Linear matrix inequalities and structured matrices

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Structured Matrix Days

May 2018 – ENS Lyon

Related talks at SMD '18

Bernard's talk : Tensor decomposition / Hankel matrices

José Henrique's talk : BTD, "minimal ranks"

Françoise's talk : Definite matrix polynomials

Stefano's talk (afternoon) : Linear matrix equations

What is a LMI ?

A **feasibility problem** defined by two objects :

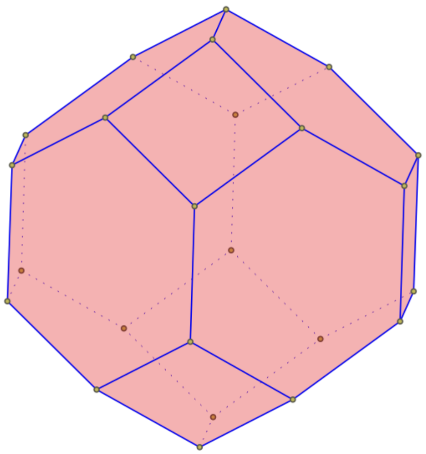
The **psd-cone** $\mathcal{S}_{+}^{m} = \{X \in \mathcal{S}^{m} : X \succeq 0\}$
An **affine space** $L = A_0 + \langle A_1, \ldots, A_n \rangle \subset \mathcal{S}^{m}$.

LMI Problem : Is $L \cap \mathcal{S}_{+}^{m}$ empty ?

Algebraically it means to compute $x \in \mathbb{R}^n$ such that

$$A_0 + x_1 A_1 + x_2 A_2 + \cdots + x_n A_n \succeq 0$$

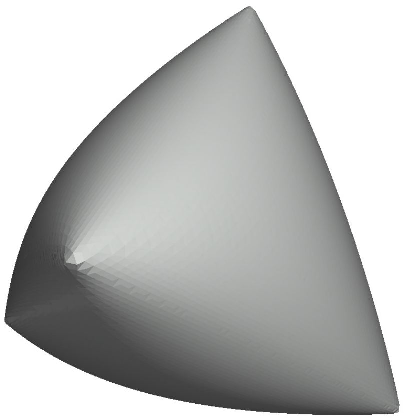
Polyhedra (LP) and Spectrahedra (SDP)



Solution of the diagonal LMI :

$$\begin{bmatrix} \ell_1(x) & & \\ & \ddots & \\ & & \ell_m(x) \end{bmatrix} \succeq 0$$

with $x = (x_1, x_2, x_3)$



Solution of the LMI :

$$\begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & x_3 \\ x_2 & x_3 & 1 \end{bmatrix} \succeq 0$$

(relaxation of MAX-CUT)

Empty spectrahedra

The following LMI has no solution ($L \cap \mathcal{S}^2_+$) :

$$\begin{bmatrix} 0 & 1 \\ 1 & x \end{bmatrix} \succeq 0 \qquad \left(\text{here } L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \left\langle \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle \right)$$

Very BAD instance of infeasible LMI (third figure below) :

The general goal (still open problem in general) is :

1. Compute a solution, if it exists (complexity, efficient algo.)
2. Certify the non-existence of a solution, if it is the case.

Difference from LP and SDP

Existence of irrational spectrahedra

$$\begin{bmatrix} 1 & x_1 \\ x_1 & 2 \end{bmatrix} \bigoplus \begin{bmatrix} 2x_1 & 2 \\ 2 & x_1 \end{bmatrix} \succeq 0 \;\Rightarrow\; x_1 = \sqrt{2}.$$

Exponential bit-size spectrahedra

$$\mathscr{S}_n = \left\{ x \in \mathbb{R}^n : \begin{bmatrix} 1 & 2 \\ 2 & x_1 \end{bmatrix} \bigoplus \begin{bmatrix} 1 & x_1 \\ x_1 & x_2 \end{bmatrix} \bigoplus \cdots \bigoplus \begin{bmatrix} 1 & x_{n-1} \\ x_{n-1} & x_n \end{bmatrix} \succeq 0 \right\}.$$

Here the size is $m = 2n$. For every $x^* \in \mathscr{S}_n$ it holds

$$x_n^* \geq (x_{n-1}^*)^2 \geq \cdots \geq (x_1^*)^{2^{n-1}} \geq 2^{2^n}.$$

Motivations

Lyapunov stability for $\dot{y} = My$: find $P \in \mathcal{S}^m$ such that

$$P \succ 0 \quad \text{and} \quad M^T P + PM \prec 0$$

SOS polynomials (sums of squares): $f(u) = f_1(u)^2 + \cdots + f_t(u)^2$, with $u = (u_1, \ldots, u_n)$, is equivalent to a LMI of type

$$f(u) = v(u)^T \cdot G \cdot v(u), \quad G \succeq 0$$

Polynomial Optimization: $f^* = \inf f(u) = \sup \lambda : \; f - \lambda \geq 0$

$$\text{SOS relaxation} : \; f - \lambda = g_1^2 + \cdots + g_t^2$$

Irrational SOS certificates

The polynomial

$$f = u_1^4 + u_1 u_2^3 + u_2^4 - 3u_1^2 u_2 u_3 - 4u_1 u_2^2 u_3 + 2u_1^2 u_3^2 + u_1 u_3^3 + u_2 u_3^3 + u_3^4$$

(has coefficients in $\mathbb{Q}$ and) is globally positive. It is SOS in $\mathbb{R}[u]$.

But f is not SOS in $\mathbb{Q}[u]$ (Scheiderer 2016) $\rightarrow$ no int point

It admits by the way a cubic certificate

$$f = \tfrac{1}{4}(2u_1^2 - \beta u_2^2 + u_2 u_3 + (2 - \beta^{-1})u_3^2)^2 + \tfrac{\beta}{4}(2u_1 u_2 + \beta^{-1} u_2^2 - 2\beta^{-1} u_1 u_3 - \beta u_2 u_3 - u_3^2)^2$$

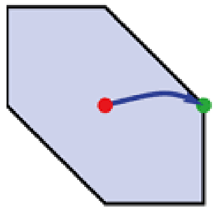
where β satisfies a cubic equation. This certificate is a solution of a LMI $A(x) \succeq 0$ of size $m = 6$ and in $n = 6 (= 21 - 15)$ variables.

IPM and the central curve

Central curve in SDP: given by the solutions (x_μ, s_μ) to

$$A(x)S(s) = \mu\,\mathrm{Id}_m \qquad A(x) \succeq 0 \qquad S(s) \succeq 0.$$

When $\mu \to 0^+$, one gets a solution (x^*, s^*).



Efficient implementations in several solvers

Assume existence of interior point

No certification of the solution (rank? alg degree?)

No convergence guarantees without Slater's conditions

8 Determinantal varieties and low rank solutions

Consider a generic SDP, given by data $A(x)$ and $\ell : \mathbb{R}^n \to \mathbb{R}$

$$\inf \ell(x) \quad \text{s.t. } A(x) \succeq 0.$$

Any solution x^* satisfies $\det A(x^*) = 0$ (boundary of $\mathcal{S}_+^m$).

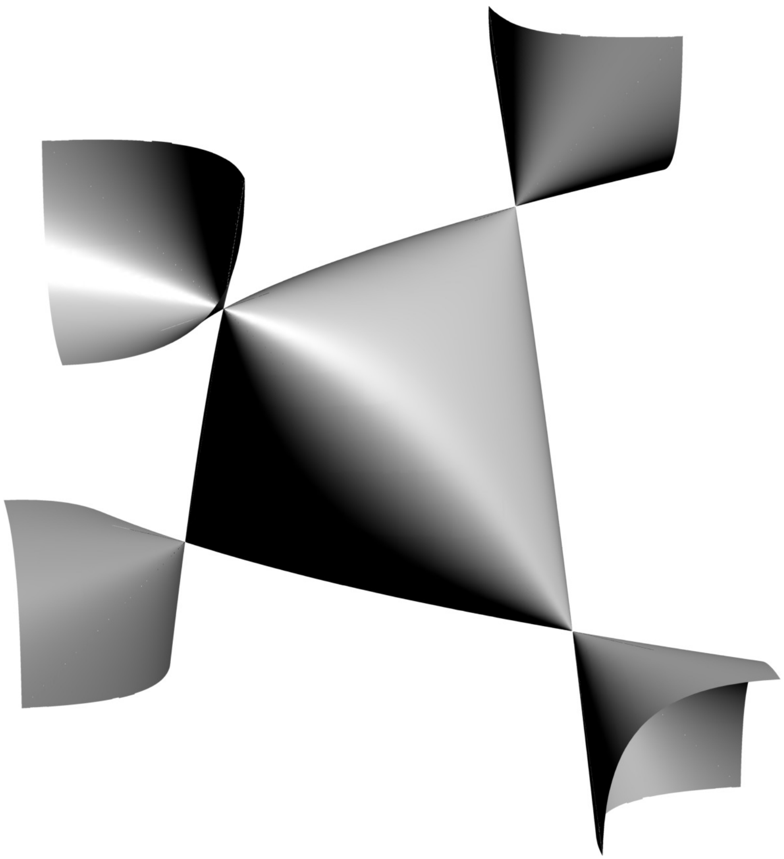
More precisely: if $r = \operatorname{rank} A(x^*)$, then x^* belongs to the set

$$\mathcal{D}_r = \{x \in \mathbb{R}^n : \operatorname{rank} A(x) \leq r\}$$

$\mathcal{D}_r$ is singular (Jacobian not full rank) along $\mathcal{D}_{r-1}$, and perturbing A will not resolve this singularity (ex. $(ax+b)^2$ has a double root even after a perturbation).

A solution of the SDP is a critical point of ℓ on $\mathcal{D}_r$, for some r. So how to remove these singularities ?

9 Cayley's nodal cubic surface (MAXCUT relaxation)



$$A = \begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & x_3 \\ x_2 & x_3 & 1 \end{bmatrix} \succeq 0$$

$\mathcal{D}_1$: 4 singular points

$\mathcal{D}_2$: whole surface.

Incidence condition

1st step. Lifting of the determinantal variety $\mathcal{D}_r$:

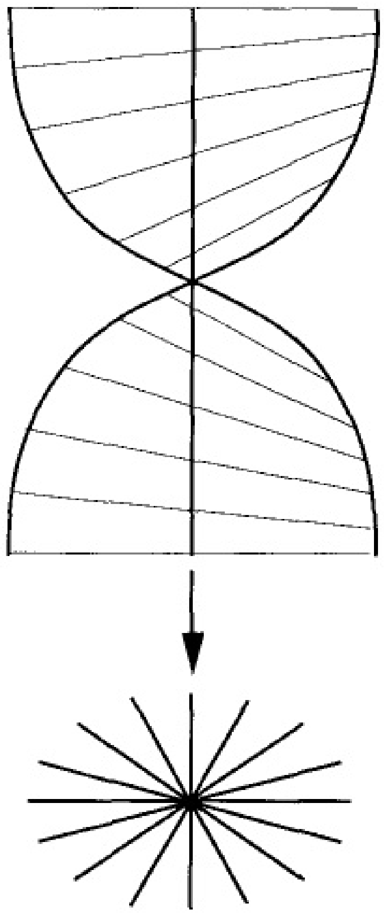
$$A(x)\,Y(y) = A(x)\begin{bmatrix} y_{1,1} & \cdots & y_{1,m-r} \\ \vdots & & \vdots \\ y_{m,1} & \cdots & y_{m,m-r} \end{bmatrix} = 0.$$

If A is generic, the lifted algebraic set $\mathcal{V}_r(A) = \{A(x)\,Y(y) = 0\}$ is **smooth** and **equidimensional**. Its projection on $\mathbb{R}^n$ is $\mathcal{D}_r$.

2nd step Compute critical points of the map $(x, y) \mapsto \ell(x)$ on $\mathcal{V}_r(A)$ (adding Lagrange multipliers).

When A and ℓ are generic, there are **finitely many** critical points, one is a solution of the LMI (and of the SDP with data A, ℓ).

Incidence condition



$$\mathcal{V}_r(A) = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^N : A(x)Y(y) = 0\}$$

$$\downarrow$$

$$\mathcal{D}_r(A) = \{x \in \mathbb{R}^n : \text{rank}\, A(x) \leq r\}$$

A structured case : Hankel LMI

Symmetric linear matrix :

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{12} & x_{22} & x_{23} & x_{24} & x_{25} \\ x_{13} & x_{23} & x_{33} & x_{34} & x_{35} \\ x_{14} & x_{24} & x_{34} & x_{44} & x_{45} \\ x_{15} & x_{25} & x_{35} & x_{45} & x_{55} \end{bmatrix} \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \\ y_{41} & y_{42} & y_{43} \\ y_{51} & y_{52} & y_{53} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{11} & \mathbf{f}_{12} & \mathbf{f}_{13} \\ \mathbf{f}_{21} & \mathbf{f}_{22} & \mathbf{f}_{23} \\ \mathbf{f}_{31} & \mathbf{f}_{32} & \mathbf{f}_{33} \\ \mathbf{f}_{41} & \mathbf{f}_{42} & \mathbf{f}_{43} \\ \mathbf{f}_{51} & \mathbf{f}_{52} & \mathbf{f}_{53} \end{bmatrix}$$

Above $\mathbf{f}_{12} - \mathbf{f}_{21} \in \langle \mathbf{f}_{41}, \mathbf{f}_{42}, \mathbf{f}_{43}, \mathbf{f}_{51}, \mathbf{f}_{52}, \mathbf{f}_{53} \rangle$!

Hankel linear matrix :

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ x_2 & x_3 & x_4 & x_5 & x_6 \\ x_3 & x_4 & x_5 & x_6 & x_7 \\ x_4 & x_5 & x_6 & x_7 & x_8 \\ x_5 & x_6 & x_7 & x_8 & x_9 \end{bmatrix} \begin{bmatrix} y_1 & 0 & 0 \\ y_2 & y_1 & 0 \\ y_3 & y_2 & y_1 \\ 0 & y_3 & y_2 \\ 0 & 0 & y_3 \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 & \mathbf{f}_2 & \mathbf{f}_3 \\ \mathbf{f}_2 & \mathbf{f}_3 & \mathbf{f}_4 \\ \mathbf{f}_3 & \mathbf{f}_4 & \mathbf{f}_5 \\ \mathbf{f}_4 & \mathbf{f}_5 & \mathbf{f}_6 \\ \mathbf{f}_5 & \mathbf{f}_6 & \mathbf{f}_7 \end{bmatrix}$$

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A structured case : Hankel LMI

Representation of the solution:

$$x_i = q_i(t)/q_0(t), \quad q(t) = 0 \text{ with } q_i, q \in \mathbb{Q}[t]$$

Complexity in the generic symmetric case :

$$\approx \text{quadratic in } \binom{n+m^2/2}{n}$$

Hankel LMI :

$$\approx \text{quadratic in } \binom{n+2m}{n}$$

Thanks !