Perverse Sheaves and the Decomposition Theorem

slides (adaptés) de Dragoș Frățilă
Throughout a variety will mean a quasi projective algebraic variety defined over the complex field. A topological space will be a paracompact, Hausdorff space.
Definition
A *stratification* on a space $X$ is a finite collection $\mathcal{X}$ of locally closed subspaces of $X$ called *strata* such that $X = \bigsqcup_{S \in \mathcal{X}} S$ and the closure of each stratum is a union of strata.

Definition
A *filtered space* $X$ is a space together with a filtration by closed subsets $X = X_n \supseteq X_{n-1} \supseteq \ldots \supseteq X_0 \supseteq X_{-1} = \emptyset$. 
Definition
A stratification $\mathcal{X}$ is a Whitney stratification if it satisfies the following conditions for any $S \neq S' \in \mathcal{X}$:

(i) Assume that a sequence $x_i \in S$ of points converges to a point $y \in S'$ and the limit $T$ of the tangent spaces $T_{x_i} S$ exists. Then we have $T_y S' \subseteq T$.

(ii) Let $x_i \in S$ and $y_i \in S'$ be two sequences of points which converge to the same point $y \in S'$. Assume further that the limit $l$ (resp. $T$) of the lines $l_i$ jointing $x_i$ and $y_i$ (resp. of the tangent spaces $T_{x_i} S$) exists. Then we have $l \subseteq T$.

Definition
If $\mathcal{X}, \mathcal{Y}$ are stratifications of $X$ we say that $\mathcal{Y}$ is a refinement of $\mathcal{X}$ and we write $\mathcal{X} \leq \mathcal{Y}$ if every stratum $S \in \mathcal{X}$ is a union of strata from $\mathcal{Y}$.

Proposition
Let $X$ be an algebraic variety. For any algebraic stratification $\mathcal{X}$ of $X$ there exists a refinement $\mathcal{Y}$ which is an algebraic Whitney stratification.
If not otherwise specified $X$ will denote a complex algebraic variety with an algebraic Whitney stratification $\mathcal{X}$. Throughout a sheaf will mean a sheaf of $\mathbb{Q}$-vector spaces.

**Definition**
A sheaf $\mathcal{F}$ on $X$ is called constructible if $\mathcal{F}|_S$ is locally constant with stalks of finite dimension over $\mathbb{Q}$ for every $S \in \mathcal{X}$.

**Definition**
The bounded derived category of constructible sheaves on $X$ relative to $\mathcal{X}$ is defined to be the full subcategory of the bounded derived category of sheaves on $X$ such that their cohomology (as complexes) is constructible. We will denote this category by $D^b_{\text{X-}c}(X)$. 
Intermediate extension relative to a filtration

Let $X$ be an algebraic variety with a filtration
\[ \overline{\mathcal{F}} : X = X_n \supseteq \ldots \supseteq X_0 \supseteq X_{-1} = \emptyset. \]
We denote by
\[ \begin{align*}
  & \bullet \quad S_k = X_{n-k} \setminus X_{n-k-1} \\
  & \bullet \quad U_k = X \setminus X_{n-k-1}, \ k = 0, \ldots, n.
\end{align*} \]

We have inclusions
\[ \begin{align*}
  & \bullet \quad i_k : U_k \hookrightarrow U_{k+1} \quad \text{and} \\
  & \bullet \quad j_{k+1} : S_{k+1} \hookrightarrow U_{k+1}
\end{align*} \]

Remark that $U_k$ is open in $U_{k+1}$ and $S_{k+1}$ is closed in $U_{k+1}$.

**Definition**

If $X$ is an algebraic variety with a filtration $\overline{\mathcal{F}}$ as above, we define the intermediate extension relative to $\overline{\mathcal{F}}$ to be the functor
\[ \iota^{\overline{\mathcal{F}}} ! : Sh(S_0) \to D^b(X) \]
defined by
\[ \iota^{\overline{\mathcal{F}}} ! \ast := \tau_{\leq n-1} Ri_{n-1} \ast \circ \ldots \circ \tau_{\leq 0} Ri_0 \ast \]
From now on we will denote by $X$ a complex quasi-projective variety and by $\mathfrak{X}$ and $\mathfrak{F}$ an algebraic Whitney stratification and the associated filtration respectively. For a stratum $S$ we will denote by $d_S$ the complex dimension of the stratum. We have the following theorem of Deligne:

**Theorem**

The intermediate extension functor $\iota_{\mathfrak{F}}^! [d_X]$ establishes an equivalence of categories between $\text{Loc}(S_0)$ and the full subcategory of complexes $\mathcal{F}^\bullet$ in $\mathcal{D}_{\mathfrak{X}-c}(X)$ verifying the following conditions:

- $H^m(\mathcal{F}^\bullet) = 0$, $\forall m < -d_X$
- $H^m(\mathcal{F}^\bullet)|_{S_0} = 0$, $\forall m > -d_X$ and $H^{-d_X}(\mathcal{F}^\bullet)|_{S_0} \in \text{Loc}(S_0)$
- $(S) \ H^m(\mathcal{F}^\bullet)|_S = 0$, $\forall m \geq -d_S$
- $(S') \ H^m(\mathcal{D}_X \mathcal{F}^\bullet)|_S = 0$, $\forall m \geq -d_S$. 
Intersection homology complexes

We can depict the degrees/strata where we can have non-zero cohomology for $\mathcal{F}^\bullet$, $\mathbb{D}_X \mathcal{F}^\bullet$ as in the theorem. Namely

| $\mathcal{H}^m(\mathcal{F}^\bullet)|s_0$ | $-d_X - 1$ | $-d_X$ | $-d_X + 1$ | $-d_X + 2$ | $-d_X + 3$ | $-d_X + 4$ | .... |
|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|
| $\mathcal{H}^m(\mathcal{F}^\bullet)|s_1$ | 0 | ● | 0 | 0 | 0 | 0 | 0 |
| $\mathcal{H}^m(\mathcal{F}^\bullet)|s_2$ | 0 | ● | 0 | 0 | 0 | 0 | 0 |
| $\mathcal{H}^m(\mathcal{F}^\bullet)|s_3$ | 0 | ● | ● | 0 | 0 | 0 | 0 |
| $\mathcal{H}^m(\mathcal{F}^\bullet)|s_4$ | 0 | ● | ● | ● | 0 | 0 | 0 |
| $\mathcal{H}^m(\mathcal{F}^\bullet)|s_5$ | 0 | ● | ● | ● | ● | 0 | 0 |
We will denote by $IC^\bullet_\mathcal{X}(X; \mathcal{L})$ the complex $\iota_{!\ast}(\mathcal{L})$ where $\mathcal{L}$ is a local system on $S_0$. This is called the *intersection homology complex* of $X$ with coefficients in $\mathcal{L}$.

We can now deduce some corollaries

**Corollary**

Let $\mathcal{X} \subset \mathcal{Y}$ and let $S_0$ respectively $T_0$ be codimension zero strata and $\iota : T_0 \hookrightarrow S_0$ be the inclusion. Then we have $\mathcal{IC}_{\mathcal{X}}(X) \subseteq \mathcal{IC}_{\mathcal{Y}}(X)$ and moreover the following diagram commutes:

\[
\begin{array}{ccc}
\text{Loc}(S_0) & \xrightarrow{\iota_{!\ast}} & \mathcal{IC}_{\mathcal{X}}(X) \\
\downarrow & & \downarrow \\
\text{Loc}(T_0) & \xrightarrow{\iota_{!\ast}} & \mathcal{IC}_{\mathcal{Y}}(X)
\end{array}
\]
Corollary

Let $X$ be a complex algebraic variety.

1. For any algebraic Whitney stratification $\mathcal{X}$ we have that $\mathcal{IC}_\mathcal{X}(X)$ is abelian and stable under the action of the Verdier duality $\mathbb{D}_X$.

2. For any refinement of algebraic Whitney stratifications $\mathcal{X} < \mathcal{Y}$ we have that the inclusion $\mathcal{IC}_\mathcal{X}(X) \subseteq \mathcal{IC}_\mathcal{Y}(X)$ is faithfully full and exact.

3. For any local system $\mathcal{L} \in \text{Loc}(S_0)$ we have a canonical isomorphism

$$\mathbb{D}_X(\mathcal{IC}_\mathcal{X}^\bullet(X; \mathcal{L})) \simeq \mathcal{IC}_\mathcal{X}^\bullet(X; \mathcal{L}^\vee)$$
Let $Z$ be a closed subset of $X$ which is a union of strata.

For any local system $\mathcal{L}$ on an open dense Zarisky subset of $Z$ we have the complex $\mathcal{IC}_X^\bullet(Z, \mathcal{L})$ in $\mathcal{D}^b_{\mathcal{X}-c}(Z)$ and we can consider its pushforward $i_Z^!(\mathcal{IC}_X^\bullet(Z, \mathcal{L})) \in \mathcal{D}^b_{\mathcal{X}-c}(X)$.

The following table illustrates/resumes the properties of the above complex:

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<tr>
<th></th>
<th>$-d_X$</th>
<th>$-d_X + 1$</th>
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<th>$-d_Z$</th>
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<th>$-d_Z + 2$</th>
<th>$-d_Z + 3$</th>
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<tr>
<td>$\mathcal{H}_m^m(F^\bullet)</td>
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<td>$\mathcal{H}_m^m(F^\bullet)</td>
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<td>$\mathcal{H}_m^m(F^\bullet)</td>
<td><em>{S</em>{d_X-d_Z+1}}$</td>
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<td>$\mathcal{H}_m^m(F^\bullet)</td>
<td><em>{S</em>{d_X-d_Z+2}}$</td>
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<tr>
<td>$\mathcal{H}_m^m(F^\bullet)</td>
<td><em>{S</em>{d_X-d_Z+3}}$</td>
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where $F^\bullet = i_Z^! \mathcal{IC}_X^\bullet(Z, \mathcal{L})[d_Z]$ or its Verdier dual.
Definition

A complex $\mathcal{F}^\bullet \in D^{b}_{\mathcal{X} - c}(X)$ is called a \textit{DGM-complex relative to $\mathcal{X}$} if there exists some closed irreducible subvariety $Z \subseteq X$ which is a union of strata from $\mathcal{X}$ and an \textbf{irreducible} local system on a non-singular dense open subset of $Z$ such that $\mathcal{F}^\bullet \simeq i_{Z*}(IC_{\mathcal{X}}^\bullet(Z, \mathcal{L}))[d_Z]$. We denote them by $DGM_{\mathcal{X}}(X)$.
**Definition**
Let $X$ be a complex algebraic variety and $\mathcal{X}$ a stratification. A complex of sheaves $\mathcal{F}^\bullet \in D_{\mathcal{X} - c}^b(X)$ is called $\mathcal{X}$-perverse if for each stratum $S \in \mathcal{X}$ we have:

- (S) $\mathcal{H}^m(\mathcal{F}^\bullet)|_S = 0$, $\forall m > -d_S$
- (S) $\mathcal{H}^m(\mathbb{D}_X(\mathcal{F}^\bullet))|_S = 0$, $\forall m > -d_S$.

The full subcategory of $\mathcal{X}$-perverse sheaves of $D_{\mathcal{X} - c}^b$ is denoted $Perv_{\mathcal{X}}(X)$.

**Remark**
From the previous discussion we deduce that if $Z \subseteq X$ is a closed set which is a union of strata then

$$i_{Z\ast}(\mathcal{IC}_{\mathcal{X}}(Z))[d_Z] \subseteq Perv_{\mathcal{X}}(X).$$
From the support and cosupport conditions we can prove that a perverse sheaf has the cohomology concentrated in degrees $[-d_X, 0]$. So we have the following picture:
Proposition

Let $Z$ be a locally closed subset of a variety $X$ which is a union of strata of $\mathfrak{X}$.

1. If $Z$ is closed then $i_Z^*(\text{Perv}_\mathfrak{X}(Z)) \subseteq \text{Perv}_\mathfrak{X}(X)$

2. If $Z$ is open then $i_Z^{-1}(\text{Perv}_\mathfrak{X}(X)) \subseteq \text{Perv}_\mathfrak{X}(Z)$

3. Let $Z = \bigsqcup_i S_i$ be a union of open strata of $\mathfrak{X}$. Then for any $\mathcal{F}^\bullet \in \text{Perv}_\mathfrak{X}(X)$ we have $i_Z^{-1}\mathcal{F}^\bullet = \bigoplus_i \mathcal{L}_i[d_{S_i}]$ where $\mathcal{L}_i$ are local systems on $S_i$.

4. If $\mathcal{F}^\bullet \in \text{Perv}_\mathfrak{X}(X)$ has the property that $|\mathcal{F}^\bullet| \subseteq Z$ then $i_Z^{-1}\mathcal{F}^\bullet = i_Z^!\mathcal{F}^\bullet \in \text{Perv}_\mathfrak{X}(Z)$

5. If $Z$ is an open affine non-singular subvariety of $X$ then $Ri_Z^*(\text{Loc}(Z)[d_Z]) \subseteq \text{Perv}_\mathfrak{X}(X)$ and $i_Z^!(\text{Loc}(Z)[d_Z]) \subseteq \text{Perv}_\mathfrak{X}(X)$. 
Structure theorems for $\mathcal{X}$-perverse sheaves

Theorem (BBD)
For any variety $X$ and any algebraic Whitney stratification $\mathcal{X}$ the full subcategory $\text{Perv}_{\mathcal{X}}(X)$ of $\mathcal{D}_X^{b,c}(X)$ is an abelian, admissible category that is stable by extensions and by Verdier duality.
Recall that a DGM-complex relative to a stratification $\mathcal{X}$ is a complex isomorphic to $i_{Z*}IC_\mathcal{X}^\bullet(Z, \mathcal{L})$ where $Z$ is an irreducible closed subvariety of $X$ union of strata and $\mathcal{L}$ is an irreducible local system on a dense open subset of $Z$. We have the following theorem

**Theorem**
The DGM-complexes are simple objects in the category $Perv_\mathcal{X}(X)$. 

Definition
The category of \textit{perverse sheaves} on $X$ is the full subcategory of $\mathcal{D}^b(X)$ consisting of objects that are $\mathcal{X}$-perverse for some algebraic Whitney stratification $\mathcal{X}$. We denote it by $\text{Perv}(X)$. In other words

$$\text{Perv}(X) = \lim_{\mathcal{X}} \text{Perv}_{\mathcal{X}}(X).$$

Definition
A \textit{DGM}-complex on $X$ is a perverse sheaf that is a \textit{DGM}-complex relative to $\mathcal{X}$ for an algebraic Whitney stratification. Again, we can express this by

$$\text{DGM}(X) = \lim_{\mathcal{X}} \text{DGM}_{\mathcal{X}}(X).$$
Theorem (BBD)

Let $X$ be a variety. We have

- The category $Perv(X)$ is a full subcategory of $\mathcal{D}_c^b(X)$ that is abelian, stable by extensions and by Verdier duality.
- The simple objects of $Perv(X)$ are precisely the $DGM$-complexes.
- All the objects of $Perv(X)$ are finite successive extensions of simple objects: the category of perverse sheaves is artinian and noetherian.
Proposition
Let \((X, \mathcal{X}), (Y, \mathcal{Y})\) be algebraic varieties with Whitney stratifications and \(f : X \to Y\) a stratified map. Then the following hold:

\[
\begin{align*}
    f^{-1} D^{b}_{\mathcal{Y} - c}(Y) & \subseteq D^{b}_{\mathcal{X} - c}, \\
    f^! D^{b}_{\mathcal{Y} - c}(Y) & \subseteq D^{b}_{\mathcal{X} - c} \\
    Rf_* D^{b}_{\mathcal{X} - c}(X) & \subseteq D^{b}_{\mathcal{Y} - c}, \\
    Rf! D^{b}_{\mathcal{X} - c}(X) & \subseteq D^{b}_{\mathcal{Y} - c}
\end{align*}
\]

Therefore, we have all the above stability properties for \(D^{b}_{c}(X)\) and \(D^{b}_{c}(Y)\).
The Decomposition Theorem

**Theorem**

Let $X$, $Y$ be two complex algebraic varieties and $f : X \to Y$ a **proper** algebraic map. For any simple perverse sheaf $i_Z^*(IC^\bullet(Z, \mathcal{L}))$ on $X$ there exist a finite number of irreducible closed sets $Z_i \subseteq Y$, irreducible local systems $\mathcal{L}_i$ on open subsets of $Z_i$ and integers $c_i$ such that

$$Rf^*(i_Z^*IC^\bullet(Z, \mathcal{L})[d_Z]) = \bigoplus_i i_{Z_i}^*(IC^\bullet(Z_i, \mathcal{L}_i))[c_i].$$

**Remark**

If the map is stratified with respect to the stratifications $(\mathcal{X}, \mathcal{Y})$ then we can choose $Z_i$ to be strata from $\mathcal{Y}$.