Diagonalisation of Para-hermitian Matrix : is it always possible ?

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- PEVD : eigen vectors/eigen values
- An example of application : blind equalization
- Some useful definitions on Laurent polynomials
- Smith form over the ring of Laurent polynomial matrices
- Properties of para-hermitian and para-unitary matrices
- Order vs degree of a polynomial matrix
- Example of a non diagonalisable PH matrix
- and then ...

Polynomial Eigenvalue Decomposition

Let $\boldsymbol{M}(z)$ be a Laurent polynomial matrix $\boldsymbol{M}(z) \in \mathbb{C}^{n \times n}[z, z^{-1}]$:

$$\boldsymbol{M}(z) = \sum_{k=m}^{p} \boldsymbol{M}_k z^k, \boldsymbol{M}_k \in \mathbb{C}^{n \times n}, m, p \in \mathbb{Z}, m \leq p$$

PEVD [McWhirter] :

Given $\boldsymbol{M}(z) \in \mathbb{C}^{n \times n}[z, z^{-1}]$, find $\lambda(z) \in \mathbb{C}[z, z^{-1}]$, $\underline{v}(z) \in \mathbb{C}^n[z, z^{-1}]$ s.t.

$$\boldsymbol{M}(z)\underline{v}(z) = \lambda(z)\underline{v}(z), \forall z$$

Here, the "eigenvalues" are (Laurent) polynomial.

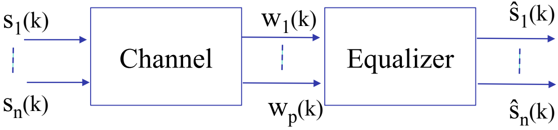
Eigenvalue of a polynomial matrix [Lancaster] :
roots of $\det(\sum_{k=0}^{m} \lambda^k \boldsymbol{M}_k) = 0$, so $\lambda \in \mathbb{C}$.

PEVD for para-hermitian matrices and "orthonormal" eigenvectors
(analog : each hermitian matrix can be diagonalized by an unitary matrix)

Blind Equalization

Problem setting

n sources p observations



Blind equalization : channel and sources are unknown

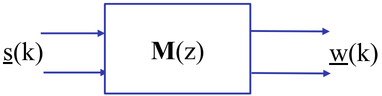
Find approximation $\hat{s}_i$ of sources s_i

Hypotheses

- white i.i.d. sources
- number of sources = number of observations
- convolutive mixture, FIR channel : $\boldsymbol{C}(z) \in \mathbb{C}^{n \times n}[z^{-1}]$

Non unique solution : up to $\boldsymbol{\Delta}(z)\boldsymbol{P}$

with $\boldsymbol{\Delta}(z) = diag\{z^{-i}\}$ and $\boldsymbol{P}$ permutation



Let $\boldsymbol{\Gamma}_w(z)$ be the Cross Spectral Density of observations :

$$\boldsymbol{\Gamma}_w(z) = \boldsymbol{M}(z)\boldsymbol{\Gamma}_s(z)\boldsymbol{M}^H(\frac{1}{z^*})$$

- if $\boldsymbol{\Gamma}_s(z) = \boldsymbol{I}$, then $\boldsymbol{\Gamma}_w(z) = \boldsymbol{M}(z)\boldsymbol{M}^H(\frac{1}{z^*})$, and

$$\boldsymbol{\Gamma}_w^H(\frac{1}{z^*}) = \boldsymbol{\Gamma}_w(z)$$

  $\boldsymbol{\Gamma}_w(z)$ is a para-hermitian matrix
  (even if $\boldsymbol{M}(z) \in \mathbb{C}^{n\times n}[z^{-1}]$, $\boldsymbol{\Gamma}_w(z) \in \mathbb{C}^{n\times n}[z, z^{-1}]$)
- if moreover $\boldsymbol{\Gamma}_w(z) = \boldsymbol{I}$ (whitened observations), then

$$\boldsymbol{M}(z)\boldsymbol{M}^H(\frac{1}{z^*}) = \boldsymbol{I}$$

  $\boldsymbol{M}(z)$ is a para-unitary matrix

Some useful definitions and properties of polynomials

Polynomials Let $\mathbb{C}[z]$ be the ring of polynomials

$$p(z) = \sum_{i=0}^{n} p_i z^i \text{ with } n \in \mathbb{N}, p_i \in \mathbb{C}$$

if $p_n \neq 0$, $\deg(p) = n$, moreover if $p_n = 1, p$ is monic.
if $p_n = 1$ and $p_0 \neq 0$, p is L-monic (no roots in 0)
(analog definitions for $p \in \mathbb{C}[z^{-1}]$)
Laurent polynomials Let $\mathbb{C}[z, z^{-1}]$ be the ring of Laurent polynomials :

$$p(z) = \sum_{i=m}^{n} p_i z^i \text{ with } m, n \in \mathbb{Z}, m \leq n, p_i \in \mathbb{C}$$

if $p_m p_n \neq 0$, $d(p) = n - m$ is the L-degree of p.
if $p \in \mathbb{C}[z], d(p) = \deg(p)$ iff $z = 0$ is not a root of p.

$\mathbb{C}[z, z^{-1}]$ is an Euclidean ring, so a Principal Ideal Domain (PID)
Units of $\mathbb{C}[z, z^{-1}]$: non-zero monomials

$$p(z) = az^{\alpha}, a \in \mathbb{C}^*, \alpha \in \mathbb{Z}$$

Unicity of gcd : impose the gcd to be L-monic (no roots in 0 neither ∞)

Para-conjugation :

$$\widetilde{p}(z) = p^*(\frac{1}{z^*}), \forall z \in \mathbb{C}^*$$

rmq : if $p(z) \in \mathbb{C}[z]$ then $\widetilde{p}(z) \in \mathbb{C}[z^{-1}]$.

Para-hermitian polynomial : $\widetilde{p} = p \Rightarrow p(z) = \sum_{i=-d}^{d} p_i z^i, p_{-i} = p_i^*$.

Para-unitary polynomial : $\widetilde{p}p = 1 \Rightarrow p(z) = e^{\jmath\theta} z^{\alpha}, \theta \in \mathbb{R}, \alpha \in \mathbb{Z}$.

Laurent polynomial matrix $\mathbb{C}^{n\times n}[z, z^{-1}]$

- a L-polynomial with matrix-valued coefficients

$$\boldsymbol{M}(z) = \sum_{k=m}^{p} \boldsymbol{M}_k z^k, \boldsymbol{M}_k \in \mathbb{C}^{n\times n}, m, p \in \mathbb{Z}, m \leq p$$

- or... a matrix with L-polynomial entries

$$\boldsymbol{M}(z) = (m_{ij}(z)) \text{ with } m_{ij}(z) = \sum_{k=m}^{p} m_{ijk} z^{-k}$$

i and j "space"-indices and k time-indice

order of $\boldsymbol{M} = p - m$ if $\boldsymbol{M}_p$ and $\boldsymbol{M}_m$ are non zero

$\mathbb{C}^{n\times n}[z,z^{-1}]$ is the ring of L-polynomial matrices

L-unimodular matrices : Units of $\mathbb{C}^{n\times n}[z,z^{-1}]$

$$\det(\boldsymbol{M}(z)) = az^{\alpha}, a \in \mathbb{C}^*, \alpha \in \mathbb{Z}$$

$d\left(\det(\boldsymbol{M}(z))\right) = 0$: zeros at 0 or infinity

Smith form over $\mathbb{C}^{n\times n}[z,z^{-1}]$:
Let $\boldsymbol{M} \in \mathbb{C}^{n\times n}[z,z^{-1}]$, $\exists$ L-unimodular $\boldsymbol{U}_1, \boldsymbol{U}_2 \in \mathbb{C}^{n\times n}[z,z^{-1}]$

$$\boldsymbol{U}_1(z)\boldsymbol{M}(z)\boldsymbol{U}_2(z) = \boldsymbol{\Lambda}(z)$$

with $\boldsymbol{\Lambda}(z)$ diagonal

Invariant polynomials

$$\mathbf{\Lambda}(z) = \begin{bmatrix} \lambda_1(z) & & & \\ & \ddots & & 0 \\ & & \lambda_r(z) & \\ & 0 & & 0 \end{bmatrix}$$

- λ_i *unique up to a multiplication by a monomial.*
- λ_i divides λ_{i+1}.
- r normal rank of $\boldsymbol{M}$.

To ensure unicity : λ_i L-monic (monic polynomial and no roots in 0)
Invariant polynomials can be computed as

$$\lambda_i(z) = \frac{\Delta_i(\boldsymbol{M}(z))}{\Delta_{i-1}(\boldsymbol{M}(z))}$$

$\Delta_i(\boldsymbol{M}(z))$ L-monic gcd of $i \times i$ minors of $\boldsymbol{M}$.

Examples

$$\boldsymbol{A}(z) = \begin{bmatrix} z & 0 \\ z & z \end{bmatrix}, \text{ order}(\boldsymbol{A}){=}0$$

$\gcd\{z, z, z\} = z, \det \boldsymbol{A}(z) = z^2$, so $\boldsymbol{A}(z) \overset{S}{\sim} z\boldsymbol{I}$

L-gcd$\{z, z, z\} = 1$, L-gcd$\{z^2\} = 1$, so $\boldsymbol{A}(z) \overset{LS}{\sim} \boldsymbol{I}$.

$$\boldsymbol{B}(z) = \begin{bmatrix} z & 0 \\ z-1 & z \end{bmatrix}, \text{ order}(\boldsymbol{B}){=}1\ ,\ \boldsymbol{B}(z) \overset{S}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & z^2 \end{bmatrix} \text{ and } \boldsymbol{B}(z) \overset{LS}{\sim} \boldsymbol{I}$$

$\boldsymbol{U}_1(z)\boldsymbol{B}(z)\boldsymbol{U}_2(z) = \boldsymbol{I}$ with

$$\boldsymbol{U}_1(z) = \begin{bmatrix} 1 & -1 \\ -z^{-1} + z^{-2} & z^{-1} \end{bmatrix} \text{ and } \boldsymbol{U}_2(z) = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} \text{ (non unique).}$$

Para-conjugation :

$$\widetilde{\boldsymbol{M}}(z) = \boldsymbol{M}^H(\frac{1}{z^*}), \forall z \in \mathbb{C}^*$$

then, on the unit circle : $\widetilde{\boldsymbol{M}}(z) = \boldsymbol{M}^H(z)$.

Para-hermitian property :

$$\boldsymbol{M}(z) = \widetilde{\boldsymbol{M}}(z)$$

extension of symmetric ($\boldsymbol{M} \in \mathbb{R}^{n \times n}$) or hermitian ($\boldsymbol{M} \in \mathbb{C}^{n \times n}$) property

Para-unitary property :

$$\boldsymbol{M}(z)\widetilde{\boldsymbol{M}}(z) = \boldsymbol{I}$$

extension of unitary property

Some properties of PH and PU matrices

- if $\boldsymbol{H}(z)$ is PH $\boldsymbol{H}(z) = \widetilde{\boldsymbol{H}}(z)$

$$\boldsymbol{H}(z) = \sum_{k=-d}^{d} \boldsymbol{H}_k z^k \text{ with } \boldsymbol{H}_{-k} = \boldsymbol{H}_k^H, \forall k$$

  the order of $\boldsymbol{H}(z)$ is even $(2d)$

- if $\boldsymbol{U}$ is PU $\widetilde{\boldsymbol{U}}(z)U(z) = \boldsymbol{I}$ of order l, $\boldsymbol{U}(z) = z^m \sum_{k=0}^{l} \boldsymbol{U}_k z^k$

$$\begin{cases} \sum_{k=0}^{l} \boldsymbol{U}_k \boldsymbol{U}_k^H = \boldsymbol{I} \\ \sum_{j=k}^{l} \boldsymbol{U}_j \boldsymbol{U}_{j-k}^H = \mathbf{0} \ \forall \mathbf{k} \in \{\mathbf{1}, \ldots, \mathbf{l}\} \end{cases}$$

L-invariant polynomials properties

- L-unimodular matrix ($\det(\boldsymbol{U}) = az^{\alpha}$) : $\lambda_i(z) = 1$, $\forall i$
  $\det \boldsymbol{U}(z) = az^{\alpha} = \det \boldsymbol{U}_1(z) \det \boldsymbol{\Lambda}(z) \det \boldsymbol{U}_2(z)$,
  so $\det \boldsymbol{\Lambda}(z) = cz^{\beta}$, but λ_i are L-monic, so $\lambda_i(z) = 1, \forall i$.
- Para-unitary matrix ($\boldsymbol{U}\widetilde{\boldsymbol{U}} = \boldsymbol{I}$) : $\lambda_i(z) = 1, \forall i$
  $\det \boldsymbol{U}(z)\widetilde{\boldsymbol{U}}(z) = 1 = cz^{\alpha} \prod_{i=1}^{n} \lambda_i(z) c^* z^{-\alpha} \prod_{i=1}^{n} \widetilde{\lambda}_i(z)$
  so λ_i invertible and as λ_i L-monic, $\lambda_i = 1$.

Para-unitary matrix $\overset{LS}{\sim}$ L-unimodular matrix

- Para-hermitian matrix ($\boldsymbol{H} = \widetilde{\boldsymbol{H}}$) : λ_i self-inversive

$$\lambda_i(z) = e^{\jmath\theta_i} z^{m_i} \widetilde{\lambda}_i(z) \text{ with } \theta_i \in \mathbb{R}, m_i = \deg(\lambda_i)$$

$\boldsymbol{\Lambda}_{\widetilde{H}} \overset{LS}{\sim} \widetilde{\boldsymbol{\Lambda}_H}$ but $\boldsymbol{\Lambda}_{\widetilde{H}}$ is not the L-Smith form of $\boldsymbol{H}$
because $\widetilde{\boldsymbol{\Lambda}_H} \in \mathbb{C}^{n\times n}[z^{-1}]$ (and $\boldsymbol{\Lambda}_H = \boldsymbol{\Lambda}_{\widetilde{H}} \in \mathbb{C}^{n\times n}[z]$)

Order vs Degree

- Order defined for polynomial matrices [Kailath, Vaidyanathan],
- Degree defined for proper rational matrices : sum of the degrees of the denominators of the Smith Mc Millan form, it is the minimum number of delays to implement $\boldsymbol{M}$.

Example :

$$\boldsymbol{H}(z) = \begin{bmatrix} \frac{1}{z} & \frac{1}{z} \\ \frac{1}{z} & \frac{1+z^2}{z^2} \end{bmatrix} = \boldsymbol{H}_{-2}z^{-2} + \boldsymbol{H}_{-1}z^{-1} + \boldsymbol{H}_0, \text{ order of } \boldsymbol{H}(z) \text{ is } 2$$

Let $\boldsymbol{N}(z) = z^2\boldsymbol{H}(z)$ polynomial.

Smith form of $\boldsymbol{N}$: $\boldsymbol{S}(z) = \begin{bmatrix} 1 & 0 \\ 0 & z(1+z^2) - z^2 \end{bmatrix}$

Smith Mc Millan form of $\boldsymbol{H}(z)$: $\boldsymbol{SM}(z) = \frac{1}{z^2}\boldsymbol{S}(z) = \begin{bmatrix} \frac{1}{z^2} & 0 \\ 0 & \frac{1-z+z^2}{z} \end{bmatrix}$

Mc Millan degree of $\boldsymbol{H}(z)$ is 2+1=3.

- Degree of Laurent polynomial matrices (proposition)

Let $\boldsymbol{H}(z) = \sum_{k=m}^{p} \boldsymbol{H}_k z^k, m \leq p, \boldsymbol{H}_m$ and $\boldsymbol{H}_p \neq \mathbf{0}$.
Define the associated causal matrix (polynomial in z^{-1}) :

$$\overline{\overline{\boldsymbol{H}}}(z) = z^{-p}\boldsymbol{H}(z)$$

order of $\boldsymbol{H} = p - m$
L-degree of $\boldsymbol{H}$: McMillan degree of $\overline{\overline{\boldsymbol{H}}}$.
Example : $\boldsymbol{H}(z) = \begin{bmatrix} 1 & 1 \\ 1 & z^{-1} + z \end{bmatrix} = \boldsymbol{H}_{-1}z^{-1} + \boldsymbol{H}_0 + \boldsymbol{H}_1 z$, order 2
$\overline{\overline{\boldsymbol{H}}}(z) = z^{-1}\boldsymbol{H}(z) = \begin{bmatrix} z^{-1} & z^{-1} \\ z^{-1} & 1 + z^{-2} \end{bmatrix}$ (previous example), L-degree 3

Property

If $\boldsymbol{U}(z)$ is para-unitary, its order and its L-degree are equal.

Proof : based on

- Vaidyanathan's factorization of a FIR paraunitary matrix :

$$\overline{\overline{\boldsymbol{U}}}(z) = \boldsymbol{R}_0 \boldsymbol{Z}(z) \boldsymbol{R}_1 \ldots \boldsymbol{Z}(z) \boldsymbol{R}_{N-1} \boldsymbol{Z}(z) \boldsymbol{R}_N \begin{bmatrix} 1 & 0 \\ 0 & \pm 1 \end{bmatrix}$$

$$\boldsymbol{Z}(z) = \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix}, \boldsymbol{R}_i = \begin{bmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{bmatrix}, \ N = \text{MacMillan degree.}$$

- and that, by definition $\overline{\overline{\boldsymbol{U}}}_0 \neq \boldsymbol{0}$

Diagonalization of a PH matrix

Proposition :

Let $\boldsymbol{H}(z)$ be a PH matrix, does there exist $\boldsymbol{U}(z)$ a PU matrix such that :

$$\widetilde{\boldsymbol{U}}(z)\boldsymbol{H}(z)\boldsymbol{U}(z) = \boldsymbol{\Lambda}(z)$$

with $\boldsymbol{\Lambda}(z)$ a Laurent polynomial diagonal matrix ?

(an hermitian matrix is diagonalizable by a unitary matrix)

- not always true (see example)
- if exist, "eigen vectors" are orthonormal w.r.t

$$\langle \underline{u}(z), \underline{v}(z) \rangle = \widetilde{\underline{u}}(z)\underline{v}(z) = \underline{u}^H(\frac{1}{z^*})\underline{v}(z), \forall z \in \mathbb{C}^*$$

- $\boldsymbol{\Lambda}(z)$ can be approximated by a rational matrix, and by truncation, a polynomial matrix [Weiss]

Example

Let $\boldsymbol{H}(z) = \begin{bmatrix} 1 & 1 \\ 1 & -2z^{-1} + 6 - 2z \end{bmatrix}, \boldsymbol{H} = \widetilde{\boldsymbol{H}}, \det \boldsymbol{H}(z) = -2z^{-1} + 5 - 2z$

L-Smith form : $\boldsymbol{S}(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 - \frac{5}{2}z + z^2 \end{bmatrix}$

Suppose $\exists$ $\boldsymbol{U}$ paraunitary s.t. $\widetilde{\boldsymbol{U}}\boldsymbol{H}\boldsymbol{U} = \boldsymbol{\Lambda}$; $\boldsymbol{S} \overset{LS}{\sim} \boldsymbol{\Lambda}$,

So, "eigen values" have to verify :

$$\text{L-gcd}\{\lambda_1, \lambda_2\} = 1 \text{ and } \lambda_1(z)\lambda_2(z) = c' z^{\alpha'} \det \boldsymbol{H}(z).$$

But λ_i, if exist, are parahermitian, so (up to a permutation) :

$\lambda_1(z) = c, c \in \mathbb{R}^*$ and $\lambda_2(z) = dz^{\beta}(1 - \frac{5}{2}z + z^2)$.

Now, parametrizing $\boldsymbol{H}(z)\underline{v}(z) = c\underline{v}(z), c \in \mathbb{R}^*$ leads to a system without any solution.

There exists no polynomial unitary matrix s.t. $\widetilde{\boldsymbol{U}}\boldsymbol{H}\boldsymbol{U} = \boldsymbol{\Lambda}$.

($\boldsymbol{H}$ as order 2 but degree 4)

- The PEVD has not always an exact solution.
- There exits approximated solution at least on the unit circle.
- There exit iterative algorithms (Jacobi type, Extended Givens rotation with delay).

Open problem

- On which condition does there exist a polynomial solution ? (degree vs order) ?
- Can we write this problem with tensors of order 3 (coefficient relations of PH and PU matrices) ?

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