Introduction to elliptic curve cryptography
Attacks on the discrete logarithm problem

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The discrete logarithm problem (DLP)

The Discrete Logarithm Problem (DLP) in $\mathbb{G}$

Given $P, Q \in \mathbb{G}$ find (if it exists) λ such that

$$Q = \lambda P$$

The simplest attack: exhaustive search can find $\lambda \in \{1, \ldots, \#G\}$ in time $O(\#G)$

Generic attacks

Pohlig-Hellman reduction

Let $\mathbb{G}$ be a group such that $\mathbb{G} \simeq \prod_{i=1}^{N} \mathbb{Z}/p_i^{\alpha_i}\mathbb{Z}$.

Let $n = \prod p_i^{\alpha_i}$ be the order of $\mathbb{G}$.

- For every i, solve the discret logarithm in $\mathbb{G}_i = \mathbb{Z}/p_i^{\alpha_i}\mathbb{Z}$.
- Iteratively compute $\lambda \pmod{p^{\alpha_i}}$, for $i \in \{1 \dots N\}$
- Use the Chinese Remainder Theorem to compute the the discrete log in $\mathbb{G}$.

Pohlig-Hellman reduction : an example

Consider $E : y^2 = x^3 + 225x + 239$ over a finite finite of char 359.

The number of points is $\#E(\mathbb{F}_q) = 2^4 \cdot 23$ and $P = (2, 112)$ is a point of order $2^4 \cdot 23$.

The point $Q = (327, 174)$ is such that there is $x \in \mathbb{Z}$ with $xP = Q$.

- Mod 23: Solve $x(16P) = 16Q \Rightarrow x = 12 \pmod{23}$.
- Mod 2^4: Solve $x(23P) = 23Q$.

We write $x = x_0 + 2x_1 + 2^2x_2 + 2^3x_3$ and compute $23P = (71, 117)$.

$$\begin{aligned} (x_0 + 2x_1 + 2^2x_2 + 2^3x_3)(71, 117) &= (90, 343) \\ \Rightarrow 2^3x_0(71, 117) &= 2^3(90, 343) \\ \Rightarrow x_0 &= 1. \end{aligned}$$

Pohlig-Hellman reduction : an example

$$\begin{aligned}(1+2x_1+2^2x_2+2^3x_3)(71,117) &= (90,343) \Rightarrow \\ (2x_1+2^2x_2+2^3x_3)(71,117) &= (90,343)-(71,117) \Rightarrow \\ \Rightarrow x_1+2x_2+2^2x_3(93,189) &= (93,189) \Rightarrow x_1=1\end{aligned}$$

$$2x_2+2^2x_3(93,189)=0 \Rightarrow x_2=0, x_3=0.$$

To sum up, we have

$$\begin{array}{l} x \equiv 12 \pmod{23} \\ x \equiv 3 \pmod{2^4} \end{array} \Rightarrow \quad x \equiv 35 \pmod{2^4 \cdot 23}$$

The Pohlig-Hellman reduction : complexity

Let $\mathbb{G} \simeq \prod_{i=1}^{N} \mathbb{Z}/p_i^{\alpha_i}\mathbb{Z}$ and $n = \prod p_i^{\alpha_i}$.

Denote by $DLP(k)$ the complexity of solving DLP in a group of order k.

The complexity of the Pohlig-Hellman algorithm is

$$O(\sum_{i=1}^{N} \alpha_i DLP(p_i)).$$

The DLP in $\mathbb{G}$ is as hard as the DLP in its largest group of prime order.

Le paradoxe des anniversaires

- Si on choisit k balles parmi N balles de manière aléatoire et independante, alors la probabilité que deux soient identiques est $> \frac{1}{2}$ si $k \geq \sqrt{N}$.
- Si $N = 365$, alors la probilité que parmi k personnes se trouvant dans une salle à un moment donné il y ait 2 avec le même jour d'anniversaire est $> \frac{1}{2}$ si $k > 23$.

Pollard's rho method

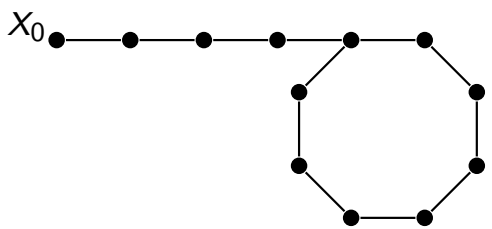
- Let $(\mathbb{G}, +)$ be a group of prime order r and $P, Q \in \mathbb{G}$. Find x s.t. $xP = Q$.
- Partition $\mathbb{G}$ into three sets S_1, S_2, S_3.
- Define the following sequence: $X_0 = P$,

$$X_{i+1} = f(X_i) = \begin{cases} X_i + P & \text{if } X_i \in S_1 \\ 2X_i & \text{if } X_i \in S_2 \\ X_i + Q & \text{if } X_i \in S_3 \end{cases}$$

Every $X_i = u_i P + v_i Q$, with $u_i, v_i \in \mathbb{Z}$ given by

$$u_{i+1} = \begin{cases} u_i + 1 & \text{if } X_i \in S_1 \\ 2u_i & \text{if } X_i \in S_2 \\ u_i & \text{if } X_i \in S_3 \end{cases} \qquad v_{i+1} = \begin{cases} v_i & \text{if } X_i \in S_1 \\ 2v_i & \text{if } X_i \in S_2 \\ v_i + 1 & \text{if } X_i \in S_3 \end{cases}$$

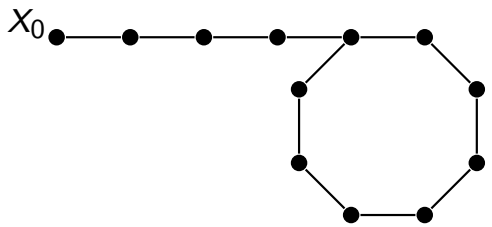
Pollard's rho method



There are m and n s.t. $X_n = X_{m+n}$.

$$u_n P + v_n Q = u_{m+n} P + v_{m+n} Q \Rightarrow x = -\frac{u_n - u_{m+n}}{v_n - v_{m+m}}$$

Pollard's rho method



There are m and n s.t. $X_n = X_{m+n}$.

By the birthday paradox $m + n \sim \sqrt{r}$.

Time complexity $O(\sqrt{r})$

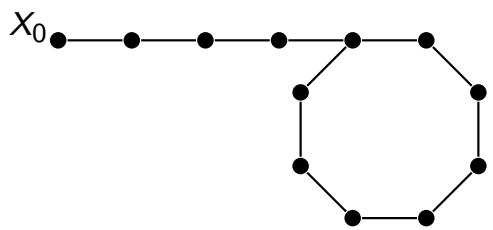
Memory complexity $O(\sqrt{r})$

Floyd's algorithm

$X_0 = P,\ Y_0 = f(P)$

$X_{i+1} = f(X_i)$

$Y_{i+1} = f(f(Y_i))$



Actually $Y_i = X_{2i}$.

There exists e s.t. $X_e = X_{2e}$ and that $n \leq e \leq n + m$.

Floyd's cycle finding algorithm

1. Compute $X = P$, $(u_X, v_x) = (1, 0)$.
2. Compute $Y = f(P)$, $(u_Y, v_Y) = f(1, 0)$.
3. while $(X \neq Y)$ do
$$X \leftarrow f(X); (u_X, v_X) = f(u_X, v_X)$$
$$Y \leftarrow f(f(Y)); (u_Y, v_Y) = f(f(u_Y, v_Y))$$
4. Return $-(u_X - u_Y)/(v_X - v_Y)$

Complexity $O(\sqrt{r})$ in time and $O(1)$ in memory