Singularities of symplectic reductions, in honor of Michel Brion

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Symplectic Quotients

- Joint work with H.-C. Herbig and C. Seaton.
- Let K be a compact Lie group with Lie algebra $\mathfrak{k}$ and let V be a unitary K-module with hermitian form $\langle\,,\rangle$.
- We have a moment mapping $\rho\colon V \to \mathfrak{k}^*$, $\rho(v)(A) = (-i/2)\langle A(v), v\rangle$, $v \in V$, $A \in \mathfrak{k}$.
- Let M denote $\rho^{-1}(0)$ and let M_0 denote M/K. Then M_0 is the Marsden-Weinstein or symplectic reduction of V.
- The imaginary part of $\langle\,,\rangle$ is a symplectic form on V and induces a symplectic form on a dense open subset of M_0 of smooth points.
- Interested in properties of M_0 and $\mathbb{R}[M_0] = \mathbb{R}[M]^K$.

Complexification

- We complexify: As K-module, $V \otimes_{\mathbb{R}} \mathbb{C} \simeq V \oplus V^*$. Let $G = K_{\mathbb{C}}$, acts naturally on $V \oplus V^*$ with usual symplectic form.
- Let $\mu\colon V \oplus V^* \to \mathfrak{g}^*$ where $\mu(v, v^*)(A) = v^*(A(v))$, $v \in V$, $v^* \in V^*$, $A \in \mathfrak{g}$. Set $N := \mu^{-1}(0)$. Call it the "shell."
- Restricted to M, $\mu = \rho$. So (N, μ) is a kind of complexification of (M, ρ). Most of time N is Zariski closure of M.
- Example: K is finite, so $K = G$. Then $\mu = 0$, $M_0 = V/K$ is an orbifold and $N/K \simeq (V \oplus V^*)/K$.
- Need to put some conditions on (V, G) for N and $N/\!\!/G := \mathrm{Spec}(\mathcal{O}(N)^G)$ to be nice.

Conditions on V and stratification of $N/\!\!/G$

- $q_V \colon V \to Z := V/\!\!/G$ dual to inclusion of $\mathcal{O}(V)^G \subset \mathcal{O}(V)$.
- Stratification: Let Gv be closed orbit with isotropy group G_v. Then G_v is reductive.
- We label $z = q_V(v) \in Z$ with the conjugacy class (G_v) of G_v. The $Z_{(G_v)}$ give a finite stratification of Z (Luna stratification).
- Unique open stratum, called principal stratum, denoted Z_{pr}. $V_{\text{pr}} := q_V^{-1}(Z_{\text{pr}})$.
- V has FPIG if the isotropy groups associated to Z_{pr} are finite.
- V is stable if V contains an open subset of closed orbits.
- V is k-principal if $V \setminus V_{\text{pr}}$ has codimension at least k in V.
- Let V_m denote $\{v \in V \mid \dim G_v = m\}$.
- V is k-modular if it has FPIG and codim $V_m \geq k + m$ for all $m \geq 1$.
- V is k-large if k-modular and k-principal.

Conditions on V

- Ex. $\mathbb{C}^2 \oplus \cdots \oplus \mathbb{C}^2$, $(k+2)$-times, is k-large, $G = \mathrm{SL}(2, \mathbb{C})$.
- If G is a torus, then V stable iff 1-principal iff 1-large.

Proposition

Suppose that V is 1-large. Then M is Zariski dense in N and N is reduced and irreducible. If G is a torus, then the converse holds.

Theorem

Let G be simple. Consider G-modules V with $V^G = (0)$. Then, up to isomorphism, all but finitely many V are k-large for any fixed k.

- We will study cases where (V, G) is at least 1-large.

Conditions on V.

- Let I denote ideal of components ρ_i of ρ in $\mathbb{R}[V]$.

Theorem

If V is 1-large then I is radical, M is coherent and the Koszul complex of the ρ_i in $C^\infty(V)$ is a resolution of $C^\infty(M)$. Conversely, if the Koszul complex is a resolution as above, then V is 1-large.

- T. Becker, R. Terpereau, (M. Bulois, C. Lehn, M. Lehn, R. Terpereau) and M. Bulois have considered "small" G-modules, say polar representations.
- Study properties of N and $N/\!\!/G$. Often $N/\!\!/G$ is an orbifold.

Theorem

Suppose that (V, G) is 2-large, $\dim G > 0$. Then $N/\!\!/G$ is not isomorphic to any quotient $(W \oplus W^*)/H$ where H is finite.

Some singularities

- For V 1-large, the G-invariant symplectic form on $V \oplus V^*$ induces a symplectic form on $(N/\!\!/G)_{\text{pr}}$.
- Let Y be a normal variety. We say that Y has symplectic singularities if there is a holomorphic symplectic form on Y_{reg} whose pullback extends to a smooth form ω on X for any resolution of singularities $X \to Y$. We say that a particular X is a symplectic resolution if ω is a symplectic form on X.
- Y has symplectic singularities implies that Y is rational Gorenstein.
- By a theorem of Flenner, in the definition of symplectic singularities, one can remove the condition on resolutions if the singularities of Y are in codimension 4.

Main Theorem

Theorem (Main)

Suppose that V is 2-large. Then $N/\!\!/G$ has symplectic singularities and $N/\!\!/G$ is graded Gorenstein.

- (G. Bellamy and T. Schedler): If $G/[G,G]$ is finite, then $N/\!\!/G$ has no symplectic resolution.
- We conjecture that $\mathrm{Spec}(\mathbb{R}[M]^K \otimes_{\mathbb{R}} \mathbb{C})$ always has properties of theorem. This is true for any module in case $K = S^1$ or $K = \mathrm{SU}(2,\mathbb{C})$.
- Let R be an $\mathbb{N}$-graded Gorenstein $\mathbb{C}$-algebra of dimension d with Poincaré series $P(t)$. Then R is graded Gorenstein if $P(1/t) = (-1)^d t^d P(t)$.

Ingredients in the proof

- If V is k-large, then singularities of $N/\!\!/G$ are in codimension $2k$. For this we need a symplectic slice theorem (more anon).
- If V is 1-large, then the smooth locus of $N/\!\!/G$ is precisely the image of the principal orbits in N and inherits a symplectic form from the one on $V \oplus V^*$.
- If V is 2-large, then N and $N/\!\!/G$ are normal and irreducible.
- So $N/\!\!/G$ is symplectic if V is 2-large.
- To show graded Gorenstein one has to explicitly compute the generator of the dualizing modules of N and $N/\!\!/G$.

Tori

Theorem

Suppose that G^0 is a torus and V is a G-module.

- *(Bulois). N is a reduced complete intersection, which is irreducible if and only if it is normal. The quotient $N/\!\!/G$ is normal.*
- *Suppose that V is stable. Then N and $N/\!\!/G$ have rational singularities and $N/\!\!/G$ has symplectic singularities. If G is connected, then $N/\!\!/G$ is graded Gorenstein.*

Symplectic slice theorem

- Let $x \in N$ where Gx is closed. Let L denote G_x and E denote $T_x(Gx)$.
- As L-module, $V \oplus V^* = E \oplus E^* \oplus W \oplus W^*$ where W is an L-module (not unique). The symplectic form ω on $V \oplus V^*$ restricts to a symplectic form ω_S on $S = W \oplus W^*$ where W and W^* are isotropic (can be arranged).
- Call S the symplectic slice representation of L at x.
- Recall that a morphism of G-varieties $\varphi\colon X \to Y$ is excellent if
  - φ is étale.
  - $\varphi /\!\!/ G\colon X /\!\!/ G \to Y /\!\!/ G$ is étale.
  - the morphism $(\varphi, q_X) : X \to Y \times_{Y/\!\!/G} X /\!\!/ G$ is an isomorphism.

Symplectic slice theorem

Theorem

Let x, L, $E = T_x(Gx)$ and S be as above. Then there is an L-saturated neighborhood Q of $0 \in E^* \oplus S$ and an excellent morphism $\varphi \colon G \times_L Q \to V \oplus V^*$ sending $[e, 0]$ to x. Give $G \times_L Q$ the induced Hamiltonian structure via φ. By appropriate choice of φ, the shell of $G \times_L Q$ is $G \times_L (Q \cap (\{0\} \times N_S))$. The induced mapping

$$\tilde{\varphi} \colon G \times_L (Q \cap (\{0\} \times N_S)) \to N$$

is excellent.

- Here N_S is the shell of S.
- Version of theorem in joint work of M. Bulois, C. Lehn, M. Lehn, R. Terpereau. Goes back to unpublished work of B. Jung.

Consequences of symplectic slice theorem

- Near $x \in N$, the quotient $N/\!\!/G$ looks like $(N_S)/\!\!/L$.
- If V is 1-large, then $N/\!\!/G \setminus (N/\!\!/G)_{\text{reg}}$ is the union of the $(N/\!\!/G)_{(L)}$ for L not principal isotropy group.
- Let $S = W \oplus W^*$ be slice representation of L as above. Then W is also 1-large.
- The codimension of $(N/\!\!/G)_{(L)}$ is the codimension of the L-fixed points in $N_S/\!\!/L$.
- Show that $\dim S - \dim S^L - 2\dim L \geq 2k$ if V is k-large. Hence we get estimate that the singularities of $N/\!\!/G$ are in codimension $2k$.