# Perverse Sheaves and the Decomposition Theorem

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# Conventions

Throughout a variety will mean a quasi projective algebraic variety defined over the complex field. A topological space will be a paracompact, Hausdorff space.

# Review on stratifications

**Definition**

A *stratification* on a space $X$ is a finite collection $\mathfrak{X}$ of locally closed subspaces of $X$ called *strata* such that $X = \coprod_{S \in \mathfrak{X}} S$ and the closure of each stratum is a union of strata.

**Definition**

A *filtered space* $X$ is a space together with a filtration by closed subsets $X = X_n \supseteq X_{n-1} \supseteq ... \supseteq X_0 \supseteq X_{-1} = \emptyset$.

# Review on stratifications

**Definition**

A stratification $\mathfrak{X}$ is a *Whitney stratification* if it satisfies the following conditions for any $S \neq S' \in \mathfrak{X}$:

(i) Assume that a sequence $x_i \in S$ of points converges to a point $y \in S'$ and the limit $T$ of the tangent spaces $T_{x_i}S$ exists. Then we have $T_yS' \subseteq T$.

(ii) Let $x_i \in S$ and $y_i \in S'$ be two sequences of points which converge to the same point $y \in S'$. Assume further that the limit $l$ (resp. $T$ ) of the lines $l_i$ jointing $x_i$ and $y_i$ (resp. of the tangent spaces $T_{x_i}S$) exists. Then we have $l \subseteq T$.

**Definition**

If $\mathfrak{X}, \mathfrak{Y}$ are stratifications of $X$ we say that $\mathfrak{Y}$ is a refinement of $\mathfrak{X}$ and we write $\mathfrak{X} \leq \mathfrak{Y}$ if every stratum $S \in \mathfrak{X}$ is a union of strata from $\mathfrak{Y}$.

**Proposition**

Let $X$ be an algebraic variety. For any algebraic stratification $\mathfrak{X}$ of $X$ there exists a refinement $\mathfrak{Y}$ which is an algebraic Whitney stratification.

# Review on stratifications

If not otherwise specified $X$ will denote a complex algebraic variety with an algebraic Whitney stratification $\mathfrak{X}$. Throughout a sheaf will mean a sheaf of $\mathbb{Q}$-vector spaces.

**Definition**

A sheaf $\mathcal{F}$ on $X$ is called constructible if $\mathcal{F}|_S$ is locally constant with stalks of finite dimension over $\mathbb{Q}$ for every $S \in \mathfrak{X}$.

**Definition**

The bounded derived category of constructible sheaves on $X$ relative to $\mathfrak{X}$ is defined to be the full subcategory of the bounded derived category of sheaves on $X^{an}$ such that their cohomology (as complexes) is constructible. We will denote this category by $\mathcal{D}^b_{\mathfrak{X}-c}(X)$.

# Intermediate extension relative to a filtration

Let $X$ be an algebraic variety with a filtration $\mathfrak{F} : X = X_n \supseteq ... \supseteq X_0 \supseteq X_{-1} = \emptyset$. We denote by

- $S_k = X_{n-k} \setminus X_{n-k-1}$
- $U_k = X \setminus X_{n-k-1}$, $k = 0, ..., n$.

We have inclusions

- $i_k : U_k \hookrightarrow U_{k+1}$ and
- $j_{k+1} : S_{k+1} \hookrightarrow U_{k+1}$

Remark that $U_k$ is open in $U_{k+1}$ and $S_{k+1}$ is closed in $U_{k+1}$.

**Definition**

If $X$ is an algebraic variety with a filtration $\mathfrak{F}$ as above, we define the intermediate extension relative to $\mathfrak{F}$ to be the functor

$$\iota_{!*}^{\mathfrak{F}} : Sh(S_0) \to \mathcal{D}^b(X)$$

defined by

$$\iota_{!*}^{\mathfrak{F}} := \tau_{\leq n-1} Ri_{n-1*} \circ ... \circ \tau_{\leq 0} Ri_{0*}$$

From now on we will denote by $X$ a complex quasi-projective variety and by $\mathfrak{X}$ and $\mathfrak{F}$ an algebraic Whitney stratification and the associated filtration respectively. For a stratum $S$ we will denote by $d_S$ the complex dimension of the stratum. We have the following theorem of Deligne:

**Theorem**

*The intermediate extension functor $\iota^{\mathfrak{F}}_{!*}[d_X]$ establishes an equivalence of categories between $Loc(S_0)$ and the full subcategory of complexes $\mathcal{F}^\bullet$ in $\mathcal{D}^b_{\mathfrak{X}-c}(X)$ verifying the following conditions:*

- $\mathcal{H}^m(\mathcal{F}^\bullet) = 0, \forall m < -d_X$
- $\mathcal{H}^m(\mathcal{F}^\bullet)|_{S_0} = 0, \forall m > -d_X$ *and* $\mathcal{H}^{-d_X}(\mathcal{F}^\bullet)|_{S_0} \in Loc(S_0)$
- *(S)* $\mathcal{H}^m(\mathcal{F}^\bullet)|_S = 0, \forall m \geq -d_S$
- *(S')* $\mathcal{H}^m(\mathbb{D}_X\mathcal{F}^\bullet)|_S = 0, \forall m \geq -d_S$.

We can depict the degrees/strata where we can have non-zero cohomology for $\mathcal{F}^\bullet, \mathbb{D}_X\mathcal{F}^\bullet$ as in the theorem. Namely

| | $-d_X - 1$ | $-d_X$ | $-d_X + 1$ | $-d_X + 2$ | $-d_X + 3$ | $-d_X + 4$ | .... |
|---|---|---|---|---|---|---|---|
| $\mathcal{H}^m(\mathcal{F}^\bullet)\vert_{S_0}$ | 0 | • | 0 | 0 | 0 | 0 | 0 |
| $\mathcal{H}^m(\mathcal{F}^\bullet)\vert_{S_1}$ | 0 | • | 0 | 0 | 0 | 0 | 0 |
| $\mathcal{H}^m(\mathcal{F}^\bullet)\vert_{S_2}$ | 0 | • | • | 0 | 0 | 0 | 0 |
| $\mathcal{H}^m(\mathcal{F}^\bullet)\vert_{S_3}$ | 0 | • | • | • | 0 | 0 | 0 |
| $\mathcal{H}^m(\mathcal{F}^\bullet)\vert_{S_4}$ | 0 | • | • | • | • | 0 | 0 |
| $\mathcal{H}^m(\mathcal{F}^\bullet)\vert_{S_5}$ | 0 | • | • | • | • | • | 0 |

We will denote by $IC^{\bullet}_{\mathfrak{X}}(X;\mathcal{L})$ the complex $\iota^{\mathfrak{F}}_{!*}(\mathcal{L})$ where $\mathcal{L}$ is a local system on $S_0$. This is called the *intersection homology complex* of $X$ with coefficients in $\mathcal{L}$.

We can now deduce some corollaries

**Corollary**
*Let $\mathfrak{X} < \mathfrak{Y}$ and let $S_0$ respectively $T_0$ be codimension zero strata and $\iota : T_0 \hookrightarrow S_0$ be the inclusion. Then we have $\mathfrak{IC}_{\mathfrak{X}}(X) \subseteq \mathfrak{IC}_{\mathfrak{Y}}(X)$ and moreover the following diagram commutes:*

$$\begin{array}{ccc} Loc(S_0) & \xrightarrow{\iota^{\mathfrak{X}}_{!*}} & \mathfrak{IC}_{\mathfrak{X}}(X) \\ \downarrow{\scriptstyle \iota^{-1}} & & \downarrow \\ Loc(T_0) & \xrightarrow{\iota^{\mathfrak{Y}}_{!*}} & \mathfrak{IC}_{\mathfrak{Y}}(X) \end{array}$$

**Corollary**

*Let $X$ be a complex algebraic variety.*

1. *For any algebraic Whitney stratification $\mathfrak{X}$ we have that $\mathfrak{IC}_{\mathfrak{X}}(X)$ is abelian and stable under the action of the Verdier duality $\mathbb{D}_X$.*
2. *For any refinement of algebraic Whitney stratifications $\mathfrak{X} < \mathfrak{Y}$ we have that the inclusion $\mathfrak{IC}_{\mathfrak{X}}(X) \subseteq \mathfrak{IC}_{\mathfrak{Y}}(X)$ is faithfully full and exact.*
3. *For any local system $\mathcal{L} \in Loc(S_0)$ we have a canonical isomorphism*

$$\mathbb{D}_X(IC^\bullet_{\mathfrak{X}}(X;\mathcal{L})) \simeq IC^\bullet_{\mathfrak{X}}(X;\mathcal{L}^\vee)$$

# Deligne-Goresky-MacPherson Complexes

Let $Z$ be a closed subset of $X$ which is a union of strata.

For any local system $\mathcal{L}$ on an open dense Zarisky subset of $Z$ we have the complex $IC^\bullet_{\mathfrak{X}}(Z,\mathcal{L})$ in $\mathcal{D}^b_{\mathfrak{X}-c}(Z)$ and we can consider its pushforward $i_{Z*}(IC^\bullet_{\mathfrak{X}}(Z,\mathcal{L})) \in \mathcal{D}^b_{\mathfrak{X}-c}(X)$.

The following table illustrates/resumes the properties of the above complex:

| | $-d_X$ | $-d_X+1$ | ... | $-d_Z$ | $-d_Z+1$ | $-d_Z+2$ | $-d_Z+3$ |
|---|---|---|---|---|---|---|---|
| $\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_0}$ | 0 | 0 | ... | 0 | 0 | 0 | 0 |
| $\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_1}$ | 0 | 0 | ... | 0 | 0 | 0 | 0 |
| ... | ... | ... | ... | ... | ... | ... | ... |
| $\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_{d_X-d_Z}}$ | 0 | 0 | 0 | $\bullet$ | 0 | 0 | 0 |
| $\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_{d_X-d_Z+1}}$ | 0 | 0 | 0 | $\bullet$ | 0 | 0 | 0 |
| $\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_{d_X-d_Z+2}}$ | 0 | 0 | 0 | $\bullet$ | $\bullet$ | 0 | 0 |
| $\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_{d_X-d_Z+3}}$ | 0 | 0 | 0 | $\bullet$ | $\bullet$ | $\bullet$ | 0 |

where $\mathcal{F}^\bullet = i_{Z*}IC^\bullet_{\mathfrak{X}}(Z,\mathcal{L})[d_Z]$ or its Verdier dual.

**Definition**

A complex $\mathcal{F}^\bullet \in \mathcal{D}^b_{\mathfrak{X}-c}(X)$ is called a *DGM-complex relative to* $\mathfrak{X}$ if there exists some closed irreducible subvariety $Z \subseteq X$ which is a union of strata from $\mathfrak{X}$ and an **irreducible** local system on a non-singular dense open subset of $Z$ such that $\mathcal{F}^\bullet \simeq i_{Z*}(IC^\bullet_{\mathfrak{X}}(Z, \mathcal{L}))[d_Z]$. We denote them by $DGM_{\mathfrak{X}}(X)$.

**Definition**

Let $X$ be a complex algebraic variety and $\mathfrak{X}$ a stratification. A complex of sheaves $\mathcal{F}^\bullet \in \mathcal{D}^b_{\mathfrak{X}-c}(X)$ is called $\mathfrak{X}$-perverse if for each stratum $S \in \mathfrak{X}$ we have:

- (S) $\mathcal{H}^m(\mathcal{F}^\bullet)|_S = 0$, $\forall m > -d_S$
- (S) $\mathcal{H}^m(\mathbb{D}_X(\mathcal{F}^\bullet))|_S = 0$, $\forall m > -d_S$.

The full subcategory of $\mathfrak{X}$-perverse sheaves of $\mathcal{D}^b_{\mathfrak{X}-c}$ is denoted $Perv_{\mathfrak{X}}(X)$.

**Remark**

From the previous discussion we deduce that if $Z \subseteq X$ is a closed set which is a union of strata then

$$i_{Z*}(\mathfrak{I}\mathcal{C}_{\mathfrak{X}}(Z))[d_Z] \subseteq Perv_{\mathfrak{X}}(X).$$

# $\mathfrak{X}$-perverse sheaves

From the support and cosupport conditions we can prove that a perverse sheaf has the cohomology concentrated in degrees $[-d_X, 0]$. So we have the following picture:

| | $-d_X - 1$ | $-d_X$ | $-d_X + 1$ | $-d_X + 2$ | $-d_X + 3$ | $-d_X + 4$ | .... |
|---|---|---|---|---|---|---|---|
| $\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_0}$ | 0 | $\bullet$ | 0 | 0 | 0 | 0 | 0 |
| $\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_1}$ | 0 | $\bullet$ | $\bullet$ | 0 | 0 | 0 | 0 |
| $\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_2}$ | 0 | $\bullet$ | $\bullet$ | $\bullet$ | 0 | 0 | 0 |
| $\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_3}$ | 0 | $\bullet$ | $\bullet$ | $\bullet$ | $\bullet$ | 0 | 0 |
| $\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_4}$ | 0 | $\bullet$ | $\bullet$ | $\bullet$ | $\bullet$ | $\bullet$ | 0 |
| $\mathcal{H}^m(\mathcal{F}^\bullet)\|_{S_5}$ | 0 | $\bullet$ | $\bullet$ | $\bullet$ | $\bullet$ | $\bullet$ | $\bullet$ |

**Proposition**

Let $Z$ be a locally closed subset of a variety $X$ which is a union of strata of $\mathfrak{X}$.

1. If $Z$ is closed then $i_{Z*}(Perv_{\mathfrak{X}}(Z)) \subseteq Perv_{\mathfrak{X}}(X)$
2. If $Z$ is open then $i_Z^{-1}(Perv_{\mathfrak{X}}(X)) \subseteq Perv_{\mathfrak{X}}(Z)$
3. Let $Z = \coprod_i S_i$ be a union of open strata of $\mathfrak{X}$. Then for any $\mathcal{F}^\bullet \in Perv_{\mathfrak{X}}(X)$ we have $i_Z^{-1}\mathcal{F}^\bullet = \oplus_i \mathcal{L}_i[d_{S_i}]$ where $\mathcal{L}_i$ are local systems on $S_i$.
4. If $\mathcal{F}^\bullet \in Perv_{\mathfrak{X}}(X)$ has the property that $|\mathcal{F}^\bullet| \subseteq Z$ then $i_Z^{-1}\mathcal{F}^\bullet = i_Z^! \mathcal{F}^\bullet \in Perv_{\mathfrak{X}}(Z)$
5. If $Z$ is an **open affine non-singular** subvariety of $X$ then $Ri_{Z*}(Loc(Z)[d_Z]) \subseteq Perv_{\mathfrak{X}}(X)$ and $i_{Z!}(Loc(Z)[d_Z]) \subseteq Perv_{\mathfrak{X}}(X)$.

**Theorem (BBD)**
For any variety $X$ and any algebraic Whitney stratification $\mathfrak{X}$ the full subcategory $Perv_{\mathfrak{X}}(X)$ of $\mathcal{D}^b_{\mathfrak{X}-c}(X)$ is an *abelian*, *admissible category* that is *stable by extensions and by Verdier duality*.

# Structure theorems for $\mathfrak{X}$-perverse sheaves

Recall that a DGM-complex relative to a stratification $\mathfrak{X}$ is a complex isomorphic to $i_{Z*}IC^{\bullet}_{\mathfrak{X}}(Z, \mathcal{L})$ where $Z$ is an irreducible closed subvariety of $X$ union of strata and $\mathcal{L}$ is an irreducible local system on a dense open subset of $Z$. We have the following theorem

**Theorem**

The DGM-complexes are simple objects in the category $Perv_{\mathfrak{X}}(X)$.

**Definition**

The category of *perverse sheaves* on $X$ is the full subcategory of $\mathcal{D}^b(X)$ consisting of objects that are $\mathfrak{X}$-perverse for some algebraic Whitney stratification $\mathfrak{X}$. We denote it by $Perv(X)$. In other words

$$Perv(X) = \varinjlim_{\mathfrak{X}} Perv_{\mathfrak{X}}(X).$$

**Definition**

A *DGM*-complex on $X$ is a perverse sheaf that is a *DGM*-complex relative to $\mathfrak{X}$ for an algebraic Whitney stratification. Again, we can express this by

$$DGM(X) = \varinjlim_{\mathfrak{X}} DGM_{\mathfrak{X}}(X)$$

# Structure theorems for perverse sheaves

**Theorem (BBD)**

Let $X$ be a variety. We have

- The category $Perv(X)$ is a full subcategory of $\mathcal{D}_c^b(X)$ that is abelian, stable by extensions and by Verdier duality.
- The simple objects of $Perv(X)$ are precisely the *DGM*-complexes.
- All the objects of $Perv(X)$ are finite successive extensions of simple objects: the category of perverse sheaves is artinian and noetherian.

**Proposition**

Let $(X, \mathfrak{X}), (Y, \mathfrak{Y})$ be algebraic varieties with Whitney stratifications and $f : X \to Y$ a stratified map. Then the following hold:

$$f^{-1}\mathcal{D}^b_{\mathfrak{Y}-c}(Y) \subseteq \mathcal{D}^b_{\mathfrak{X}-c}, \quad f^!\mathcal{D}^b_{\mathfrak{Y}-c}(Y) \subseteq \mathcal{D}^b_{\mathfrak{X}-c}$$

$$Rf_*\mathcal{D}^b_{\mathfrak{X}-c}(X) \subseteq \mathcal{D}^b_{\mathfrak{Y}-c}, \quad Rf_!\mathcal{D}^b_{\mathfrak{X}-c}(Y) \subseteq \mathcal{D}^b_{\mathfrak{Y}-c}$$

Therefore, we have all the above stability properties for $\mathcal{D}^b_c(X)$ and $\mathcal{D}^b_c(Y)$.

**Theorem**

Let $X$, $Y$ be two complex algebraic varieties and $f : X \to Y$ a **proper** algebraic map. For any simple perverse sheaf $i_{Z*}(IC^\bullet(Z, \mathcal{L}))$ on $X$ there exist a finite number of irreducible closed sets $Z_i \subseteq Y$, irreducible local systems $\mathcal{L}_i$ on open subsets of $Z_i$ and integers $c_i$ such that

$$Rf_*(i_{Z*}IC^\bullet(Z, \mathcal{L})[d_Z]) = \bigoplus_i i_{Z_i*}(IC^\bullet(Z_i, \mathcal{L}_i))[c_i].$$

**Remark**

If the map is stratified with respect to the stratifications $(\mathfrak{X}, \mathfrak{Y})$ then we can choose $Z_i$ to be strata from $\mathfrak{Y}$.