Existence of traveling waves for the nonlocal Gross–Pitaevskii equation in dimension one

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We consider the nonlocal Gross–Pitaevskii equation in dimension one

$$i\partial_t u = \Delta u + u\left(W * (1 - |u|^2)\right), \text{ on } \mathbb{R} \times \mathbb{R}, \tag{NGP}$$

where u is a complex-valued function and W is a tempered distribution. In the Bose–Einstein condensate, u represents a wave function whereas W describes the interaction between bosons.
If W is a real-valued even distribution, (NGP) is a Hamiltonian equation whose energy given by

$$E(u(t)) = \frac{1}{2}\int_{\mathbb{R}} |u'(t)|^2 dx + \frac{1}{4}\int_{\mathbb{R}} (W * (1 - |u(t)|^2))(1 - |u(t)|^2) dx, \tag{1}$$

is formally conserved. If one considers finite energy solution, then u should not vanish at infinity and should in some sense tend to 1 when $|x| \to +\infty$. Thus, we will consider the Cauchy problem for (NGP) with an initial date $u(0) = u_0$ verifying $|u_0(x)| \underset{|x|\to+\infty}{\longrightarrow} 1$. We recall the concept of physical momentum

$$P(u) = \int_{\mathbb{R}} \langle iu', u\rangle dx, \tag{2}$$

which is also formally conserved but not always well-defined.

A traveling wave of speed $c \in \mathbb{R}$ is a solution of (NGP) of the form

$$u_c(t, x) = v(x - ct).$$

Hence, the profile v satifies

$$icv' + \Delta v + v\left(W * (1 - |v|^2)\right) = 0 \text{ in } \mathbb{R} \tag{NTWc}$$

and by using complex conjugation, we can restrict ourselves to the case $c \geq 0$. Note that any constant complex-valued function v of modulus one verifies (NTWc), so that we refer to them as the trivial solutions. In the case $W = \delta$, the explicit formula of finite energy travelling waves is known (see [1]).

We will present a constraint minimization approaches to prove the existence of (non trivial) traveling waves for a wide class of tempered distributions. An important part of the proof is based on the study of the long-wave transonic limit of (NGP) which leads to the Korteweg-de-Vries equation. We will also present a numerical method based on projected gradient descent which will give us an approximation of traveling waves and energy-momentum diagrams.

Références

[1] Béthuel F., Gravejat P., Saut J-C., *Existence and properties of traveling waves for the Gross-Pitaevskii equation*, Stationary and time dependent Gross–Pitaevskii equations, 473, American Mathematical Society, pp.55-104, 2008, Contemporary Mathematics, 978-0-8218-4357.

[2] DE LAIRE A., *Global well-posedness for a nonlocal Gross-Pitaevskii equation with non-zero condition at infinity*, Communications in Partial Differential Equations. 35(11):2021-2058, 2010.

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