Is an affine spherical variety determined by its automorphism group?

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Group of symmetries of a finite set S determines the set S (in the category of finite sets).

Erlangen Program

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In 1872 Felix Klein in his Erlangen Program proposed that group theory, a branch of mathematics that uses algebraic methods to abstract the idea of symmetry, was the most useful way of organizing geometrical knowledge.

Study geometrical objects via their transformation (diffeomorphisms, isometries, automorphism, etc.) groups.

Erlangen Program

This approach was very fruitful in many areas of mathematics, for example, to study manifolds via their diffeomorphism groups

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Theorem(R. P. Filipkewicz, 1982)

Let M and N be smooth (i.e. C^∞) manifolds without boundary and let $\mathrm{Diff}(M)$ and $\mathrm{Diff}(N)$ denote the groups of C^∞ diffeomorphisms of M and N respectively. If $\phi : \mathrm{Diff}(M) \xrightarrow{\sim} \mathrm{Diff}(N)$ is a group isomorphism then there is a C^∞ diffeomorphism $w : M \xrightarrow{\sim} N$.

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In algebraic geometry there are at least two natural possibilities for the group of symmetries:

Regular automorphism group $\mathrm{Aut}(X)$

Birational automorphism group $\mathrm{Bir}(X)$

$\mathrm{Aut}(X) \subset \mathrm{Bir}(X)$

Theorem(Cantat, Xie)

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$\mathbb{P}^n$ is uniquely determined (up to birational equivalence) among n-dimensional varieties by its group of birational transformations.

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Most toric projective varieties have automorphism group isomorphic to algebraic torus. Hence, projective toric variety is not determined by its automorphism group.

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Definition

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Let G be a reductive, $B \subset G$ a Borel subgroup. An affine normal G-variety X is called spherical if B acts on X with an open orbit.

There exists an affine surface and an isomorphism $\psi : \mathrm{Aut}(S) \xrightarrow{\sim} \mathrm{Aut}(S)$ such that for any algebraic subgroup $H \subset \mathrm{Aut}(S)$, $\psi(H)$ is not an algebraic subgroup of $\mathrm{Aut}(S)$.

Ind-group structure

An ind-variety is a set V together with an ascending filtration $V_0 \subset V_1 \subset V_2 \subset \cdots \subset V$ such that the following conditions are satisfied

- $V = \cup_{k \geq 0} V_k$;
- each V_k has is an algebraic variety;
- for every $k \in \mathbb{Z}_{\geq 0}$, the embedding $V_k \subset V_{k+1}$ is closed in the Zariski topology.

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Morphisms of ind-varieties: $V = \cup_k V_k$ and $W = \cup_m W_m$. A map $\psi : V \to W$ such that for any k there exists $m \in \mathbb{Z}_{\geq 0}$ such that $\psi(V_k) \subset W_m$ and such that the induced map $V_k \to W_m$ is a morphism of algebraic varieties.

Ind-group structure

Definition

An ind-group is a group object in the category of ind-varieties, i.e., an ind-variety endowed with a group structure such that multiplication and inversion maps are morphisms.

A subgroup H of an ind-group V is called an algebraic subgroup if $H \subset V_k$ is a closed subset for some $k \in \mathbb{N}$.

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Theorem(Furter, Kraft)

Let X be an affine variety. Then $\mathrm{Aut}(X)$ has a natural structure of an ind-group such that for any algebraic group G , a regular G-action on X induces an ind-group homomorphism $G \to \mathrm{Aut}(X)$.

Characterization of spherical varieties

Weight Monoid

$\mathcal{O}(X)$ is a multiplicity free G-module, that is, the multiplicity of every irreducible module in $\mathcal{O}(X)$ is at most 1. By the weight monoid $\Lambda^+(X)$ of X we mean the set of all highest weights of the G-module $\mathcal{O}(X)$.

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Theorem(van Santen, R.)

Let X be a spherical affine variety different from algebraic torus and Y be an affine irreducible normal variety. If $\mathrm{Aut}(Y) \simeq \mathrm{Aut}(X)$ as an ind-group, then Y is also spherical and $\Lambda^+(X) = \Lambda^+(Y)$.

Corollary

- if X is toric, then $Y \simeq X$.
- if X and Y are smooth, then $Y \simeq X$.
- in general, for a given X there are finitely many spherical varieties $Y_1, \ldots, Y_l$ such that $\mathrm{Aut}(Y_j) \simeq \mathrm{Aut}(X)$.

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Example (the case of algebraic torus)

Let T be an algebraic torus and let C be a smooth affine curve. If C has trivial automorphism group and no invertible global functions, then $\mathrm{Aut}(T)$ and $\mathrm{Aut}(C \times T)$ are isomorphic as ind-groups.

Definition

Let X be an affine spherical G-variety. A unipotent subgroup $H \subset \mathrm{Aut}(X)$ is called a generalized root subgroup (with respect to B) if H is commutative and every one-dimensional subgroup of H is normalized by B.

the weights of all the one-dimensional subgroups of a generalized root subgroups are the same. This weight, we call the weight of the generalized root group.

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Proposition

Let Y be an irreducible normal affine G-variety. The following statements are equivalent:

- Y is G-spherical;
- there exists a constant C such that $\dim H \leq C$ for each generalized root subgroup $H \subset \mathrm{Aut}(Y)$.

Idea of the proof

Let $\varphi : \mathrm{Aut}(X) \xrightarrow{\sim} \mathrm{Aut}(Y)$ be an isomorphism of ind-groups.

For every algebraic subgroup $K \subset \mathrm{Aut}(X)$, the isomorphism φ restricts to an isomorphism of algebraic groups K and $\varphi(K)$. In particular, a generalized root subgroup of weight λ is maped to generalized root subgroup with the same weight.

The set of weights of generalized root subgroups determines the weight monoid of spherical variety.

Example

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Theorem(Leuenberger, R.)

For generic D_p and D_q there exists an isomorphism $\varphi : \mathrm{Aut}(\mathrm{D}_p) \xrightarrow{\sim} \mathrm{Aut}(\mathrm{D}_q)$ and the restriction of φ to any algebraic subgroup $H \subset \mathrm{Aut}(\mathrm{D}_p)$ is an isomorphism of algebraic groups of H and $\varphi(H)$. But $\mathrm{Aut}(\mathrm{D}_p)$ is not isomorphic to $\mathrm{Aut}(\mathrm{D}_q)$ as an ind-group. Moreover, if Y is an irreducible affine normal variety such that $\mathrm{Aut}(Y) \simeq \mathrm{Aut}(\mathrm{D}_p)$ as an ind-group, then $Y \simeq \mathrm{D}_p$.

What if $\mathrm{Aut}(X)$ and $\mathrm{Aut}(Y)$ are isomorphic only as abstract groups?

Characterization of affine spherical surfaces

Theorem(Liendo, Urech, R.)

Let X and Y be affine surfaces and there is an isomorphism $\varphi : \mathrm{Aut}(X) \xrightarrow{\sim} \mathrm{Aut}(Y)$ of groups. Then the following is satisfied.
(1) If H is a connected non-unipotent algebraic subgroup of $\mathrm{Aut}(S)$, then $\varphi(H)$ is an algebraic subgroup of $\mathrm{Aut}(Y)$ isomorphic to H.
(2) If X is spherical G-variety, then Y is also spherical G-variety that is isomorphic to X.

Topology on $\mathrm{Bir}(S)$

Let A be a variety and $f : A \times S \to A \times S$ be an A-birational map, i.e.,

- f is the identity in the first factor
- f induces an isomorphism between open subsets U and V of $A \times S$ such that the projections from U and from V to A are both surjective.

Each $a \in A$ defines an element in $\mathrm{Bir}(S)$ and hence we obtain a map $A \to \mathrm{Bir}(S)$ that we call a *morphism*.

The Zariski topology on $\mathrm{Bir}(S)$ is the finest topology making all such morphisms continuous.

Algebraic elements in $\mathrm{Bir}(S)$

Definition

An algebraic subgroup of $\mathrm{Bir}(S)$ is the image of an algebraic group G by a morphism $G \to \mathrm{Bir}(S)$ that is also an injective homomorphism of groups. An element $g \in \mathrm{Bir}(S)$ is called algebraic if it is contained in an algebraic subgroup.

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Lemma

Let S be a surface and $f \in \text{Bir}(S)$.
Then the following two conditions are equivalent:

- There exists a $k > 0$ such that f^k is divisible,
- f is algebraic.

Algebraic elements in Aut(S)

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Lemma

Let S be an affine surface and let $g \in \mathrm{Aut}(S)$ be an automorphism. Then g is an algebraic element in $\mathrm{Bir}(S)$ if and only if g is an algebraic element in $\mathrm{Aut}(S)$.

Algebraic elements are preserved

Proposition

Let X and Y be affine surfaces, $\varphi : \mathrm{Aut}(X) \to \mathrm{Aut}(Y)$ a group homomorphism, and $g \in \mathrm{Aut}(X)$ an algebraic element. Then $\varphi(g)$ is an algebraic element in $\mathrm{Aut}(Y)$.

Spherical surfaces

Any spherical surface different from toric surface is isomorphic either to $\mathrm{SL}(2, \mathbb{C})/D$ or to $\mathrm{SL}(2, \mathbb{C})/N$.

Therefore, we have to deal, mainly, with toric surfaces.

Torus goes to 2-dimensional torus

Lemma

Let X and Y be normal affine surfaces with X toric and $\varphi : Aut(X) \xrightarrow{\sim} Aut(Y)$ a group isomorphism. Then $\varphi(T)$ is a maximal subtorus in $Aut(Y)$.

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Lemma

Let X and Y be normal affine surfaces with X toric, $\varphi : Aut(X) \xrightarrow{\sim} Aut(Y)$ a group isomorphism, and $U \subset Aut(X)$ a root subgroup. Then $\varphi(U)$ is a root subgroup in $Aut(Y)$ with respect to $\varphi(T)$.

End of the proof

We know now that Y is a toric surface and we have a bijection on the root subgroups of $\mathrm{Aut}(X)$ and $\mathrm{Aut}(Y)$ with respect to T and $\varphi(T)$.

To finish the proof, it is enough to show that we can retrieve a toric surface X from the abstract group structure of its root subgroups and their relationship with the torus.

Recall that any affine toric surface X without torus factor is isomorphic to $X_{d,e}$, the quotient of $\mathbb{A}^2$ under the $\mu_d = \{\xi \in \mathbb{C}^* \mid \xi^d = 1\}$-action

$$g : (x, y) \mapsto (\xi^e x, \xi y)$$

where ξ is a d-th primitive root of unity $0 \leq e < d$, $(e, d) = 1$.

$X_{d,e}$ is isomorphic to $X_{d',e'}$ if and only if $d = d'$ and $e = e'$ or $d = d'$ and $e \cdot e' = 1$ mod d.

End of the proof

The center of $G_a \rtimes T$ is $\{0\} \times \ker \chi$, so we can recover $\ker \chi$.

There are two families $\mathcal{K}$ and $\mathcal{L}$ of commuting root subgroups in $\mathrm{Aut}(X)$. We define the following subsets of $\mathbb{Z}_{\geq 0}$:

$$K_U = \{|\ker \chi \cap \ker \chi'|, \forall U' \in \mathcal{L}\} \ \ \forall U \in \mathcal{K}$$

$$L_U = \{|\ker \chi \cap \ker \chi'|, \forall U' \in \mathcal{K}\} \ \ \forall U \in \mathcal{L}$$

After some finite part, they form arithmetic progressions.

The two shortest common differences in this arithmetic progressions are

$$d \text{ and } d + e \text{ or } d \text{ and } d + e' \text{ with } e \cdot e' = 1 \bmod d.$$

Hence, these sets uniquely determine X.

What about higher dimensional case?

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Theorem(Cantat, Xie, R.)

Let X be an irreducible affine variety such that $\text{Aut}(X) \simeq \text{Aut}(\mathbb{A}^n)$, then $X \simeq \mathbb{A}^n$ as a variety.

THANK YOU FOR YOUR ATTENTION

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HAPPY BIRTHDAY