A decomposition of vector fields in $\mathbb{R}^{d+1}$

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Given a vector field $\rho(1, \mathbf{b}) \in L^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d, \mathbb{R}^{d+1})$ such that $\mathrm{div}_{t,x}(\rho(1, \mathbf{b}))$ is a measure, we consider the problem of uniqueness of the representation η of $\rho(1, \mathbf{b})\mathcal{L}^{d+1}$ as a superposition of characteristics $\gamma : (t^-_\gamma, t^+_\gamma) \to \mathbb{R}^d$, $\dot\gamma(t) = \underline{(}t, \gamma(t))$. We give conditions in terms of a local structure of the representation η on suitable sets in order to prove that there is a partition of $\mathbb{R}^{d+1}$ into disjoint trajectories $\wp_{\mathfrak{a}}$, $\mathfrak{a} \in \mathfrak{A}$, such that the PDE

$$\mathrm{div}_{t,x}\big(u\rho(1, \mathbf{b})\big) \in \mathcal{M}(\mathbb{R}^{d+1}), \qquad u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d),$$

can be disintegrated into a family of ODEs along $\wp_{\mathfrak{a}}$ with measure r.h.s.. The decomposition $\wp_{\mathfrak{a}}$ is essentially unique. We finally show that $\mathbf{b} \in L^1_t(\mathrm{BV}_x)_{\mathrm{loc}}$ satisfies this local structural assumption and this yields, in particular, the renormalization property for nearly incompressible BV vector fields.