Existence of traveling waves for the nonlocal Gross–Pitaevskii equation in dimension one

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We consider the nonlocal Gross–Pitaevskii equation in dimension one

\[ i \partial_t u = \Delta u + u (W * (1 - |u|^2)) , \text{ on } \mathbb{R} \times \mathbb{R}, \]  

(NGP)

where \( u \) is a complex-valued function and \( W \) is a tempered distribution. In the Bose–Einstein condensate, \( u \) represents a wave function whereas \( W \) describes the interaction between bosons.

If \( W \) is a real-valued even distribution, (NGP) is a Hamiltonian equation whose energy given by

\[ E(u(t)) = \frac{1}{2} \int_{\mathbb{R}} |u'(t)|^2 dx + \frac{1}{4} \int_{\mathbb{R}} (W * (1 - |u(t)|^2))(1 - |u(t)|^2) dx, \]  

(1)

is formally conserved. If one considers finite energy solution, then \( u \) should not vanish at infinity and should in some sense tend to 1 when \( |x| \to +\infty \). Thus, we will consider the Cauchy problem for (NGP) with an initial date \( u(0) = u_0 \) verifying \( |u_0(x)| \to 1 \) as \( |x| \to +\infty \). We recall the concept of physical momentum

\[ P(u) = \int_{\mathbb{R}} (iu', u) dx, \]  

(2)

which is also formally conserved but not always well-defined.

A traveling wave of speed \( c \in \mathbb{R} \) is a solution of (NGP) of the form

\[ u_c(t, x) = v(x - ct). \]

Hence, the profile \( v \) satisfies

\[ icv' + \Delta v + v (W * (1 - |v|^2)) = 0 \text{ in } \mathbb{R} \]  

(NTWc)

and by using complex conjugation, we can restrict ourselves to the case \( c \geq 0 \). Note that any constant complex-valued function \( v \) of modulus one verifies (NTWc), so that we refer to them as the trivial solutions. In the case \( W = \delta \), the explicit formula of finite energy travelling waves is known (see [1]).

We will present a constraint minimization approaches to prove the existence of (non trivial) traveling waves for a wide class of tempered distributions. An important part of the proof is based on the study of the long-wave transonic limit of (NGP) which leads to the Korteweg-de-Vries equation. We will also present a numerical method based on projected gradient descent which will give us an approximation of traveling waves and energy-momentum diagrams.

Références
