term of $Y^\pm$, for $\mu > 0$, for all $\eta \in H^1$,

$$\langle L_+ \Re\eta, \Re\eta\rangle + \langle L_- \Im\eta, \Im\eta\rangle \ge \mu \|\eta\|_{H^1}^2 - \frac{1}{\mu}\left(\langle \eta, iY^+\rangle^2 + \langle \eta, iY^-\rangle^2 + |\langle \eta, xQ\rangle|^2 + \langle \eta, i\Lambda Q\rangle^2\right).$$

In the above three coercivity relations, one can modify the directions chosen in the second member to ensure coercivity. However, we will see that these choices are especially relevant because of their relations to (4) (see §3.2).

2.6. **Orbital stability of the soliton in the sub-critical case.** For a function u close to Q in H^1, there exist unique $\sigma(u)$ and $\gamma(u)$ small such that u can be decomposed as $u = e^{i\gamma}(Q + \varepsilon)(x - \sigma)$ where the function ε satisfies $|\langle \varepsilon, yQ\rangle| = \langle \varepsilon, i\Lambda Q\rangle = 0$. This is one of the standard *modulation results* that we will use throughout these notes. The proof is elementary by the implicit function theorem and we omit it. Of course, the choice of the orthogonality conditions on ε is related to property (5). To obtain some control on $\langle \varepsilon, Q\rangle$, we

assume in addition that the solution is such that $\int |u_0|^2 = \int Q^2$. Then, by mass conservation, it holds
$$0 = \int |u|^2 - \int Q^2 = \int |Q+\varepsilon|^2 - Q^2 = 2\langle Q,\varepsilon\rangle + \langle \varepsilon,\varepsilon\rangle.$$
Thus, by (5), we have $\langle L_+\Re\varepsilon,\Re\varepsilon\rangle + \langle L_-\Im\varepsilon,\Im\varepsilon\rangle \gtrsim \|\varepsilon\|_{H^1}^2 - C\|\varepsilon\|_{L^2}^4 \gtrsim \|\varepsilon\|_{H^1}^2$, for ε small.

For time close to 0, by continuity, $u(t)$ is close to Q, and we can perform the above decomposition. For later times, the energy argument below proves that $u(t)$ stays close to Q up to translation and phase, and thus the decomposition still uniquely exists, even if the related parameters $\sigma(t)$ and $\gamma(t)$ are no longer small.

Define $W(u) = 2E(u)+\int |u|^2$. Using $u = e^{i\gamma}(Q+\varepsilon)(x-\sigma)$, we have the following expansion, integrating by parts and using the equation of Q,
$$\begin{aligned} W(u) = W(Q+\varepsilon) &= W(Q)+2\left(\langle \nabla Q,\nabla\varepsilon\rangle + \langle Q,\varepsilon\rangle - \langle Q^p,\varepsilon\rangle\right)+\langle L_+\Re\varepsilon,\Re\varepsilon\rangle+\langle L_-\Im\varepsilon,\Im\varepsilon\rangle+K\\ &= W(Q) + \langle L_+\Re\varepsilon,\Re\varepsilon\rangle + \langle L_-\Im\varepsilon,\Im\varepsilon\rangle + K,\end{aligned}$$
where we have set
$$K = \frac{2}{p+1}\int \Big(|Q+\varepsilon|^{p+1} - Q^{p+1} - (p+1)Q^p\Re\varepsilon \\ - \frac12 (p+1)pQ^{p-1}(\Re\varepsilon)^2 - \frac12(p+1)Q^{p-1}(\Im\varepsilon)^2\Big).$$
Note that by Taylor expansion and Gagliardo-Nirenberg inequality, $K = O(\|\varepsilon\|_{H^1}^{\bar p+1})$ where $\bar p = \min(2,p)$. Since $W(u(t)) = W(u_0)$ and since $|W(u_0)-W(Q)| \lesssim \delta^2$ by the assumption on u_0, we obtain $\langle L_+\Re\varepsilon,\Re\varepsilon\rangle + \langle L_-\Im\varepsilon,\Im\varepsilon\rangle \lesssim \delta^2 + \|\varepsilon\|_{H^1}^{\bar p+1}$. Combining these computations with the coercivity property, we find $\|\varepsilon\|_{H^1} \lesssim \delta$ for δ small enough. Finally, we note that we can reduce to the special case $\int |u_0|^2 = \int Q^2$ without loss of generality by a scaling argument.

In practice, to argue rigorously, for a constant $A \gg 1$, we define
$$T = \sup\{t \geq 0,\ \forall t' \in [0,t],\ \inf_{\sigma,\gamma} \|u(t') - e^{i\gamma}Q(\cdot-\sigma)\|_{H^1} \leq A\delta\} > 0,$$
and we perform the above decomposition and energy arguments on $[0,T]$. For A large enough, we obtain $T=+\infty$.

Note that another way of dealing with the direction $\langle\varepsilon, Q\rangle$ is to modulate the scaling parameter to impose $\langle\varepsilon,Q\rangle = 0$ for all time; see §4. Of course, by the invariances of the equation, Theorem 1 implies stability for all the ground state solitary waves. Alternatively, to prove directly stability of the soliton $R(t,x) = Q_\lambda(x-\sigma-\beta t)e^{i\Gamma(t,x)}$, we can use the modified functional
$$W(u) = E(u) + \nu\int |u|^2 - \Im\int (\beta\cdot\nabla u)\bar u, \quad \nu = \lambda^{-2} + \frac{|\beta|^2}{4}.$$
Writing $u = R+w$, we can also use a functional in w
$$W(w) = \int |\nabla w|^2 - (F(R+w) - F(R) - dF(R)(w)) + \nu\int |w|^2 - \Im\int(\beta\cdot\nabla w)\bar w.$$

References. For existence and uniqueness results for the nonlinear elliptic equation (3), see [3, 5, 24, 34]. For the stability issue and the properties of the linearized operator, see [2, 6, 7, 19, 26, 50, 51, 71, 75, 76]. For the proof of a typical decomposition by modulation, see *e.g.* §2 of [46].

2.7. **Global existence versus blow up in the mass critical case.** In the mass critical case $p = 1 + \frac{4}{d}$, it follows from variational arguments (see [74, 5]) that the best constant C in the Gagliardo–Nirenberg inequality

$$\forall v \in H^1(\mathbb{R}^d), \quad \|v\|_{L^{2+\frac{4}{d}}}^{2+\frac{4}{d}} \leq C\|\nabla v\|_{L^2}^2 \|v\|_{L^2}^{\frac{4}{d}} \tag{6}$$

is related to Q as follows $C = \frac{p+1}{2}\|Q\|_{L^2}^{-\frac{4}{d}}$. As a consequence, one has

$$\forall u \in H^1(\mathbb{R}^d), \quad E(u) \geq \frac{1}{2}\|\nabla u\|_{L^2}^2 \left(1 - \frac{\|u\|_{L^2}^{\frac{4}{d}}}{\|Q\|_{L^2}^{\frac{4}{d}}}\right).$$

Together with the conservation of mass and energy and the blow up criterion, this implies the global existence of any solution with initial data $\|u_0\|_{L^2} < \|Q\|_{L^2}$. Actually it is also known that in this case, the solution *scatters*, i.e. behaves asymptotically in large time as a solution of the linear equation, see [30, 18] and references therein.

We also know that $\|u\|_{L^2} = \|Q\|_{L^2}$ corresponds to the mass threshold for global existence. Indeed, the pseudo–conformal symmetry of the mass critical NLS equation

$$v(t,x) = \frac{1}{|t|^{\frac{d}{2}}} u\left(\frac{1}{|t|}, \frac{x}{|t|}\right) e^{-i\frac{|x|^2}{4|t|}} \tag{7}$$

applied to the solitary wave solution $u(t,x) = e^{it}Q(x)$ yields the existence of an explicit single bubble blow up solution $S(t)$ with minimal mass ($T^\star$ is arbitrary)

$$S(t,x) = \frac{e^{\frac{i}{|T^\star - t|}}}{|T^\star - t|^{\frac{d}{2}}} Q\left(\frac{x}{|T^\star - t|}\right) e^{-i\frac{|x|^2}{4|T^\star - t|}}, \quad \|S(t)\|_{L^2} = \|Q\|_{L^2}, \quad \|\nabla S(t)\|_{L^2} \underset{t \sim T^\star}{\sim} \frac{1}{|T^\star - t|}.$$

We refer to [5] for more properties of the pseudo-conformal transform. From [49], this is the only minimal mass blow up solution up to the invariances of the equation.

Recall also the following well-known criterion for finite time blow up: for initial data $u_0 \in \Sigma$, the virial identity

$$\frac{d^2}{dt^2}\int |x|^2 |u(t,x)|^2 dx = 16E(u_0) \tag{8}$$

implies blow up in finite time provided $E(u_0) < 0$. This observation is enough to prove Theorem 1 in the critical case, since the functions $u_{0,n} = (1 + \frac{1}{n})Q$ for $n \geq 1$ have negative energy (recall that $E(Q) = 0$).

The case of H^1 initial data with mass slightly above the threshold, i.e.

$$\|Q\|_{L^2} < \|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha_0, \quad 0 < \alpha_0 \ll 1, \tag{9}$$

has known a lot of progress the last fifteen years. We first recall in this context that a large class of (instable) finite time blow up solutions was constructed in [4, 33, 58] by weak (but subtle) perturbation of the minimal mass solution S. In particular, these solutions blow up with the pseudo–conformal blow up rate

$$\|\nabla u(t)\|_{L^2} \underset{t \sim T^\star}{\sim} \frac{1}{T^\star - t},$$

which is expected to correspond in some sense to a threshold case between the log–log blow up (see below) and the scattering case.

Second, recall that the series of works [66, 50, 51, 69, 52, 53] provides a thorough study of the *stable* blow up dynamics under condition (9), corresponding to the *log–log* blow up regime

$$\|\nabla u(t)\|_{L^2} \underset{t\sim T^\star}{\sim} c^\star \sqrt{\frac{\log|\log(T^\star - t)|}{T^\star - t}}.$$

Finally, recall that a universal gap on the possible blow up speeds was proved in [69]: given a finite time blow up solution satisfying (9), either it blows up in the log–log regime, or it blows up faster than the pseudo–conformal rate, i.e. $\|\nabla u(t)\|_{L^2} \gtrsim \frac{1}{T^\star - t}$. However, the existence of solutions with blow up rate different than the conformal speed and the log-log rate is an open problem, which is equivalent, by the pseudo–conformal symmetry (7), to the existence of global solutions blowing up in infinite time.

For more information on blow up for the mass critical case, we refer to [5, 70] and to references therein.

2.8. **Finite time blow up for the super-critical case.** In this case, the virial identity also proves blow up for negative energy solutions in Σ. However, since the energy of Q is positive, this is not enough to prove the strong instability result stated in Theorem 1. We refer to [2, 5]. Few other general results on blow up for the super-critical case exist, the most notable ones concern (i) the blow up of the critical norm for radial solutions [55], (ii) the construction of a class of solutions blowing up with the self-similar blow up rate [56].

3. Construction of multi-solitary waves with weak interactions

3.1. **General existence result.** We present a general result of existence of solutions of (1) behaving in large time exactly as a sum of $K \geq 2$ arbitrarily given solitary waves with different speeds. This is a typical case of weak interaction. The result holds for any nonlinearity, up to technical changes in the proof in the critical and the super-critical cases due to different properties of the linearized operator. In particular, for the super-critical case, an additional topological argument is needed to control one exponential instability direction for each wave.

Theorem 2 (Existence of multi-solitary waves)**.** *Let $1 < p < \frac{d+2}{d-2}$. Let $K \geq 1$ and for any $k \in \{1, \ldots, K\}$, let $\lambda_k > 0$, $\beta_k \in \mathbb{R}^d$, $\sigma_k \in \mathbb{R}^d$ and $\gamma_k \in \mathbb{R}$. Assume that, for any $k \neq k'$, $\beta_k \neq \beta_{k'}$. Let $\mathbf{R} = \sum R_k$ where*

$$R_k(t,x) = Q_{\lambda_k}(x - \sigma_k - \beta_k t)e^{i\Gamma_k(t,x)}, \quad \Gamma_k(t,x) = \frac{1}{2}(\beta_k \cdot x) - \frac{1}{4}|\beta_k|^2 t + \lambda_k^{-2} t + \gamma_k. \tag{10}$$

Then there exist $\gamma > 0$ and a solution u of (1) *such that, for all $t \geq 0$, $\|u(t) - \mathbf{R}(t)\|_{H^1} \lesssim e^{-\gamma t}$.*

Such solutions for the nonlinear Schrödinger equations correspond to an exceptional behavior. Indeed, the solution $u(t)$ as constructed in Theorem 1 is a non-dispersive solution and by strong H^1 convergence

$$\int u^2(t) = \sum_{k=1}^{K} \int R_k^2(t) \quad \text{and} \quad E(u(t)) = \sum_{k=1}^{K} E(R_k(t)),$$

which means that all the mass and all the energy of the solution is due to the solitary waves.

Solutions such as in Theorem 2 were known to exist for the integrable case (2). See [77, 20] for a derivation of their explicit expression. Moreover, these solutions have very special properties: they describe globally in time the perfectly elastic interaction between several solitary waves. In the non-integrable cases, the only known property of the solution $u(t)$

constructed in Theorem 2 concerns $t \to +\infty$, and what happens in general to the K solitary waves backwards in time is not clear. See §6.4 for the collision problem.

References. For the sub-critical case, see [37]. For the critical case $p = 1 + \frac{4}{d}$, the result of Theorem 2 was first proved by Merle [48]. It was obtained as a consequence of a blow up result and the conformal invariance (7). For the super-critical case, we refer to [14] and we recall that for a given set of parameters, there exists several multi-solitary wave solutions due to exponential instability directions, see [9] for the 1D case.

It is expected that a method by fixed point with weighted norms in time could give the same existence result, see [67, 35, 27]. The proof by compactness is rather flexible and allows to treat the case of power-like interactions (see references in next sections and [32]).

3.2. **Sketch of the proof in the sub-critical case.** First, note that the function $\mathbf{R}$ satisfies $i\dot{\mathbf{R}} + \Delta\mathbf{R} + f(\mathbf{R}) = \Psi_{\mathbf{R}}$ where the error term $\Psi_{\mathbf{R}} = f(\mathbf{R}) - \sum f(R_k)$ is such that $\|\Psi_{\mathbf{R}}\|_{H^1} \lesssim e^{-\gamma t}$ for some $\gamma > 0$. We consider a sequence $t_n \to +\infty$ and for any n, we define the global solution u_n of (1) with $u(t_n) = \mathbf{R}(t_n)$. We claim the following uniform estimate, for some $t_0 > 0$, $C > 0$, $\gamma > 0$, for all n and $t \in [t_0, t_n]$,

$$\|u_n(t) - \mathbf{R}(t)\|_{H^1} \leq Ce^{-\gamma t}. \tag{11}$$

The proof of (11) is a variant of the stability of a single soliton using extra localization arguments to deal with several solitons. We set $w_n = u_n - \mathbf{R}$, so that w_n satisfies

$$i\dot{w}_n + \Delta w_n + f(w_n + \mathbf{R}) - f(\mathbf{R}) + \Psi_{\mathbf{R}} = 0, \quad w_n(t_n) \equiv 0.$$

Recall $\nu_k = \lambda_k^{-2} + \frac{1}{4}|\beta_k|^2$, and define $\varphi_k = \chi(A^{-1}(x - \sigma_k - \beta_k t))$, where $A \gg 1$ is to be fixed later. Also set $\varphi = 1 - \sum \varphi_k$. Note that the functions φ_k satisfy, for $\gamma > 0$, $k' \neq k$, for t large,

$$|\nabla\varphi_k| + |\nabla\varphi| \lesssim A^{-1}, \quad |R_k|(\varphi_{k'} + \varphi) \lesssim e^{-\gamma t}, \quad |R_k|(1 - \varphi_k) \lesssim e^{-\gamma A}, \quad \varphi_k\varphi_{k'} \equiv 0.$$

We define the following functional

$$\begin{aligned} W_n = \int \left\{|\nabla w_n|^2 + |w_n|^2\varphi - 2(F(\mathbf{R} + w_n) - F(\mathbf{R}) - dF(\mathbf{R})(w_n))\right\} \\ + \sum \nu_k \int |w_n|^2\varphi_k - \sum \Im \int (\beta_k \cdot \nabla w_n)\bar{w}_n\varphi_k \end{aligned}$$

We compute

$$\begin{aligned} \frac{1}{2}\dot{W}_n &= \langle \dot{w}_n, -\Delta w_n + w_n(\varphi + \sum \nu_k\varphi_k) - (f(\mathbf{R} + w_n) - f(\mathbf{R}))\rangle \\ &\quad - \langle \dot{\mathbf{R}}, f(\mathbf{R} + w_n) - f(\mathbf{R}) - df(\mathbf{R})(w_n)\rangle - \sum \langle i\dot{w}_n, (\beta_k \cdot \nabla w_n)\varphi_k + \frac{1}{2}(\beta_k \cdot \nabla\varphi_k)w_n\rangle \\ &\quad + \int |w_n|^2\dot{\varphi} + \sum \nu_k \int |w_n|^2\dot{\varphi}_k - \sum \Im \int (\beta_k \cdot \nabla w_n)\bar{w}_n\dot{\varphi}_k, \end{aligned}$$

and next using the equation of w_n and the explicit expression $\dot R_k = -(\beta_k\cdot\nabla R_k) + i\nu_k R_k$, we decompose $\frac12 \dot W_n = K_1+K_2+K_3+K_4$, where

$$\begin{aligned}
K_1 &= \langle i\nabla\Psi_{\mathbf R}, \nabla w_n\rangle + \sum\langle i\Psi_{\mathbf R}, w_n\varphi\rangle + \sum\langle i\Psi_{\mathbf R}, w_n\nu_k\varphi_k\rangle - \langle i\Psi_{\mathbf R}, f(\mathbf R + w_n) - f(\mathbf R)\rangle\\
&\quad + \sum\langle \Psi_{\mathbf R}, (\beta_k\cdot\nabla w_n)\varphi_k + \frac12(\beta_k\cdot\nabla\varphi_k)w_n\rangle,\\
K_2 &= -\sum(\nu_k-1)\langle i(\nabla w_n\cdot\nabla\varphi_k), w_n\rangle + \frac12\sum\langle f(\mathbf R + w_n) - f(\mathbf R), (\beta_k\cdot\nabla\varphi_k)w_n\rangle\\
&\quad - \frac12\int|\nabla w_n|^2(\beta_k\cdot\nabla\varphi_k) - \frac12\langle\nabla w_n, \nabla((\beta_k\cdot\nabla\varphi_k)w_n)\rangle\\
&\quad - \sum(\nu_k-1)\int|w_n|^2(\beta_k\cdot\nabla\varphi_k) + \sum\Im\int(\beta_k\cdot\nabla w_n)\bar w_n(\beta_k\cdot\nabla\varphi_k),\\
K_3 &= \sum\langle(\beta_k\cdot\nabla R_k), f(\mathbf R+w_n) - f(\mathbf R) - df(\mathbf R)(w_n)\rangle + \langle(\beta_k\cdot\nabla w_n)\varphi_k, f(\mathbf R + w_n) - f(\mathbf R)\rangle,\\
K_4 &= -\sum\nu_k\left\{\langle iw_n\varphi_k, f(\mathbf R+w_n)-f(\mathbf R)\rangle + \langle iR_k, f(\mathbf R+w_n) - f(\mathbf R) - df(\mathbf R)(w_n)\rangle\right\}.
\end{aligned}$$

By the properties of $\Psi_{\mathbf R}$, φ and φ_k, we have $|K_1|\lesssim e^{-\gamma t}\|w_n\|_{H^1}$ and $|K_2|\lesssim A^{-1}\|w_n\|_{H^1}^2$. Note also that by integration by parts,

$$\begin{aligned}
K_3 = -\sum\langle(\beta_k\cdot\nabla\varphi_k), F(\mathbf R + w_n) - F(\mathbf R) - dF(\mathbf R)(w_n)\rangle&\\
+ \sum\langle(\beta_k\cdot\nabla R_k)(1-\varphi_k), f(\mathbf R + w_n) - f(\mathbf R) - df(\mathbf R)(w_n)\rangle&,
\end{aligned}$$

and thus $|K_3|\lesssim A^{-1}\|w_n\|_{H^1}^2$. Last, since for any k,

$$\langle iw_n, f(R_k+w_n) - f(R_k) - f(w_n)\rangle + \langle iR_k, f(R_k+w_n) - f(R_k) - df(R_k)(w_n)\rangle = 0,$$

by the properties of φ_k, we obtain similarly $|K_4|\lesssim(e^{-\gamma t}+e^{-\gamma A})\|w_n\|_{H^1}^2$. Therefore, we have obtained the uniform estimate $|\dot W_n|\lesssim e^{-\gamma t}\|w_n\|_{H^1} + A^{-1}\|w_n\|_{H^1}^2$.

Since the solitons R_k are decoupled, one can check from (5) that for A large enough, the following coercivity property holds

$$W_n \geq \mu\|w_n\|_{H^1}^2 - \frac1\mu\sum\left(\langle\eta_k, Q\rangle^2 + |\langle\eta_k, xQ\rangle|^2 + \langle\eta_k, i\Lambda Q\rangle^2\right),$$

where we have defined η_k by

$$w_n(t,x) = \lambda_k^{-\frac{2}{p-1}}\eta_k\left(t, \lambda_k^{-1}(x-\sigma_k t-\beta_k t)\right)e^{i\Gamma_k(t,x)}.$$

From the equation of w_n, we check that $\eta_k = a_k + ib_k$ satisfies

$$\frac{\dot a_k}{\lambda_k^2} = L_- b_k + \Psi_{a_k}, \quad \frac{\dot b_k}{\lambda_k^2} = -L_+ a_k + \Psi_{b_k},$$

where the error terms satisfy the following weighted estimates

$$\|\Psi_{a_k}e^{-\frac{|x|}{10}}\|_{L^1} + \|\Psi_{b_k}e^{-\frac{|x|}{10}}\|_{L^1} \lesssim \|w_n\|_{L^2}^2 + e^{-\gamma t}.$$

Using the special relations (4), we thus find

$$\begin{aligned}
&\langle\dot\eta_k, Q\rangle = O(\|w_n\|_{L^2}^2 + e^{-\gamma t}), \quad \langle\dot\eta_k, xQ\rangle = -2\langle\eta_k, i\nabla Q\rangle + O(\|w_n\|_{L^2}^2 + e^{-\gamma t}),\\
&\langle\dot\eta_k, i\nabla Q\rangle = O(\|w_n\|_{L^2}^2 + e^{-\gamma t}), \quad \langle\dot\eta_k, i\Lambda Q\rangle = -2\langle\eta_k, Q\rangle + O(\|w_n\|_{L^2}^2 + e^{-\gamma t}).
\end{aligned}$$

By standard ODE arguments, taking A large enough, the previous estimates are enough to prove the uniform exponential estimate (11) for some $C, \gamma > 0$ (note that $W_n(t_n) = 0$ since $w_n(t_n) \equiv 0$).

Once (11) is proved, we define $u_0 \in H^1$ as the weak limit of a subsequence of the bounded H^1 sequence $(u_n(t_0))$. Then, we define the solution $u(t)$ of (1) corresponding to the data $u(t_0) = u_0$. Using an additional simple localization argument (see Lemma 2 of [37]), we also obtain the strong compactness of $(u_n(t_0))$ in H^s, for any $0 \le s < 1$. In particular, by the Cauchy theory in H^s for $0 \le s < 1$ (see [5]), we obtain that $u(t)$ exists on $[t_0, +\infty)$ and, for any $t \ge t_0$, $u_n(t) \to u(t)$ strongly in H^s and weakly in H^1 (up a subsequence). Finally, passing to the weak limit in (11), we see that the solution $u(t)$ satisfies $\|u(t) - \mathbf{R}(t)\|_{H^1} \lesssim e^{-\gamma t}$ for all $t \ge t_0$.

4. Multi-solitons with strong interactions and logarithmic distances

This section, devoted to sub-critical and super-critical case, and the next one, devoted to the critical case, deal with strong interactions. Indeed, we look for multi-solitary wave solutions in situations where the distance between the waves are of lower order, typically, logarithmic in t. Then, the long time dynamics of the solitons is substantially changed by the interactions. Such situations seem to be related to symmetry properties.

4.1. General existence result.

Theorem 3 (Two-solitary waves with logarithmic distance)**.** *Let $1 < p < \frac{d+2}{d-2}$, $p \neq 1 + \frac{4}{d}$. There exists a solution $u(t)$ of (1) such that,*

$$\text{for all } t \ge 0, \qquad \left\| u(t) - e^{-i\gamma(t)} \sum_{k=1}^{2} Q(\cdot - z_k(t)) \right\|_{H^1} \lesssim t^{-1}, \tag{12}$$

where the translation parameters satisfy $|z_1(t) - z_2(t)| = 2(1 + o(1)) \log t$ as $t \to +\infty$.

In the integrable case (2), the existence and the properties of a *double pole* solution, corresponding to the regime of Theorem 3, is studied in [77, 65]. Note that it is a very special solution, with a similar global behavior as $t \to -\infty$. In the general non-integrable case, as in Theorem 2, the behavior of the solution for $t < 0$ is not known.

It is important in the proof that the two solitons have same scaling and phase. A similar statement holds for $K \ge 2$ solitary waves with symmetry properties (see the statement of Theorem 5 below). Note finally that such a solution cannot exist for the critical case $p = 1 + \frac{4}{d}$, see next Section.

4.2. Sketch of the proof for $d = 2$ and $1 < p < 3$.

The main ingredient of the proof is the construction of an approximate solution made of solitons whose geometrical parameters (notably, the speeds and positions of the solitons) satisfy an appropriate system of ODE up to error terms. Once this is done, the construction of an actual solution of (1) follows the same lines as in the proof of Theorem 2, by a localized energy method and compactness. We omit this last part and we focus on the determination of the leading order dynamics of the geometrical parameters of the solitons.

Let $(\lambda, z, \gamma, \beta)$, $\lambda > 0$ be free parameters such that $|\beta| + \lambda \ll 1$ and $z \gg 1$. Renormalize the flow (scaling and phase)

$$u(t,x) = \frac{e^{i\gamma(s)}}{\lambda^{\frac{2}{p-1}}(s)} w(s,y), \quad dt = \lambda^2(s)ds, \quad x = \lambda(s)y$$

We look for a symmetric 2-bubble approximate solution $\mathbf{P}$ of the rescaled equation

$$i\dot{w} + \Delta w - w + |w|^{p-1}w - i\frac{\dot{\lambda}}{\lambda}\Lambda w + (1-\dot{\gamma})w = 0.$$

For $z_1 = \frac{z}{2}\mathbf{e}_1$, $z_2 = -\frac{z}{2}\mathbf{e}_1$, $\beta_1 = \frac{\beta}{2}\mathbf{e}_1$, $\beta_2 = -\frac{\beta}{2}\mathbf{e}_1$ (here $\mathbf{e}_1$ is the first vector of the canonical basis of $\mathbb{R}^d$), let

$$\mathbf{P} = \sum_{k=1}^{2} P_k, \qquad P_k(s,y) = e^{i\beta_k(s)\cdot(y-z_k(s))}Q(y - z_k(s)).$$

Let M_k be defined by

$$i\dot{P}_k + \Delta P_k - P_k + |P_k|^{p-1}P_k - i\frac{\dot{\lambda}}{\lambda}\Lambda P_k + (1-\dot{\gamma})P_k = -e^{i\beta_k\cdot(y-z_k)}M_k(y - z_k),$$

Then, by direct computations,

$$\begin{aligned} M_k = i\frac{\dot{\lambda}}{\lambda}\Lambda Q + i(\dot{z}_k - 2\beta_k + \frac{\dot{\lambda}}{\lambda}z_k)\cdot\nabla Q \\ + (\dot{\gamma} - 1 + |\beta_k|^2 - \frac{\dot{\lambda}}{\lambda}(\beta_k\cdot z_k) - (\beta_k\cdot\dot{z}_k))Q + (\dot{\beta}_k - \frac{\dot{\lambda}}{\lambda}\beta_k)\cdot yQ. \end{aligned}$$

(Observe that if the parameters satisfy $\dot{\lambda} = 0$, $\dot{z}_k = 2\beta_k$, $\dot{\gamma} = 1 - \frac{1}{4}|\beta|^2$, $\dot{\beta}_k = 0$, then $\Psi_k = 0$.) Thus $\mathbf{P}$ satisfies

$$i\dot{\mathbf{P}} + \Delta\mathbf{P} - \mathbf{P} + |\mathbf{P}|^{p-1}\mathbf{P} - i\frac{\dot{\lambda}}{\lambda}\Lambda\mathbf{P} + (1-\dot{\gamma})\mathbf{P} = \Psi_{\mathbf{P}} - \sum e^{i\beta_k\cdot(y-z_k)}M_k(y - z_k),$$

where we have defined the nonlinear interaction term $\Psi_{\mathbf{P}} = f\left(\sum P_k\right) - \sum f(P_k)$. Now, we look for a solution u of (1) of the form

$$u(t,x) = \frac{e^{i\gamma(s)}}{\lambda^{\frac{2}{p-1}}(s)}(\mathbf{P} + \varepsilon)(s,y), \quad dt = \lambda^2(s)ds, \quad x = \lambda(s)y$$

where η_k defined by $\varepsilon(s,y) = e^{i\beta_k(s)\cdot(y-z_k(s))}\eta_k(s, y - z_k(s))$ satisfies the orthogonality relations $\langle\eta_k(s), Q\rangle = |\langle\eta_k(s), yQ\rangle| = \langle\eta_k(s), i\Lambda Q\rangle = |\langle\eta_k(s), i\nabla Q\rangle| = 0$. Note that the coercivity property (5) requires only 3 directions but there are 4 geometrical parameters and thus 4 orthogonality directions to impose for ε. Then, ε satisfies the equation

$$i\dot{\varepsilon} + \Delta\varepsilon + f(\mathbf{P} + \varepsilon) - f(\mathbf{P}) + \Psi_{\mathbf{P}} - \sum e^{i\beta_k\cdot(y-z_k)}M_k(y - z_k) = 0.$$

Therefore, it remains to find the equations for the parameters $(\lambda, z, \gamma, \beta)$ by projecting the above equation on the special directions of orthogonality for ε. By symmetry, it is enough to project on the soliton P_1. We change the equation by setting $\varepsilon(s,y) = e^{i\beta_1\cdot(y-z_1)}\eta(s, y - z_1)$, $\theta Q = e^{-i\beta(y_1+z)}Q(y + z\mathbf{e}_1)$ and $\Psi_1 = f(Q + \theta Q) - f(Q) - f(\theta Q)$, so that

$$i\dot{\eta} + \Delta\eta + f(Q + \theta Q + \eta) - f(Q + \theta Q) = \Psi_1 - M_1 - \theta M_2.$$

We claim that this yields, at the main order, the following differential inequalities

$$\left|\frac{\dot\lambda}{\lambda}\right| + \left|\dot z - 2\beta + \frac{\dot\lambda}{\lambda} z\right| \lesssim z^{-\frac12} e^{-z}, \quad \left|\dot\beta - \frac{\dot\lambda}{\lambda}\beta + c\, z^{-\frac12} e^{-z}\right| \lesssim z^{-\frac32} e^{-z}.$$

We justify the form of the source term $c\, z^{-\frac12} e^{-z}$ in the equation for $\dot\beta$, which is decisive in the derivation of the desired regime. Note that similar terms appear in the equations of $\dot\lambda$ and $\dot z$, but they are treated as error terms, which can be justified rigorously for the specific regime that we are interested in. Recall that we choose $d = 2$. Anticipating a regime where $z \gg 1$ and $\beta z \ll 1$, using $Q(x) \sim c_Q |x|^{-\frac12} e^{-|x|}$ as $|x| \to +\infty$, for some constant $c_Q > 0$, we observe that for large z,

$$|y + z\mathbf{e}_1| = z + y_1 + O(z^{-1}) \quad \text{and thus} \quad \theta Q \sim Q(y + z\mathbf{e}_1) \sim c_Q |z|^{-\frac12} e^{-y_1 - z}.$$

Therefore,

$$\nabla Q \Psi_1 \sim \nabla Q f'(Q)(\theta Q) \sim p c_Q z^{-\frac12} e^{-z} e^{-y_1} \nabla Q(y) Q^{p-1}(y).$$

Integrating by parts, we have $p \int \nabla Q Q^{p-1} e^{-y_1} dy = \int Q^p e^{-y_1} dy > 0$, $\int y_1 (\mathbf{e}_1 \cdot \nabla Q) Q = -\frac12 \int Q^2$, and thus by taking the scalar product of the equation of η by ∇Q, neglecting all terms in ε (second order in ε) and in M_2 (decoupling), we obtain

$$\langle \Psi_1 - M_1, \nabla Q\rangle \sim 0 \quad \text{and so} \quad \frac12 (\dot\beta - \frac{\dot\lambda}{\lambda}\beta) \int Q^2 + z^{-\frac12} e^{-z} c_Q \int Q^p e^{-y_1} \sim 0.$$

Thus, $c = 2c_Q (\int Q^p e^{-y_1})(\int Q^2)^{-1} > 0$. An (unstable) formal solution exists with the following asymptotics for s large

$$\lambda(s) \sim 1, \quad \beta(s) \sim s^{-1}, \quad z(s) \sim 2\log(s), \quad z^{\frac12} e^z \sim c\, s^2.$$

References. This result is proved in [63]. Technically, note from [63] that the case $1 < p \le 2$ requires additional computations, and $p > 1 + \frac4d$ requires a specific control of the instability directions as in Theorem 2.

5. The L^2 critical case

In this section, we focus on the 2D mass critical case

$$i\partial_t u + \Delta u + |u|^2 u = 0, \quad t \in \mathbb{R},\ x \in \mathbb{R}^2, \tag{13}$$

but similar results hold in the critical case for other dimensions. We start by recalling the following result from [48].

Theorem 4. *Let $\{z_1, \dots, z_K\}$ be K given points of $\mathbb{R}^2$. There exists a solution v of (13) that blows up at $T^\star = 0$ at the points $\{z_1, \dots, z_K\}$. Moreover,*

$$\|\nabla v(t)\|_{L^2} = K^{\frac12} \|\nabla Q\|_{L^2} (1 + o(1)) \frac{1}{|t|}, \quad |v|^2 \rightharpoonup \|Q\|_{L^2}^2 \sum_k \delta_{z_k} \quad \text{as} \quad t \uparrow 0. \tag{14}$$

This result is proved directly in [48] by using a compactness argument which later inspired the strategy used in [36, 37].

By the pseudo conformal law (7), the soliton $e^{it} Q(x)$ transforms into the blow up solution $S(t)$ and similarly the above result corresponds to the existence of multi-solitons as given by Theorem 2 for the mass critical case. This was duly observed in [48].

For multi-bubbling with the log-log rate, see a construction in [21]; see also [68] for the related question of blow up in a bounded domain.

5.1. **Existence result for strong interactions.** We first note that for the mass critical NLS, the existence of bounded multi-solitons with logarithmic relative distances as in Theorem 3 is ruled out by the virial law (8), which would formally give $\int |x|^2 |u^2(t,x)| dx \sim C(\log t)^2$ for such a solution. It turns out that the scaling instability direction of the critical case is excited by the interactions which leads to infinite time concentration of solitons at fixed distance, as shown by the following theorem.

Theorem 5 (Infinite time blow up)**.** *For any $K \geq 1$, there exists a solution u of* (13) *on* $[0,+\infty)$ *such that*

$$\left\| u(t) - e^{i\gamma(t)} \sum_{k=1}^{K} \frac{1}{\lambda(t)} Q\left(\frac{\cdot - z_k(t)}{\lambda(t)}\right) \right\|_{H^1} \to 0, \quad \lambda(t) = \frac{1+o(1)}{\log t} \quad \textit{as} \quad t \to +\infty, \tag{15}$$

where the parameters $z_k(t)$ converge as $t \to +\infty$ to the vertices of a K-sided regular polygon. In particular,

$$\|\nabla u(t)\|_{L^2} = K^{\frac{1}{2}} \|\nabla Q\|_{L^2}(1+o(1)) \log t \quad \textit{as} \quad t \to +\infty. \tag{16}$$

5.2. **Pseudo-conformal counter part.** Using the pseudo conformal transform (7) on the solution u constructed in Theorem 5 (note that this solution belongs to the space Σ, which implies that the corresponding v is also a solution of (13) in Σ), one obtains the following corresponding finite time blow result.

Corollary 6 (Finite time collision)**.** *There exists a solution v of* (13) *that blows up at $T^\star = 0$ with*

$$\|\nabla v(t)\|_{L^2} = K^{\frac{1}{2}} \|\nabla Q\|_{L^2}(1+o(1)) \left|\frac{\log |t|}{t}\right|, \quad |v|^2 \rightharpoonup K\|Q\|_{L^2}^2 \delta_0 \quad \textit{as} \quad t \uparrow 0. \tag{17}$$

5.3. **Comments.** As before, the proof of Theorem 5 follows the strategy of constructing *minimal* dynamics by approximate solutions and compactness, initiated in [48] and extended in various ways and contexts in [36, 32, 14, 71]. We combine in a focusing context the approach developed for multi-solitary wave problems in [36, 37, 32] and a specific strategy to construct minimal blow up solutions for NLS type equations developed in [71, 31].

As in Theorem 5, a key ingredient of the proof is the precise tuning of the interactions between the waves. In particular, we observe that the K bubbles in (15) need to have the same phase and scaling. Note in contrast that the dynamics of two symmetric bubbles with *opposite phase* ($\gamma_1 = \gamma_2 + \pi$) is related to the dynamics of a single bubble on a half–plane with Dirichlet boundary condition and it is known in this context that minimal mass blow up at a boundary point (which corresponds to the collision case) does not exists, see [1].

It is also interesting to note that solutions as constructed in Theorem 5 are specific to the critical case and cannot exist in the sub and super-critical cases. Indeed, in the sub-critical case, we have seen that all H^1 solutions are bounded. In the super-critical case, any solution in Σ that is global for $t \geq 0$ satisfies

$$\liminf_{t\to+\infty} \|\nabla u(t)\|_{L^2} \lesssim 1.$$

Indeed in this case, the Virial identity $\frac{d^2}{dt^2}\int |x|^2|u|^2 = c_1 E(u_0) - c_2 \int |\nabla u|^2$, where $c_1, c_2 > 0$, integrated twice in time provides the global bound $\int_0^t \int_0^s \|\nabla u(s')\|_{L^2}^2 ds' ds \lesssim t^2$.

As for the previous constructions, an interesting issue would be to understand the global behaviour of such solutions for $t \leq 0$.

5.4. **Sketch of proof.** As for Theorem 3, we focus on the determination of the dynamics of the parameters. In the critical case, two more parameters related to scaling instability are needed; they are denoted by a and b. See [57] and references therein for previous use of such a parametrization. In this proof, for $k \in \{1, \dots, K\}$, $\mathbf{e}_k$ denotes the unit vector of $\mathbb{R}^2$ corresponding to the complex numbers $e^{i\frac{2\pi(k-1)}{K}}$.

Let $(\lambda, z, \gamma, \beta, b, a)$, be free parameters such that $\lambda > 0$, $|a| + |b| + |\beta| + |\lambda| \ll 1$ and $z \gg 1$. Set $\beta_k = \beta \mathbf{e}_k$ and $z_k = z\mathbf{e}_k$. Renormalize the flow by scaling and phase

$$u(t,x) = \frac{e^{i\gamma(s)}}{\lambda(s)} w(s,y), \quad dt = \lambda^2(s)ds, \quad x = \lambda(s)y.$$

We look for a symmetric K-bubble approximate solution $\mathbf{P}$ of the rescaled equation

$$i\dot{w} + \Delta w - w + |w|^2 w - i\frac{\dot{\lambda}}{\lambda}\Lambda w + (1-\dot{\gamma})w = 0.$$

Let

$$\mathbf{P} = \sum_{k=1}^{K} P_k \quad \text{where} \quad P_k(s,y) = e^{i\left((\beta_k(s)\cdot y) - \frac{b(s)}{4}|y|^2\right)} Q_{a(s)}(y - z_k(s))$$

and $Q_a = Q + a\rho$. Note that by the definition of ρ in §2.4, the function Q_a satisfies the following equation $\Delta Q_a - Q_a + |Q_a|^2 Q_a = -a\frac{|y|^2}{4} Q_a + O(a^2)$ (see Chapter 8 of [72]; see also [57] where an exact solution Q_a of this equation is construted for $a < 0$). Let M_k be defined by

$$i\dot{P}_k + \Delta P_k - P_k + |P_k|^2 P_k - i\frac{\dot{\lambda}}{\lambda}\Lambda P_k + (1-\dot{\gamma})P_k = -e^{i\left(\beta(\mathbf{e}_k\cdot y) - \frac{b}{4}|y|^2\right)} M_k(y - ze_k)$$

Then, by direct computations and ignoring terms of order a^2, we find

$$\begin{aligned} M_k &= i(b + \frac{\dot{\lambda}}{\lambda})\Lambda Q_a + i(\dot{z}_k - 2\beta_k + \frac{\dot{\lambda}}{\lambda}z_k)\cdot\nabla Q_a + (\dot{\gamma} - 1 + |\beta_k|^2 - \frac{\dot{\lambda}}{\lambda}(\beta_k\cdot z_k) + (\beta_k\cdot\dot{z}_k))Q_a \\ &\quad + (\dot{\beta}_k - \frac{\dot{\lambda}}{\lambda}\beta_k + \frac{b}{2}(\dot{z}_k - 2\beta_k + \frac{\dot{\lambda}}{\lambda}z_k))\cdot yQ_a + (\dot{b} + b^2 - 2b(b + \frac{\dot{\lambda}}{\lambda}) - a)\frac{|y|^2}{4}Q_a - i\dot{a}\rho. \end{aligned}$$

Looking for a solution u of (13) of the form

$$u(t,x) = \frac{e^{i\gamma(s)}}{\lambda(s)}(\mathbf{P} + \varepsilon)(s,y), \quad dt = \lambda^2(s)ds, \quad x = \lambda(s)y$$

where η_k defined by $\varepsilon(s,y) = e^{i\beta_k(s)\cdot(y - z_k(s))}\eta_k(s, y - z_k(s))$ satisfies the orthogonality relations $\langle\eta_k(s), Q\rangle = \langle\eta_k(s), |y|^2 Q\rangle = \langle\eta_k(s), i\rho\rangle = |\langle\eta_k(s), yQ\rangle| = |\langle\eta_k(s), i\nabla Q\rangle| = \langle\eta_k(s), i\Lambda Q\rangle = 0$. Then, ε satisfies the equation

$$i\dot{\varepsilon} + \Delta\varepsilon + f(\mathbf{P} + \varepsilon) - f(\mathbf{P}) + \Psi_{\mathbf{P}} - e^{i\left(\beta(\mathbf{e}_k\cdot y) - \frac{b}{4}|y|^2\right)} M_k(y - ze_k) = 0.$$

The contribution of the nonlinear interaction term $\Psi_{\mathbf{P}} = f(\sum P_k) - \sum f(P_k)$ yields the following system at the main order

$$\left|\frac{\dot{\lambda}}{\lambda} + b\right| + |\dot{z} - 2\beta - bz| + \left|\dot{\beta} + b\beta\right| + \left|\dot{b} + b^2 - a\right| \lesssim z^{-\frac{1}{2}} e^{-\kappa z}$$

and, for some constant $c > 0$,

$$\left|\dot{a} + c\, b z^{\frac{3}{2}} e^{-\kappa z}\right| \lesssim |b| z^{\frac{1}{2}} e^{-\kappa z}.$$

Perturbations due to the nonlinear interaction between the waves also appear for the other parameters, but the strongest effect is noted on the parameter a. An approximate solution can be found with the following asymptotics

$$z(s) \sim \frac{2}{\kappa}\log(s), \quad z^{-\frac{3}{2}}(s)e^{\kappa z(s)} \sim cs^2, \quad \lambda(s) \sim \log^{-1}(s), \quad |\beta(s)| \lesssim s^{-1}\log^{-\frac{3}{2}}(s),$$

$$b(s) \sim s^{-1}\log^{-1}(s), \quad a(s) \sim -s^{-2}\log^{-1}(s)$$

Going back to the original variables using $s \sim t\log^2(t)$, one justifies the regime of Theorem 5.

References. Theorem 5 is proved in [47].

6. Discussion

6.1. **The generalized Korteweg-de Vries equation.** A result of existence of multi-solitary waves similar to Theorem 2 was previously proved in [36] for the sub-critical and critical generalized KdV equations (using tools from [45]), and in [14] for the super-critical case. Note that for the generalized KdV equations, it is also proved in [36] that being given the parameters of K solitons, the solution converging to the sum of these K solitons in H^1 is unique. In the super-critical case, there is no uniqueness due to the exponential instability of the solitons, and a complete classification of multi-solitary waves was obtained in [8]. Note finally in the case of the gKdV equations that the stability and asymptotic stability of multi-soliton solutions follows from [46] in the sub-critical case. The analog of Theorem 3 for gKdV in the sub and super-critical cases is proved in [64]. Another point of view on the (repulsive) interaction of solitons of same sign for the quartic gKdV equation is given in [61]. For the water-wave system close to KdV, see [60].

The question of multi-bubble blow up for the mass critical gKdV was addressed recently in [10, 11], following works for single bubble blow up [42, 43, 44] (see also references therein).

6.2. **Stability and uniqueness issues for NLS.** Several stability and asymptotic stability results for the sum of solitary waves of some nonlinear Schrödinger equations are known, but none for the pure power nonlinearity as in (1), for any $p > 1$ (except in the integrable case, see [12, 17]). Moreover, the general uniqueness problem in Theorem 2 is open, unlike for the gKdV equations.

6.3. **Wave type equations.** The strategy of proof of existence of multi-solitary waves from [36, 37] has already been extended to several other models. For example, see [15, 13] for the case of the nonlinear Klein-Gordon equation. See also [59, 16] for multi-soliton behaviors related to blow up for the wave equation. In [27, 28, 29], solutions containing two bubbles for all positive times are studied for the energy critical wave equation in large dimensions and the wave maps. A transient regime due to strong interaction of solitons is fully described in [23] for the half-wave equation.

6.4. **Collision problem.** Understanding the behavior of multi-solitary waves for $t \to -\infty$ means understanding the very challenging question of the collision of solitons. The problem is completely open apart from the integrable cases, some numerical experiments, and the following few references: for gKdV, see [38, 39, 40]; for NLS, close to the integrable case, see [67]; for the 5D critical wave equation, see [41]. See also references in those works.

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